Eigenvalue problem on strictly pseudoconvex CR manifolds

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Strictly pseudoconvex CR manifolds

A CR manifold is a pair $(M, T^{1,0})$, where M is a (smooth) manifold and $T^{1,0}M$ is complex sub-bundle of $\mathbb{C}TM$ which satisfies

- $T^{1,0}M \cap \overline{T^{1,0}M} = \{0\}$.
- $[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M$.

A CR manifold M is of *hypersurface type* if

$$\dim_{\mathbb{R}} M = 2 \dim_{\mathbb{C}} T^{1,0}M + 1.$$

Real hypersurfaces in $\mathbb{C}^{n+1}$ are of hypersurface type.
We shall confine to hypersurface type CR manifolds. Let $H(M) = \Re T^{1,0}M$ be the codimension one, real vector sub-bundle of TM. There exists a real 1-form θ such that

$$H(M) = \ker \theta.$$

M is strictly pseudoconvex if the Levi form $d\theta$ is definite on $H(M)$.

Pseudo-Hermitian structures

Let M be a s.p.c CR manifold. We can find a 1-form θ such that $d\theta$ is positive definite on $H(M)$. Then $d\theta$ induces a metric G_θ on $H(M)$ via $G_\theta(X,Y) = d\theta(X, JY)$, J being the complex structure on $H(M)$. Then θ is called a pseudo-Hermitian structure.
The Reeb vector field T is uniquely determined by

$$d\theta \lrcorner T = 0, \quad \theta(T) = 1.$$

This T is everywhere transverse to $H(M)$. The Webster metric is

$$g_\theta = G_\theta + \theta \cdot \theta.$$

Fix such θ, any pseudo-Hermitian structure on M is given by

$$\hat{\theta} = e^\sigma \theta, \quad \sigma \in C^\infty(M, \mathbb{R}).$$

Following Webster, we call M with the data $T^{1,0}M$ and θ a pseudo-Hermitian manifold.

Tanaka-Webster connection

Let $\{\theta^\alpha\}$ be an *admissible coframe* for $(T^{1,0})^*$, $\theta^\alpha(T) = 0$. Then $\{\theta^\alpha, \theta^{\bar\alpha}, \theta\}$ is a coframe for $\mathbb{C}TM^*$. Let $\{Z_\alpha\}$ be a local frame for $T^{1,0}M$ dual to $\{\theta^\alpha\}$, $\{Z_\alpha, Z_{\bar\alpha}, T\}$ is a local frame for $\mathbb{C}TM$. Then

$$-id\theta = h_{\alpha\bar\beta}\theta^\alpha \wedge \theta^{\bar\beta}, \quad \text{(summation convention)}$$

for some Hermitian matrix $h_{\alpha\bar\beta}$, called the Levi matrix. The Tanaka-Webster connection forms ${\omega_\alpha}^\beta$ and torsion forms $\tau_\beta = A_{\beta\alpha}\theta^\alpha$ are defined by the relations

$$d\theta^\beta = \theta^\alpha \wedge {\omega_\alpha}^\beta + \theta \wedge \tau^\beta, \quad \omega_{\alpha\bar\beta} + \omega_{\bar\beta\alpha} = dh_{\alpha\bar\beta},$$

$$A_{\alpha\beta} = A_{\beta\alpha}.$$

Tanaka-Webster connection associated to θ is defined via

$$\nabla Z_\alpha = {\omega_\alpha}^\beta \otimes Z_\beta, \quad \nabla T = 0, \quad \text{etc.}$$

Tanaka-Webster vs. Levi-Civita

Let $\tilde{\nabla}$ be the (torsionless) Levi-Civita connection associated to g_θ.
The Tanaka-Webster connection always has torsion.

$$\begin{aligned}\widetilde{\nabla}_X Y =& \nabla_X Y + \theta(Y) AX + \frac{1}{2}\left(\theta(Y)\phi X + \theta(X)\phi Y\right) \\ & - \left[\langle AX, Y\rangle + \frac{1}{2} d\theta(X, Y)\right] T.\end{aligned}$$

Here ϕ is the extension of J to TM by setting $\phi(T) = 0$, and $A(X) = \tau(T, X)$ where τ being the torsion of ∇.
The relation is particularly useful when the "pseudo-Hermitian" torsion $A_{\alpha\beta}$ vanishes (Sasakian case).

Cauchy–Riemann $\bar{\partial}_b$-operator and Kohn-Laplacian

The Cauchy–Riemann operator ∂_b is defined by

$$\partial_b f = (Z_\alpha f)\theta^\alpha, \quad \bar{\partial}_b f = (Z_{\bar{\beta}} f)\theta^{\bar{\beta}}, \quad df = \partial_b f + \bar{\partial}_b f + (Tf)\theta.$$

The divergence operator δ_b takes (0,1)-forms to functions:

$$\delta_b(\sigma_\alpha \theta^\alpha) = \sigma_{\alpha,}{}^{\alpha},$$

On compact manifolds without boundary, Stokes theorem implies

$$\int_M \delta_b \sigma \theta \wedge (d\theta)^n = 0$$

Consequently, $\partial_b^* = -\delta_b$ and so

$$\Box_b f = -h^{\alpha\bar{\beta}} \nabla_\alpha \nabla_{\bar{\beta}} f.$$

By definition, $\Box_b$ depends on the "scale" θ: Let $\hat{\theta} = e^u \theta$ and $\hat{\Box}_b$ is the Kohn-Laplacian operator that corresponds to θ. Then

$$e^u \, \hat{\Box}_b f = \Box_b f - n \, \langle \partial_b u, \bar{\partial}_b f \rangle. \tag{1}$$

Theorem (Burn-Epstein)

Let (M, θ) be a compact pseudo-Hermitian manifold of dimension 3. Then the spectrum of $\Box_b$ in $(0, +\infty)$ consists of point eigenvalues of finite multiplicity. Moreover, all corresponding eigenfunctions are smooth.

If M is embedded into $\mathbb{C}^N$, then zero is an eigenvalue of infinite dimension; the kernel $\ker \Box_b$ consists of CR functions. If M is non-embeddable, then small eigenvalues converge rapidly to zero.

Theorem (Kohn)

Let (M,θ) be a compact CR manifold (not necessarily strictly pseudoconvex). Then M is CR embeddable into $\mathbb{C}^N$ if and only if the Kohn-Laplacian has closed range.

If M is embeddable,then the spectrum of $\Box_b$ consists of a discrete sequence $0=\lambda_0<\lambda_1<\cdots<\cdots$ converging to infinity.
If M is non-embeddable, then there exists a sequence of eigenvalues decreasing rapidly to zero.
If the dimension is of at least five, by a theorem of Boutet de Monvel, all compact s.p.c CR manifolds are embeddable and hence zero is an isolated eigenvalue of infinite multiplicity.
If the manifold is not embedded, the kernel of $\Box_b$ need not be of infinite dimension. Barrett constructed examples on which CR functions are just constants. There are perturbation of the standard CR structure on the sphere (Rossi's structures) such that the CR functions are "even".

Estimate for the eigenvalue of Kohn-Laplacian

Theorem (Chanillo, Chiu and Yang)

Let (M, θ) be a pseudo-Hermitian manifold of dimension 3. If the Webster scalar curvature is positive and the CR Paneitz operator is nonnegative, then any non-zero eigenvalue of $\Box_b$ satisfies

$$\lambda \geq \frac{1}{2} \min_M R.$$

The CR Paneitz operator in three dimension is given by

$$P_0 \varphi = \left(\varphi_{;\bar{1}}{}^{\bar{1}}{}_1 + i A_{11} \varphi^1 \right)^{;1}$$

This operator first appeared on the work of Graham–Lee. The nonnegativity of CR Paneitz operator is CR invariant (e.g., does not depend on the scale θ). It holds when M admits a transversal symmetry (an infinitesimal CR automorphism that transverses to $H(M)$ at all points).

Example 1

Let M be the standard sphere

$$M := \{(z, w) \in \mathbb{C}^2 \colon |z|^2 + |w|^2 = 1\}.$$

The standard pseudo-Hermitian structure $\theta := i\bar{\partial}\|Z\|^2$. Let $L = \bar{z}\partial_w - \bar{w}\partial z$. Then

$$\Box_b f = -L\overline{L}f.$$

Thus, $\Box_b \bar{z} = \bar{z}$ and $\Box_b \bar{w} = \bar{w}$. The Webster curvature $R = 2$, and the CR Paneitz operator is nonnegative, since the sphere admits a transversal symmetry. Hence, $\lambda_1 = 1$. The eigenspace is spanned by $\bar{z}$ and $\bar{w}$.

Bochner formula for Kohn-Laplacian

Lemma (Chanillo, Chiu, Yang)

Let f be a complex-valued function. Then

$$-\Box_b \left|\overline{\partial}_b f\right|^2 = \left|f_{\overline{\alpha},\overline{\beta}}\right|^2 + \left|f_{\overline{\alpha},\beta}\right|^2 - \frac{n+1}{n}(\Box_b f)_{\overline{\alpha}}\overline{f}_{\alpha} - \frac{1}{n} f_{\overline{\alpha}}\overline{(\Box_b f)_{\alpha}} + R_{\alpha\overline{\beta}} f_{\overline{\alpha}} \overline{f}_{\beta} - \frac{1}{n}\overline{f}_{\alpha}\overline{P_\alpha \overline{f}} + \frac{n-1}{n} f_{\overline{\alpha}}\left(P_\alpha \overline{f}\right).$$

The CR Paneitz operator is given by

$$P_0 f = \nabla^\alpha P_\alpha f, \quad -\int_M f_{\bar{\alpha}}(P_\alpha \bar{f}) = \int_M f P_0 \bar{f} \geq 0.$$

It is always positive if $n \geq 2$ (Graham–Lee). The proof of the estimate follows the proof of the Lichnerowicz estimate using Bochner formula for Laplacian on Riemannian manifolds.

Equality case

Theorem

Let (M, θ) be a compact strictly pseudoconvex manifold of dimension $2n + 1 \geq 5$. Suppose that there exists a positive constant κ such that

$$Ric \geq \kappa$$

then

$$\lambda_1 \geq n\kappa/(n+1).$$

The equality case occurs if and only if (M, θ) is CR equivalent to a sphere in $\mathbb{C}^{n+1}$.

The inequality is just a straightforward generalization of Chanillo-Chiu-Yang's estimate. The characterization of equality case is proved by D.-Li-Wang.

Let (M, θ) be given by $\rho = 0$ where

$$\rho := \sum_{j=1}^{n+1} (\log |z_j|^2)^2 - \epsilon^2, \quad \theta = -i\partial\rho|_M.$$

Then the Webster Ricci curvature is

$$R_{\alpha\bar{\beta}} = \left(\frac{n}{2\epsilon^2}\right) h_{\alpha\bar{\beta}}$$

Thus, $\lambda_1 \geq \frac{n^2}{2(n+1)\epsilon^2}$, provided that $n \geq 2$. On the other hand,

$$\Box_b(\log |z_j|^2) = \frac{n^2}{2\epsilon^2}(\log |z_j|^2).$$

Thus $\lambda_1 \leq \frac{n^2}{2\epsilon^2}$. We suspect that the last value is precisely the first eigenvalue of $\Box_b$ on M.
The case $n = 1$ is less understood. One needs positivity of CR Paneitz operator to obtain the lower bound.

Proof of the characterization of equality case

1. Let f be the "critical" eigenfunction. Formulate the overdetermined PDEs that must be satisfied by f:

$$\nabla_\alpha \nabla_\beta f = 0, \quad \nabla_\beta \nabla_{\bar{\alpha}} f = -cfh_{\beta\bar{\alpha}}, \quad c > 0.$$

2. Prove that $A_{\alpha\beta}$ is of rank-one: $|\bar{\partial}_b f|^2 A_{\alpha\beta} = \psi \bar{f}_\alpha \bar{f}_\beta$.
3. Prove that

$$(n-1)\int_M |\partial_b \psi|^2 |\bar{\partial}_b f|^2 = 0.$$

4. Show that the pseudo-Hermitian torsion $A_{\alpha\beta}$ must vanish.
5. Prove that $D^2 u = Cug_\theta$ where $u = \Re f$ and the Hessian is computed using the Levi-Civita connection.
6. Apply Obata's theorem.

When $n = 1$ there're several places the proof does not work. Under the weaker condition that the manifold is complete?

Let M be given by $\rho = 0$ and $\theta = -i\partial\rho|_M$. Since

$$\Box_b f = \bar{\partial}_b^* \bar{\partial}_b f = -f_{\bar{\alpha},}{}^{\bar{\alpha}} = -h^{\beta\bar{\gamma}}(Z_\beta Z_{\bar{\gamma}} f - \omega_{\bar{\gamma}}{}^{\bar{\sigma}}(Z_\beta) Z_{\bar{\sigma}} f).$$

To calculate $\Box_b$, we need to calculate the connection form $\omega_{\beta\bar{\alpha}}$. This was done by Li-Luk, Webster, etc.
Suppose that $\rho_{j\bar{k}}$ is positive definite and let $\rho^{\bar{k}j}$ be its inverse. Write $|\partial\rho|_\rho^2 = \rho^{j\bar{k}}\rho_j\rho_{\bar{k}}$. Choose a local admissible holomorphic coframe $\{\theta^\alpha\}$, $\alpha = 1, 2, \ldots, n$ on M by

$$\theta^\alpha = dz^\alpha - ih^\alpha\theta, \quad h^\alpha = |\partial\rho|_\rho^{-2}\rho^\alpha = |\partial\rho|_\rho^{-2}\rho_{\bar{j}}\rho^{\alpha\bar{j}}, \qquad (2)$$

We use summation conventions: repeated Latin indices are summed from 1 to $n+1$, while repeated Greek indices are summed from 1 to n.

Explicit formulas for $\Box_b$

This admissible coframe is valid when $\rho_{n+1} \neq 0$. At p with $\rho_{n+1} \neq 0$,

$$d\theta = i h_{\alpha\bar{\beta}} \theta^{\alpha} \wedge \theta^{\bar{\beta}},$$

where the Levi matrix $[h_{\alpha\bar{\beta}}]$ is an $n \times n$ matrix given explicitly:

$$h_{\alpha\bar{\beta}} = \rho_{\alpha\bar{\beta}} - \rho_{\alpha}\partial_{\bar{\beta}} \log \rho_{n+1} - \rho_{\bar{\beta}}\partial_{\alpha} \log \rho_{\overline{n+1}} + \rho_{n+1\overline{n+1}} \frac{\rho_{\alpha}\rho_{\bar{\beta}}}{|\rho_{n+1}|^2}.$$

The inverse $[h^{\gamma\bar{\beta}}]$ of the Levi matrix is given by

$$h^{\gamma\bar{\beta}} = \rho^{\gamma\bar{\beta}} - \frac{\rho^{\gamma}\rho^{\bar{\beta}}}{|\partial\rho|^2_{\rho}}, \quad \rho^{\gamma} = \sum_{k=1}^{n+1} \rho_{\bar{k}}\rho^{\gamma\bar{k}}.$$

Explicit formulas for $\Box_b$

The Tanaka-Webster connection forms are

$$\omega_{\bar{\beta}\alpha} = (Z_{\bar{\gamma}} h_{\alpha\bar{\beta}} - h_{\bar{\beta}} h_{\alpha\bar{\gamma}})\theta^{\bar{\gamma}} + h_\alpha h_{\gamma\bar{\beta}}\theta^\gamma + i h_{\alpha\bar{\sigma}} Z_{\bar{\beta}} h^{\bar{\sigma}}\theta,$$

where $h_\alpha = h_{\alpha\bar{\beta}} h^{\bar{\beta}}$ and the dual frame are given by

$$Z_\alpha = \frac{\partial}{\partial z^\alpha} - \frac{\rho_\alpha}{\rho_{n+1}} \frac{\partial}{\partial z_{n+1}}.$$

The relevant Christoffel is

$$\omega_{\bar{\gamma}}^{\bar{\sigma}}(Z_\beta) = h^{\bar{\sigma}} h_{\beta\bar{\gamma}}$$

Therefore,

$$h^{\beta\bar{\gamma}} \omega_{\bar{\gamma}}^{\bar{\sigma}}(Z_\beta) = n h^{\bar{\sigma}}.$$

Formulas for $\Box_b$: (a) The following holds on M.

$$\begin{aligned}\Box_b f =& h^{\beta\bar{\gamma}}(Z_\beta Z_{\bar{\gamma}} f - \omega_{\bar{\gamma}}{}^{\bar{\sigma}}(Z_\beta) Z_{\bar{\sigma}} f) \\ =& \left(|\partial\rho|_\rho^{-2}\rho^k\rho^{\bar{j}} - \rho^{\bar{j}k}\right) f_{\bar{j}k} + n|\partial\rho|_\rho^{-2}\rho^{\bar{k}} f_{\bar{k}} \\ =& - \operatorname{trace}(i\partial\bar{\partial} f) + |\partial\rho|_\rho^{-2}\langle\partial\bar{\partial} f, \partial\rho\wedge\bar{\partial}\rho\rangle + n|\partial\rho|_\rho^{-2}\langle\partial\rho, \bar{\partial} f\rangle,\end{aligned}$$

(b) Suppose that $(z^1, z^2, \ldots, z^{n+1})$ is a local coordinate system on an open set V. Define the vector fields

$$X_{jk} = \rho_k\partial_j - \rho_j\partial_k, \quad X_{\bar{j}\bar{k}} = \overline{X_{jk}}. \tag{3}$$

Then the following holds on $M \cap V$.

$$\Box_b f = -\frac{1}{2}|\partial\rho|_\rho^{-2}\rho^{p\bar{k}}\rho^{q\bar{j}} X_{pq} X_{\bar{j}\bar{k}} f. \tag{4}$$

These formulas were derived by D.-Li-Lin.

Examples 3

- If M is the sphere with $\theta := i\bar{\partial}(\|Z\|^2)$, then (5) reduces to a well-known formula (e.g., Daryl Geller)

$$\Box_b f = \sum_{j<k} X_{jk} X_{\bar{j}\bar{k}} f. \tag{5}$$

- If M is an ellipsoid: $\rho = \|Z\|^2 + \Re(a_{jk} Z^j Z^k)$, then $\rho_{j\bar{k}} = \delta_{jk}$; then

$$\Box_b f = -|\partial\rho|_\rho^{-2} \sum_{j<k} X_{jk} X_{\bar{j}\bar{k}} f. \tag{6}$$

- If M is given by $\rho = 0$ with

$$\rho = (\log|z|^2)^2 + (\log|z|^2)^2 - \epsilon^2$$

then

$$\Box_b f = (|zw|^2/(2\epsilon^2)) L\overline{L} f$$

with $L = z^{-1}\log|z|^2\partial_w - w^{-1}\log|w|^2\partial_z$.

Ritz-Rayleigh quotient

The characterization of the first positive eigenvalue of $\Box_b$ is

$$\lambda_1 = \inf\left\{\frac{\int_M |\bar{\partial}_b u|^2}{\int_M |u|^2} : u \in \ker \Box_b{}^{\perp}\right\}.$$

Finding a "test function" $u \in \ker \Box_b{}^{\perp}$ is not easy. It should have something to do with Szegö projection.
Another Ritz-Rayleigh type quotient is

$$\lambda_1 = \inf\left\{\frac{\int_M |\Box_b u|^2}{\int_M \bar{u}\,\Box_b u} : u \notin \ker \Box_b\right\}.$$

This is useful in some case. These were proved in D.-Li-Lin.

An useful observation

Let (M, θ) be compact, strictly pseudoconvex pseudohermitian manifold. If there is a smooth non-CR function f on M such that $|\Box_b f|^2 \leq B(z)\Re(\bar{f}\Box_b f)$ for some non-negative function B on M, then

$$\lambda_1 \leq \max_M B(z). \tag{7}$$

If the equality holds, then B must be a constant.
In fact, the hypothesis implies that

$$\lambda_1 \int_M \bar{f}\Box_b f \leq \int_M |\Box_b f|^2 \leq \int_M B(z)\Re(\bar{f}\Box_b f).$$

Then the estimate (6) follows from the Mean Value Theorem.

Upper estimate 1

Theorem (D.-Li-Lin 2016)

Asume that for some j*,* $\Re \rho_{\bar{j}} \tilde{\Delta}_\rho \, \rho_j + \frac{1}{n} |\partial\rho|^2_\rho \, |\tilde{\Delta}\rho_j|^2 \leq 0$ *on* M*, then*

$$\lambda_1(M, \theta) \leq n \max_M |\partial\rho|_\rho^{-2} \tag{8}$$

and the equality holds only if $|\partial\rho|^2_\rho$ *is constant along* M*.*

This theorem hold for a large class of ρ, e.g., if

$$\rho(Z, \bar{Z}) := |z_1|^2 + \psi(z_2, \bar{z}_2) + \text{pluriharmonic terms},$$

then λ_1 is bounded above by $\max |\partial\rho|^{-2}$.
The proof follows from the observation above with $B(z) = n|\partial\rho|_\rho^{-2}$.

Another interpretation

If M is strictly pseudoconvex defined by $\rho = 0$, then there is a function $r[\rho]$ such that,

$$-id\theta = h_{\alpha\bar{\beta}}\theta^{\alpha} \wedge \theta^{\bar{\beta}} + r\partial\rho \wedge \bar{\partial}\rho, \tag{9}$$

This function is called *transverse curvature* by Graham and Lee. When $\rho_{j\bar{k}}$ is definite, then

$$r = |\partial\rho|_{\rho}^{-2}.$$

The estimate above reads

$$\lambda_1 \leq \max_M r[\rho].$$

If the equality holds, then $r[\rho]$ must be a constant.

Upper estimate 2 - Reilly type estimate

Theorem (D.-Li 2017)

Let M be a compact strictly pseudoconvex hypersurface in $\mathbb{C}^{n+1}$ defined by $\rho = 0$ and let $\theta = \iota^*(i/2)(\bar{\partial}\rho - \partial\rho)$. Suppose that there are positive numbers $N > 0$, $\nu > 0$, and a pluriharmonic function ψ defined in a neighborhood of M such that

$$(\rho + \nu)^N - \psi = \sum_{\mu=1}^{K} |f^{(\mu)}|^2. \tag{10}$$

where f^μ are holomorphic for $\mu = 1, 2, \ldots, K$. Then

$$\lambda_1(M, \theta) \leq \frac{n}{v(M,\theta)} \int_M r[\rho]\, \theta \wedge (d\theta)^n + \frac{n(N-1)}{\nu}. \tag{11}$$

If $\psi = 0$ and the equality occurs, then $r[\rho] = \frac{1}{\nu}$ is a constant on M.

Proof of Estimate 2

For anti-CR function $\bar{f}$,

$$\Box_b \bar{f} = nr[\rho]\xi^{\bar{k}} f_{\bar{k}}.$$

Where $\xi = (\xi^k)$ is the transverse vector field determined by

$$\xi \lrcorner \partial\bar{\partial}\rho = r[\rho]\,\bar{\partial}\rho, \quad \partial\rho(\xi) = 1. \tag{12}$$

Compute

$$|\Box_b \bar{f}^{(\mu)}|^2 = n^2 \xi^{\bar{k}} \bar{f}^{(\mu)}_{\bar{k}} \xi^l f^{(\mu)}_l. \tag{13}$$

Summing over $\mu = 1, 2, \ldots, K$, we obtain

$$\begin{aligned} \sum_{\mu=1}^{K} |\Box_b \bar{f}^{(\mu)}|^2 &= n^2 \xi^{\bar{k}} \xi^l \sum_{\mu=1}^{K} \bar{f}^{(\mu)}_{\bar{k}} f^{(\mu)}_l \\ &= n^2 \xi^{\bar{k}} \xi^l \left(N\nu^{N-1} \rho_{l\bar{k}} + N(N-1)\nu^{N-2} \rho_j \rho_{\bar{k}} \right) \\ &= n^2 N \nu^{N-2} \left(\nu\, r + N - 1 \right). \end{aligned} \tag{14}$$

Proof of Estimate 2 continued

Next, observe that by (16) (assuming $\rho_w \neq 0$)

$$\begin{aligned}\sum_{\mu=1}^{K} Z_{\bar\gamma}\bar f^{(\mu)} Z_\sigma f^{(\mu)} &= \sum_{\mu=1}^{K}\left(\bar f^{(\mu)}_{\bar\gamma} - \frac{\rho_{\bar\gamma}}{\rho_{\bar w}}\bar f^{(\mu)}_{\bar w}\right)\left(f^{(\mu)}_\sigma - \frac{\rho_\sigma}{\rho_w} f^{(\mu)}_w\right)\\ &= N\nu^{N-1}\left(\rho_{\sigma\bar\gamma} - \frac{\rho_{\bar\gamma}\rho_{\sigma\bar w}}{\rho_{\bar w}} - \frac{\rho_\sigma\rho_{\bar\gamma w}}{\rho_w} + \frac{\rho_{\bar\gamma}\rho_\sigma\rho_{w\bar w}}{|\rho_w|^2}\right)\\ &= N\nu^{N-1} h_{\sigma\bar\gamma}. \qquad (15)\end{aligned}$$

Therefore,

$$\sum_{\mu=1}^{K} |\bar\partial_b \bar f^{(\mu)}|^2 = h^{\sigma\bar\gamma}\sum_{\mu=1}^{K} Z_{\bar\gamma}\bar f^{(\mu)} Z_\sigma f^{(\mu)} = nN\nu^{N-1}. \qquad (16)$$

Applying the Ritz-Rayleigh characterization of λ_1, we get desired estimate.

Corollary (D.-Li 2017)

Suppose that M is a compact strictly pseudoconvex manifold and $F\colon M \to \mathbb{S}^{2K+1}$ is a CR immersion. Let Θ be the standard pseudohermtian structure on the unit sphere, and let $r_F = r[\rho]$ where $\rho := \sum_{\mu=1}^{K} |F^{(\mu)}|^2 - 1$. Then

$$\lambda_1(M, F^*\Theta) \leq \frac{n}{v(M, F^*\Theta)} \int_M r_F \, F^*\Theta \wedge (dF^*\Theta)^n. \tag{17}$$

If the equality occurs, then $r_F = 1$ on M and $\bar{f}^{(\mu)}$ are eigenfunctions of $\Box_b$ for the first positive eigenvalue λ_1.

This follows immediately from the previous theorem with $N = 1$, $\nu = 1$ and $\psi = 0$.

Example 4

The unit sphere $\mathbb{S}^3$ in $\mathbb{C}^2$ can be defined by $\rho = 0$ with

$$\rho = |z^2|^2 + 2|zw|^2 + |w^2|^2 - 1. \tag{18}$$

Observe that on $\mathbb{S}^3$, $\det H[\rho] = 8$, $J[\rho] = 4$, and the transverse curvature is constant: $r[\rho] = 2$. Since $\partial\rho = 2(\bar{z}dz + \bar{w}dw)$ on $\mathbb{S}^3$, the pseudohermitian $\theta := (i/2)(\bar{\partial}\rho - \partial\rho)$ is twice of the standard pseudohermitian structure on $\mathbb{S}^3$ and hence $\lambda_1(\mathbb{S}^3, \theta) = \frac{1}{2}$. Observe that $(\mathbb{S}^3, \theta)$ is CR immersed into $\mathbb{S}^5 \subset \mathbb{C}^3$ via H. Alexander's map $F(z, w) := (z^2, \sqrt{2}zw, w^2)$ and the corollary above does apply. The constancy of $r[\rho]$ does *not* implies that the equality occurs in (17).

Upper estimate 3

Theorem (D.-Li-Lin 2016)

Let ρ be a smooth strictly plurisubharmonic function defined on an open set U of $\mathbb{C}^{n+1}$, M a compact connected regular level set of ρ, and λ_1 the first positive eigenvalue of $\Box_b$ on M. Let $\gamma(z)$ be the spectral radius of the matrix $[\rho^{j\bar{k}}(z)]$ and $s(z) = \operatorname{trace}[\rho^{j\bar{k}}] - \gamma(z)$. Then

$$\lambda_1 \leq \frac{n^2 \int_M \gamma(z) |\partial\rho|_\rho^{-2}}{\int_M s(z)}. \tag{19}$$

Dropping condition on ρ, the upper bound also depend on the eigenvalues of the complex Hessian $\rho_{j\bar{k}}$.

First, we define

$$C_j = \int_M \frac{|\rho^{\bar{j}}|^2}{|\partial\rho|_\rho^4}, \quad D_j = \int_M \left(\rho^{j\bar{j}} - \frac{|\rho^{\bar{j}}|^2}{|\partial\rho|_\rho^2}\right). \tag{20}$$

From the explicit formula for $\Box_b$, we can compute

$$\Box_b \bar{z}^j = n|\partial\rho|^{-2}\rho^{\bar{j}}. \tag{21}$$

Therefore,

$$\|\Box_b \bar{z}^j\|^2 = n^2 \int_M \frac{|\rho^{\bar{j}}|^2}{|\partial\rho|_\rho^4} = n^2 C_j. \tag{22}$$

Proof of Upper estimate 3

We can also compute

$$|\bar{\partial}_b \bar{z}^j|^2 = \delta_{j\alpha}\delta_{j\beta}\left(\rho^{\alpha\bar{\beta}} - \frac{\rho^{\alpha}\rho^{\bar{\beta}}}{|\partial\rho|^2}\right) = \rho^{j\bar{j}} - \frac{|\rho^{\bar{j}}|^2}{|\partial\rho|^2}. \tag{23}$$

Here without lost of generality, we assume $j \neq n+1$. Therefore,

$$\int_M |\bar{\partial}_b \bar{z}^j|^2 = D_j. \tag{24}$$

Thus, we obtain for all j

$$\lambda_1 \leq n^2 C_j / D_j. \tag{25}$$

Observe that $1/\gamma(z)$ is the smallest eigenvalue of the Hermitian matrix $[\rho_{j\bar{k}}(z)]$, and thus, for all $(n+1)$-vector v^j,

$$\frac{1}{\gamma(z)} \sum_{j=1}^{n+1} |v^j|^2 \leq v^j \rho_{j\bar{k}} v^{\bar{k}}. \tag{26}$$

Plugging $v^j = \rho^j$ into the inequality, we easily obtain $\sum_{j=1}^{n+1} |\rho^j|^2 \leq \gamma(z)|\partial\rho|_\rho^2$. Consequently

$$\sum_j C_j = \sum_{j=1}^{n+1} \int_M \frac{|\rho^j|^2}{|\partial\rho|_\rho^4} \leq \int_M \gamma(z)|\partial\rho|_\rho^{-2}, \tag{27}$$

and therefore,

$$\sum_j D_j = \sum_{j=1}^{n+1} \int_M \left(\rho^{j\bar{j}} - \frac{|\rho^j|^2}{|\partial\rho|_\rho^2} \right) \geq \int_M \left[\mathrm{trace}[\rho^{j\bar{k}}] - \gamma(z) \right]$$
$$= \int_M s(z).$$

Thus, from (25), (27), we obtain

$$\lambda_1 \leq n^2 \min_j (C_j/D_j) \leq \frac{n^2 \sum_j C_j}{\sum_j D_j} = \frac{n^2 \int_M \gamma(z) |\partial\rho|_\rho^{-2}}{\int_M s(z)}. \qquad (28)$$

The proof is complete.
Since $\gamma(z) \geq ns(z)$, the bound in this theorem is weaker.

Example 5

Put

$$\rho(Z) = \log(1 + \|Z\|^2) + \psi(Z, \bar{Z}), \tag{29}$$

where ψ is a real-valued pluriharmonic function. $[\rho^{j\bar{k}}]$ has eigenvalues $\lambda_{n+1} = (1 + \|Z\|^2)^2$ of multiplicity one and $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 1 + \|Z\|^2$. Thus

$$\gamma(z) = (1 + \|Z\|^2)^2, \quad s(z) = n(1 + \|Z\|^2).$$

One gets

$$\lambda_1 \leq \frac{n \int_M (1 + \|Z\|^2)^2 |\partial\rho|_\rho^{-2}}{\int_M (1 + \|Z\|^2)} \leq n \max_M (1 + \|Z\|^2) |\partial\rho|_\rho^{-2}.$$

When $\psi = 0$, M is just the sphere.

Example 6: Ellipsoids

Theorem (D.-Li.-Lin 2016)

Suppose that $\rho_{j\bar{k}} = \delta_{jk}$, *then*

$$\lambda_1 \leq \frac{n}{v(M)} \int_M |\partial\rho|_\rho^{-2} \theta \wedge (d\theta)^n.$$

The equality occurs only if $|\partial\rho|_\rho^2$ *is a constant on* M. *If furthermore,* ρ *is defined in the domain bounded by* M, *then* M *must be a sphere.*

This follows from the previous theorem, since $\gamma(z) = 1$ and $s(z) = n$.

Example 6: Ellipsoids

Let

$$\rho(Z, \bar{Z}) = \|Z\|^2 + \Re(a_{jk} z^j \bar{z}^k).$$

Compact level sets of ρ are ellipsoids. Moreover, $\rho_{j\bar{k}} = \delta_{jk}$. By either one of the theorems above,

$$\lambda_1 \leq \frac{n}{vol(M)} \int_M |\partial\rho|^{-2} \theta \wedge (d\theta)^n.$$

Lemma

If M is a compact level set $\{\rho = \nu\}$, with $\nu > 0$, then

$$\frac{n}{vol(M)} \int_M |\partial\rho|^{-2} \theta \wedge (d\theta)^n = \frac{1}{\nu}.$$

Example 6: Ellipsoids

Corollary (D.-Li-Lin)

Let $\rho(Z)$ *be a real-valued, strictly plurisubharmonic homogeneous quadratic polynomial satisfying* $\rho_{j\bar{k}} = \delta_{jk}$, $M = \rho^{-1}(\nu)$ *(*$\nu > 0$*) a compact connected regular level set of* ρ. *Then*

$$\lambda_1(M, \theta) \leq \lambda_1(\sqrt{\nu}\,\mathbb{S}^{2n+1}, \theta_0) = n/\nu. \tag{30}$$

The equality occurs if and only if $(M, \theta) = (\sqrt{\nu}\,\mathbb{S}^{2n+1}, \theta_0)$.

Here, $\sqrt{\nu}\,\mathbb{S}^{2n+1}$ is the sphere $\|Z\|^2 = \nu$ and $\theta_0 = \iota^*(i\bar{\partial}\|Z\|^2)$ is the "standard" pseudohermitian structure on the sphere.