Algebraic properties of the ring of germs of CR functions and solutions of integrable structures

Giuseppe Della Sala

In the past few years, a number of works have started to explore certain local properties of the space of CR functions of class C^∞ defined on a smooth CR manifold - in particular, on a real hypersurface. The picture that emerged is rather peculiar: while the tangential CR equations and related PDEs have been the subject of intense study for several decades, as it turns out, the algebraic side of their ring of solution is not as well understood as their analytic side, and some surprisingly elementary questions are as yet unanswered. Among these, for instance, is the characterization of the maximal ideal of the ring of germs of CR functions at a given point p of a CR manifold M.

One tool that has played an important role in these investigations is the *Borel map*, assigning to a smooth CR germ its formal power series expansion at p, suitably expressed in terms of basic solutions. The behavior of the Borel map (especially its surjectivity/injectivity) is strongly connected to the geometry of M, in particular to the presence of a peak function; a recent conjecture, suggesting a link between the Borel map and the structure of the polynomial hull of M, has been verified in several particular cases, but its full scope seems out of reach at the moment.

The purpose of this mini-course is to review some recent results that are related to this circle of ideas. An outline of the topics to be covered can be given as follows:

(1) Rings of germs of CR functions and the basic properties of the Borel map
(2) Some analytic and geometric conditions for the surjectivity/injectivity of the Borel map
(3) The structure of maximal ideals and principal manifold ideals
(4) Discussion of some open problems and (given time) of the setting of integrable structures