Quasianalyticity of classes of ultradifferentiable functions and the non-surjectivity of the Borel mapping

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Conventions - Notation

$\mathcal{E}$ denotes the class of smooth functions

$\mathcal{C}^\omega$ denotes the class of real analytic functions

$\mathcal{H}$ denotes the class of holomorphic functions

write $\mathbb{R}_{>0} := \{x \in \mathbb{R} : x > 0\}$

write $\mathbb{N}_{>0} = \{1, 2, \dots\}$ and $\mathbb{N} = \mathbb{N}_{>0} \cup \{0\}$

$f^{(k)}$ denotes the k-th order Fréchet derivative of f

write $\|f^{(k)}(x)\|_{L^k(\mathbb{R}^r,\mathbb{R}^s)} := \sup\{\|f^{(k)}(x)(v_1, \dots, v_k)\|_{\mathbb{R}^s} : \|v_i\|_{\mathbb{R}^r} \leq 1 \ \forall\, 1 \leq i \leq k\}$

we write $[\cdot]$ if either $\{\cdot\}$ or $(\cdot)$ is considered but not mixing the cases

Ultradifferentiable classes

Ultradifferentiable classes $\mathcal{E}_{[\star]}$, $\star \in \{M, \omega, \mathcal{M}\}$ - certain subclasses of smooth functions satisfying growth conditions on all their derivatives

Classically defined by using weight sequences M or weight functions ω

Historically first the classes $\mathcal{E}_{[M]}$ were considered

Classes $\mathcal{E}_{[\omega]}$: First the decay property of the Fourier transform $\hat{f}$ was measured w.r.t. to ω (Beurling)

Using a weight matrix $\mathcal{M} = \{M^x : x \in \mathcal{I}\}$ unifies/generalizes both approaches

In each setting one can distinguish between the *Roumieu case* $\mathcal{E}_{\{\star\}}$ and the *Beurling case* $\mathcal{E}_{(\star)}$

(Non)-Quasianalyticity of $\mathcal{E}_{[\star]}$

Each case can be divided into *quasianalytic* and *non-quasianalytic* classes

Non-quasianalyticity: Existence of functions with compact support of the particular case ("$\mathcal{E}_{[\star]}$-test functions")

É. Borel (ca. 1900) - Discovery of quasianalytic functions:

Explicit construction of smooth functions on the real line which are not real-analytic but nevertheless $f^{(j)}(0)$ for all $j \in \mathbb{N}$ implies $f = 0$.

Weight sequences M

$M=(M_p)_p\in\mathbb{R}_{>0}^{\mathbb{N}}$, put $m_p:=\frac{M_p}{p!}$ and get $m:=(m_p)_p$.
M is called *normalized*, if $1=M_0\le M_1$ (w.l.o.g.).
Write $M\le N$ if $M_p\le N_p$ for all $p\in\mathbb{N}$.
(1) M is called *log-convex* if

$$\forall\, j\in\mathbb{N}_{>0}:\quad M_j^2\le M_{j+1}M_{j-1}.$$

If M is normalized and log-convex, then $k\mapsto M_k$ and $k\mapsto (M_k)^{1/k}$ are increasing.
(2) M has *moderate growth* (write (mg)) if

$$\exists\, C\ge 1\,\forall\, j,k\in\mathbb{N}:\ M_{j+k}\le C^{j+k}M_jM_k.$$

(3) M is called *non-quasianalytic* (write (nq)) if

$$\sum_{p=1}^{\infty}\frac{M_{p-1}}{M_p}<+\infty$$

(4) For M, N we define

$$M \preceq N :\Leftrightarrow \sup_{p \in \mathbb{N}_{>0}} \left(\frac{M_p}{N_p}\right)^{1/p} < +\infty,$$

i.e. $\exists\, C, h > 0\ \forall\, p \in \mathbb{N}:\ M_p \le C h^p N_p$.

$$M \lhd N :\Leftrightarrow \lim_{p \to \infty} \left(\frac{M_p}{N_p}\right)^{1/p} = 0,$$

i.e. $\forall\, h > 0$ (small) $\exists\, C_h > 0\ \forall\, p \in \mathbb{N}:\ M_p \le C_h h^p N_p$.
We call two sequences equivalent, if

$$M \approx N :\Leftrightarrow\ M \preceq N \text{ and } N \preceq M$$

Example

The Gevrey-sequences $G^s = (p!^s)_{p\in\mathbb{N}}$, $s > 1$, are normalized and satisfy all properties $(1) - (3)$. If $s < t$, then $G^s \lhd G^t$.

For convenience we put

$$\mathcal{LC} := \{M \in \mathbb{R}_{>0}^{\mathbb{N}} : \text{ normalized, log-convex, } \lim_{k\to\infty}(M_k)^{1/k} = +\infty\}$$

Classes $\mathcal{E}_{[M]}$ (H. Cartan, S. Mandelbrojt, W. Rudin, H. Komatsu)

Let $r, s \in \mathbb{N}_{>0}$, $U \subseteq \mathbb{R}^r$ be non-empty open, then define the *Roumieu class*

$$\mathcal{E}_{\{M\}}(U, \mathbb{R}^s) := \{f \in \mathcal{E}(U, \mathbb{R}^s) : \forall K \subseteq U \text{ compact } \exists h > 0 : \|f\|_{M,K,h} < +\infty\}$$

and the *Beurling class*

$$\mathcal{E}_{(M)}(U, \mathbb{R}^s) := \{f \in \mathcal{E}(U, \mathbb{R}^s) : \forall K \subseteq U \text{ compact } \forall h > 0 : \|f\|_{M,K,h} < +\infty\},$$

where

$$\|f\|_{M,K,h} := \sup_{k \in \mathbb{N}, x \in K} \frac{\|f^{(k)}(x)\|_{L^k(\mathbb{R}^r, \mathbb{R}^s)}}{h^k M_k}.$$

Topology on $\mathcal{E}_{[M]}$

For compact sets K with smooth boundary

$$\mathcal{E}_{M,h}(K,\mathbb{R}^s) := \{f \in \mathcal{E}(K,\mathbb{R}^s) : \|f\|_{M,K,h} < +\infty\}$$

is a Banach space and we have the topological vector space representations

$$\mathcal{E}_{\{M\}}(U,\mathbb{R}^s) = \varprojlim_{K\subseteq U} \varinjlim_{h>0} \mathcal{E}_{M,h}(K,\mathbb{R}^s) = \varprojlim_{K\subseteq U} \mathcal{E}_{\{M\}}(K,\mathbb{R}^s) \qquad (1)$$

resp.

$$\mathcal{E}_{(M)}(U,\mathbb{R}^s) = \varprojlim_{K\subseteq U} \varprojlim_{h>0} \mathcal{E}_{M,h}(K,\mathbb{R}^s) = \varprojlim_{K\subseteq U} \mathcal{E}_{(M)}(K,\mathbb{R}^s). \qquad (2)$$

$\mathcal{E}_{(M)}$ is a *Fréchet space*, $\mathcal{E}_{\{M\}}(K,\mathbb{R}^s)$ is a *Silva space*.

We get $\mathcal{E}_{\{(p!)_p\}} = \mathcal{C}^\omega$ and $\mathcal{E}_{((p!)_p)}(U) = \mathcal{H}(\mathbb{C}^n)$ (restrictions of entire functions on open connected U)

If $M \in \mathcal{LC}$ and N arbitrary, then $M \preceq N \Leftrightarrow \mathcal{E}_{[M]} \subseteq \mathcal{E}_{[N]}$ and $M \lhd N \Leftrightarrow \mathcal{E}_{\{M\}} \subseteq \mathcal{E}_{(N)}$.

$\liminf_{p\to\infty}(m_p)^{1/p} > 0 \Longleftrightarrow \mathcal{C}^\omega \subseteq \mathcal{E}_{\{M\}} \Longleftrightarrow \mathcal{H}(\mathbb{C}^n) \subseteq \mathcal{E}_{(M)}(U)$
$\lim_{p\to\infty}(m_p)^{1/p} = \infty \Longleftrightarrow \mathcal{C}^\omega \subseteq \mathcal{E}_{(M)}$

Weight functions ω

A function $\omega : [0, \infty) \to [0, \infty)$ is called a *weight function* if:

(*i*) ω is continuous,

(*ii*) ω is increasing,

(*iii*) $\omega(x) = 0$ for $x \in [0, 1]$ (normalization - w.l.o.g.),

(*iv*) $\lim_{x \to \infty} \omega(x) = +\infty$.

$(i) - (iv)$ are denoted by (ω_0)

Sometimes ω is extended to $\mathbb{C}^n$ by $\omega(z) := \omega(|z|)$ - connection to complex analysis

Moreover we consider:

(ω_1) $\omega(2t) = O(\omega(t))$ as $t \to \infty$

(ω_2) $\omega(t) = O(t)$ as $t \to \infty$

(ω_3) $\log(t) = o(\omega(t))$ as $t \to \infty$

(ω_4) $\varphi_\omega : t \mapsto \omega(e^t)$ is convex

(ω_5) $\omega(t) = o(t)$ as $t \to +\infty$

(ω_6) $\exists\, H \geq 1\ \forall\, t \geq 0 :\ 2\omega(t) \leq \omega(Ht) + H$

(ω_7) $\exists\, H > 0\ \exists\, C > 0\ \forall\, t \geq 0 : \omega(t^2) \leq C\omega(Ht) + C$ (new!)

(ω_{nq}) $\int_1^\infty \frac{\omega(t)}{t^2} dt < \infty$.

Example

$\omega_s(t) := \max\{0, (\log(t))^s\}$, $s > 1$, satisfies all properties except (ω_6).
The weight $\omega_s(t) := t^{1/s}$, $s > 1$, yields the Gevrey class G^s. It satisfies all properties except (ω_7).

Put

$$\mathcal{W}_0 := \{\omega : [0,\infty) \to [0,\infty) : \omega \text{ has } (\omega_0), (\omega_3), (\omega_4)\},$$

$$\mathcal{W} := \{\omega \in \mathcal{W}_0 : \omega \text{ has } (\omega_1)\}.$$

For $\omega \in \mathcal{W}_0$ define the *Legendre-Fenchel-Young conjugate*

$$\varphi^*_\omega(x) := \sup_{y \geq 0}(xy - \varphi_\omega(y)),$$

which is convex, increasing, $\varphi^*_\omega(0) = 0$, $\varphi^{**}_\omega = \varphi_\omega$, $\lim_{x\to\infty} \frac{x}{\varphi^*_\omega(x)} = 0$ and finally $x \mapsto \frac{\varphi_\omega(x)}{x}$ and $x \mapsto \frac{\varphi^*_\omega(x)}{x}$ are increasing.

Classes $\mathcal{E}_{[\omega]}$ - Braun, Meise, Taylor (1990)

Let $r, s \in \mathbb{N}_{>0}$, $U \subseteq \mathbb{R}^r$ non-empty open, for $\omega \in \mathcal{W}_0$ define the *Roumieu class*

$$\mathcal{E}_{\{\omega\}}(U, \mathbb{R}^s) :=$$
$$\{f \in \mathcal{E}(U, \mathbb{R}^s) : \ \forall\, K \subseteq U \text{ compact } \exists\, l > 0 : \ \|f\|_{\omega,K,l} < +\infty\}$$

and the *Beurling class*

$$\mathcal{E}_{(\omega)}(U, \mathbb{R}^s) :=$$
$$\{f \in \mathcal{E}(U, \mathbb{R}^s) : \ \forall\, K \subseteq U \text{ compact } \forall\, l > 0 : \ \|f\|_{\omega,K,l} < +\infty\},$$

where

$$\|f\|_{\omega,K,l} := \sup_{k \in \mathbb{N}, x \in K} \frac{\|f^{(k)}(x)\|_{L^k(\mathbb{R}^r,\mathbb{R}^s)}}{\exp(\frac{1}{l}\varphi^*_\omega(lk))}.$$

Relations of weight functions

For $\sigma, \tau \in \mathcal{W}_0$ write

$$\sigma \preceq \tau :\Leftrightarrow \tau(t) = O(\sigma(t)), \text{ as } t \to +\infty$$

$$\sigma \sim \tau :\Leftrightarrow \sigma \preceq \tau \text{ and } \tau \preceq \sigma$$

$$\sigma \lhd \tau :\Leftrightarrow \tau(t) = o(\sigma(t)), \text{ as } t \to +\infty$$

If $\sigma, \tau \in \mathcal{W}$, then $\sigma \preceq \tau \Leftrightarrow \mathcal{E}_{[\sigma]} \subseteq \mathcal{E}_{[\tau]}$ and $\sigma \lhd \tau \Leftrightarrow \mathcal{E}_{\{\sigma\}} \subseteq \mathcal{E}_{(\tau)}$.

Associated function ω_M - S. Mandelbrojt, H. Cartan, H. Komatsu

For $M := (M_p)_p \in \mathbb{R}_{>0}^{\mathbb{N}}$ we define the *associated function* $\omega_M : \mathbb{R}_{\ge 0} \to \mathbb{R} \cup \{+\infty\}$ by

$$\omega_M(t) := \sup_{p \in \mathbb{N}} \log \left(\frac{t^p M_0}{M_p} \right) \quad \text{for } t > 0, \qquad \omega_M(0) := 0.$$

Lemma

If $M \in \mathcal{LC}$, then $\omega_M \in \mathcal{W}_0$.
$\liminf (m_p)^{1/p} > 0$ *implies* (ω_2), *i.e.* $\omega_M(t) = O(t)$ *as* $t \to \infty$,
$\lim (m_p)^{1/p} = +\infty$ *implies* (ω_5), *i.e.* $\omega_M(t) = o(t)$ *as* $t \to \infty$.
M has (nq) *if and only if ω_M has* (ω_{nq}).
M has (mg) *if and only if ω_M has* (ω_6).

Associating a weight matrix

A central new idea: To each $\omega \in \mathcal{W}_0$ we consider the associated weight matrix $\Omega := \{\Omega^l = (\Omega^l_j)_{j \in \mathbb{N}} : l > 0\}$ defined by

$$\Omega^l_j := \exp\left(\frac{1}{l}\varphi^*_\omega(lj)\right)$$

Motivation for this: Compare the expressions in the denominators of the defining seminorms

Lemma

Let $\omega \in \mathcal{W}_0$, then $\Omega^l \in \mathcal{LC}$ and $\omega \sim \omega_{\Omega^l}$ for each $l > 0$.

Important new representations

Theorem

Let $\omega \in \mathcal{W}$, then we have for each compact $K \subseteq \mathbb{R}^r$ and non-empty open $U \subseteq \mathbb{R}^r$:

$$\mathcal{E}_{(\omega)}(U) = \varprojlim_{l>0} \mathcal{E}_{(\Omega^l)}(U) \quad \text{and} \quad \mathcal{E}_{\{\omega\}}(K) = \varinjlim_{l>0} \mathcal{E}_{\{\Omega^l\}}(K).$$

If in addition (ω_7), i.e. $\exists\ H, C > 0\ \forall\ t \geq 0 : \omega(t^2) \leq C\omega(Ht) + C$, then

$$\mathcal{E}_{(\omega)}(U) = \varprojlim_{l>0} \mathcal{E}_{\{\Omega^l\}}(U) \quad \text{and} \quad \mathcal{E}_{\{\omega\}}(K) = \varinjlim_{l>0} \mathcal{E}_{(\Omega^l)}(K).$$

Characterizing condition (ω_6)

Proposition

Let $\omega \in \mathcal{W}_0$*, TFAE:*

(1) ω *has* (ω_6)*, i.e.* $\exists\, H \geq 1\ \forall\, t \geq 0:\ 2\omega(t) \leq \omega(Ht) + H$,

(2) *each/some* ω_{Ω^l} *has* (ω_6)

(3) $\mathcal{E}_{[\Omega^x]} = \mathcal{E}_{[\Omega^y]}$ *for all* $x, y > 0$

(4) $\Omega^x \approx \Omega^y$ *for all* $x, y > 0$

(5) Ω^x *has* (mg) *for some/for each* $x > 0$

ω *has* (ω_{nq}) *if and only if some/each* Ω^x *has* (nq)

Consequence: (ω_7) is an obstruction for (ω_6)

Definition of a weight matrix

The representations above motivate the following abstract definition:
A weight matrix $\mathcal{M} := \{M^x \in \mathbb{R}_{>0}^{\mathbb{N}} : x \in \mathcal{I} = \mathbb{R}_{>0}\}$ is a set of weight sequences, s.th.

$$(\mathcal{M}) :\Leftrightarrow \ \forall\, x : \quad M^x \text{ is normalized, increasing, } M^x \leq M^y \text{ for } x \leq y.$$

We call $\mathcal{M}$ *standard log-convex,* if

$$(\mathcal{M}_{\mathrm{sc}}) :\Leftrightarrow (\mathcal{M}) \text{ and } \forall\, x \in \mathcal{I} : \quad M^x \in \mathcal{LC}.$$

Spaces $\mathcal{E}_{[\mathcal{M}]}$

Let $r, s \in \mathbb{N}_{>0}$, let $U \subseteq \mathbb{R}^r$ be non-empty and open, for all compact $K \subseteq U$ we put

$$\mathcal{E}_{\{\mathcal{M}\}}(K, \mathbb{R}^s) := \bigcup_{x \in \mathcal{I}} \mathcal{E}_{\{M^x\}}(K, \mathbb{R}^s)$$

$$\mathcal{E}_{\{\mathcal{M}\}}(U, \mathbb{R}^s) := \bigcap_{K \subseteq U} \bigcup_{x \in \mathcal{I}} \mathcal{E}_{\{M^x\}}(K, \mathbb{R}^s)$$

and

$$\mathcal{E}_{(\mathcal{M})}(K, \mathbb{R}^s) := \bigcap_{x \in \mathcal{I}} \mathcal{E}_{(M^x)}(K, \mathbb{R}^s)$$

$$\mathcal{E}_{(\mathcal{M})}(U, \mathbb{R}^s) := \bigcap_{x \in \mathcal{I}} \mathcal{E}_{(M^x)}(U, \mathbb{R}^s)$$

$\mathcal{E}_{(\mathcal{M})}$ is a *Fréchet space*, $\mathcal{E}_{\{\mathcal{M}\}}(K, \mathbb{R}^s)$ is a *Silva space*.

Some conditions for $\mathcal{M}$

Roumieu type:

$(\mathcal{M}_{\{mg\}})$ $\forall\, x \in \mathcal{I}\ \exists\, C\ \exists\, y \in \mathcal{I}\ \forall\, j,k \in \mathbb{N} : M^x_{j+k} \le C^{j+k} M^y_j M^y_k$

$(\mathcal{M}_{\{L\}})$ $\forall\, C\ \forall\, x \in \mathcal{I}\ \exists\, D\ \exists\, y \in \mathcal{I}\ \forall\, k \in \mathbb{N} : C^k M^x_k \le D M^y_k$

$(\mathcal{M}_{\{BR\}})$ $\forall\, x \in \mathcal{I}\ \exists\, y \in \mathcal{I} : M^x \lhd M^y$

Beurling type:

$(\mathcal{M}_{(mg)})$ $\forall\, x \in \mathcal{I}\ \exists\, C\ \exists\, y \in \mathcal{I}\ \forall\, j,k \in \mathbb{N} : M^y_{j+k} \le C^{j+k} M^x_j M^x_k$

$(\mathcal{M}_{(L)})$ $\forall\, C\ \forall\, x \in \mathcal{I}\ \exists\, D\ \exists\, y \in \mathcal{I}\ \forall\, k \in \mathbb{N} : C^k M^y_k \le D M^x_k$

$(\mathcal{M}_{(BR)})$ $\forall\, x \in \mathcal{I}\ \exists\, y \in \mathcal{I} : M^y \lhd M^x$

Relations for weight matrices

If $\mathcal{M} = \{M^x : x \in \mathbb{R}_{>0}\}$, $\mathcal{N} = \{N^y : y \in \mathbb{R}_{>0}\}$, then

$$\mathcal{M}\{\preceq\}\mathcal{N} :\Leftrightarrow \forall\, x \,\exists\, y : \; M^x \preceq N^y$$

$$\mathcal{M}(\preceq)\mathcal{N} :\Leftrightarrow \forall\, x \,\exists\, y : \; M^y \preceq N^x$$

$$\mathcal{M}[\approx]\mathcal{N} :\Leftrightarrow \mathcal{M}[\preceq]\mathcal{N}, \mathcal{N}[\preceq]\mathcal{M}$$

Finally

$$\mathcal{M} \lhd \mathcal{N} :\Leftrightarrow \forall\, x \,\forall\, y : \; M^x \lhd N^y$$

In this context we introduce also:

$(\mathcal{M}_{\{\mathcal{C}^\omega\}})$ $\exists\, x \in \mathcal{I} : \; \liminf_{k\to\infty}(m_k^x)^{1/k} > 0$

$(\mathcal{M}_{\mathcal{H}})$ $\forall\, x \in \mathcal{I} : \; \liminf_{k\to\infty}(m_k^x)^{1/k} > 0$

$(\mathcal{M}_{(\mathcal{C}^\omega)})$ $\forall\, x \in \mathcal{I} : \; \lim_{k\to\infty}(m_k^x)^{1/k} = +\infty$

Characterization of relations

Let $\mathcal{M}$ and $\mathcal{N}$ be $(\mathcal{M}_{\mathrm{sc}})$, then

(i) $\mathcal{E}_{[\mathcal{M}]} \subseteq \mathcal{E}_{[\mathcal{N}]} \Leftrightarrow \mathcal{M}[\preceq]\mathcal{N}$

(ii) $\mathcal{E}_{\{\mathcal{M}\}} \subseteq \mathcal{E}_{(\mathcal{N})} \Leftrightarrow \mathcal{M} \lhd \mathcal{N}$

$$(\mathcal{M}_{\{\mathcal{C}^\omega\}}) \Longleftrightarrow \mathcal{C}^\omega \subseteq \mathcal{E}_{\{\mathcal{M}\}}$$
$$(\mathcal{M}_{\mathcal{H}}) \Longleftrightarrow \mathcal{H}(\mathbb{C}^n) \subseteq \mathcal{E}_{(\mathcal{M})}(U)$$
$$(\mathcal{M}_{(\mathcal{C}^\omega)}) \Longleftrightarrow \mathcal{C}^\omega \subseteq \mathcal{E}_{(\mathcal{M})}$$

We call $\mathcal{M}$ *constant*, if $\mathcal{M} = \{M\}$ or more generally if $M^x \approx M^y$ for all $x, y \in \mathcal{I}$.

Conditions for ω versus conditions for Ω

Let $\mathcal{M} = \Omega$ for $\omega \in \mathcal{W}_0$, then:

(1) Ω is $(\mathcal{M}_{sc})$

(2) Ω has $(\mathcal{M}_{\{mg\}})$ and $(\mathcal{M}_{(mg)})$

(3) If ω has in addition (ω_1) then $(\mathcal{M}_{\{L\}})$ and $(\mathcal{M}_{(L)})$

(4) $\mathcal{M}$ is constant if and only if (ω_6) is satisfied

(5) If ω has in addition (ω_7) then $(\mathcal{M}_{\{BR\}})$ and $(\mathcal{M}_{(BR)})$

(6) Let $\sigma, \tau \in \mathcal{W}$ with $\sigma \preceq \tau$ then $\mathcal{M}\{\preceq\}\mathcal{N}$ and $\mathcal{M}(\preceq)\mathcal{N}$. If $\sigma \lhd \tau$ then $\mathcal{M} \lhd \mathcal{N}$ ($\mathcal{M}$ is associated to σ and $\mathcal{N}$ to τ).

(7) If ω has (ω_1) and (ω_2), then $(\mathcal{M}_{\mathcal{H}})$ and if it has (ω_1) and (ω_5) then $(\mathcal{M}_{(\mathcal{C}^\omega)})$

Autonomy of $\mathcal{E}_{[\mathcal{M}]}$ - joint work with A. Rainer

Comparison of $\mathcal{E}_{[M]}$ and $\mathcal{E}_{[\omega]}$ - in general mutually distinct (Bonet, Meise, Melikhov - 2007)

Using weight matrices generalizes both approaches

But we can describe more classes: Set $\mathcal{G} := \{G^{1+s} = (p!^{s+1})_{p\in\mathbb{N}} : s > 0\}$ - the *Gevrey-matrix*.

Proposition

Neither $\mathcal{E}_{\{\mathcal{G}\}}$ nor $\mathcal{E}_{(\mathcal{G})}$ coincides with $\mathcal{E}_{\{M\}}$, $\mathcal{E}_{(M)}$, $\mathcal{E}_{\{\omega\}}$ or $\mathcal{E}_{(\omega)}$ for any $M \in \mathcal{LC}$ and any $\omega \in \mathcal{W}$.

Definition

Let $\mathcal{M}$ be $(\mathcal{M})$, then $\mathcal{E}_{[\mathcal{M}]}$ is called non-quasianalytic if $\mathcal{E}_{[\mathcal{M}]}$ contains non-trivial functions with compact support.

Importance of non-quasianalyticity: existence of $\mathcal{E}_{[\mathcal{M}]}$-testfunctions/partitions of unity

Characterization of non-quasianalyticity is given by the "Denjoy-Carleman theorem"

Regularizations of a weight M (Mandelbrojt, Cartan, Komatsu, Hörmander)

Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ with $M_0 = 1$.
$M^{\mathrm{lc}} = (M_k^{\mathrm{lc}})_k$ denotes the log-convex minorant of M given by

$$M_k^{\mathrm{lc}} := \sup_{t>0} \frac{t^k}{\exp(\omega_M(t))}.$$

Moreover put $M^I := (M_k^I)_k$ defined by

$$M_k^I := \left(\inf\{(M_j)^{1/j} : j \geq k\}\right)^k \text{ for } k \geq 1, \quad M_0^I := 1$$

$((M_k^I)^{1/k})_k$ is the increasing minorant of $((M_k)^{1/k})_k$
we have $M^{\mathrm{lc}} \leq M^I \leq M$ - if M is log-convex, then $M^{\mathrm{lc}} = M^I = M$.

Importance of M^{lc}

Theorem

Let M be arbitrary and $U \subseteq \mathbb{R}^n$ open.

(i) *If* $\liminf_{k\to\infty}(m_k)^{1/k} > 0$, *then* $\mathcal{E}_{\{M\}}(U) = \mathcal{E}_{\{M^{lc}\}}(U)$.

(ii) *If* $\lim_{k\to\infty}(m_k)^{1/k} = +\infty$, *then* $\mathcal{E}_{(M)}(U) = \mathcal{E}_{(M^{lc})}(U)$.

Roumieu case: H. Cartan (1940)
Beurling case: Rainer, S. (2014) - reduction to the Roumieu case

Denjoy-Carleman theorem for classes $\mathcal{E}_{[M]}$ (e.g. L. Hörmander, H. Komatsu, W. Rudin)

Theorem

Let $M \in \mathbb{R}_{>0}^{\mathbb{N}}$ with $M_0 = 1$. TFAE

(*i*) $\mathcal{E}_{[M]}$ *is non-quasianalytic*

(*ii*) M^{lc} *satisfies* (nq)

(*iii*) $\sum_{p \geq 1} \frac{1}{(M_p^I)^{1/p}} < +\infty$.

In this case $\mathcal{C}^\omega \subsetneq \mathcal{E}_{[M]} = \mathcal{E}_{[M^I]} = \mathcal{E}_{[M^{\mathrm{lc}}]}$ *holds.*

Denjoy-Carleman theorem for $\mathcal{E}_{[\mathcal{M}]}$

Generalizing a result by J. Schmets/M. Valdivia (2008) we prove:

Theorem

Let $\mathcal{M} = \{M^x : x \in \mathcal{I} = \mathbb{R}_{>0}\}$ be $(\mathcal{M})$.

(i) *$\mathcal{E}_{\{\mathcal{M}\}}$ is non-quasianalytic if and only if there exists $x_0 \in \mathcal{I}$ such that $\mathcal{E}_{[M^{x_0}]}$ is non-quasianalytic.*

(ii) *$\mathcal{E}_{(\mathcal{M})}$ is non-quasianalytic if and only if each $\mathcal{E}_{[M^x]}$ is non-quasianalytic.*

Attention: "Large intersections" of non-quasianalytic classes are in general NOT non-quasianalytic again! - The class $\mathcal{C}^\omega$ is the intersection of all non-quasianalytic classes (T. Bang).

Denjoy-Carleman theorem for $\mathcal{M} = \Omega$

Corollary

Let $\omega \in \mathcal{W}$ be given. TFAE:

(i) ω *has* (ω_{nq}),

(ii) $\mathcal{E}_{\{\omega\}}$ *contains functions with compact support,*

(iii) $\mathcal{E}_{(\omega)}$ *contains functions with compact support,*

(iv) *some* Ω^l *has* (nq),

(v) *each* Ω^l *has* (nq).

General assumption

- Joint work with A. Rainer -

From now on assume for each weight M resp. $M^x \in \mathcal{M}$:

$$M \in \mathcal{LC} \quad \text{and} \quad \liminf_{k\in\mathbb{N}_{>0}}(m_k)^{1/k} > 0 \Leftrightarrow \mathcal{C}^\omega \subseteq \mathcal{E}_{\{M\}}$$

This is no restriction whenever $\mathcal{C}^\omega \subseteq \mathcal{E}_{[M]}$: Make a change $M \mapsto M^{\mathrm{lc}}$ if necessary

Definitions

The space of germs at $0 \in \mathbb{R}^n$ of $\mathcal{E}_{[M]}$-type is defined by

$$\mathcal{E}^{0,n}_{\{M\}} := \varinjlim_{k \in \mathbb{N}_{>0}} \mathcal{E}_{\{M\}}((-1/k, 1/k)^n)$$

resp.

$$\mathcal{E}^{0,n}_{(M)} := \varinjlim_{k \in \mathbb{N}_{>0}} \mathcal{E}_{(M)}((-1/k, 1/k)^n).$$

The germ of real analytic functions (corresponding to $M_p = p!$) is defined by $\mathcal{O}^{0,n}$.

Analogously we introduce $\mathcal{E}^{0,n}_{[\omega]}$ and $\mathcal{E}^{0,n}_{[\mathcal{M}]}$.

Moreover we define

$$\Lambda^n_{\{M\}} := \{a = (a_j)_j \in \mathbb{C}^{\mathbb{N}^n} : \exists\, h > 0 : \ |a|_{M,h} < +\infty\}$$

resp.

$$\Lambda^n_{(M)} := \{a = (a_j)_j \in \mathbb{C}^{\mathbb{N}^n} : \forall\, h > 0 : \ |a|_{M,h} < +\infty\},$$

where

$$|a|_{M,h} := \sup_{j \in \mathbb{N}^n} \frac{|a_j|}{h^{|j|} M_{|j|}}.$$

Analogously we introduce $\Lambda^n_{[\omega]}$ and $\Lambda^n_{[\mathcal{M}]}$.

The *Borel mapping* $j^\infty : \mathcal{E}^{0,n}_{[\mathcal{M}]} \longrightarrow \Lambda^n_{[\mathcal{M}]}$ is defined by

$$f \mapsto (f^{(j)}(0))_{j \in \mathbb{N}^n}.$$

Questions concerning the Borel mapping j^∞

Let $\mathcal{E}^{0,n}_{[\mathcal{M}]}$ be quasianalytic $\Longleftrightarrow j^\infty$ is injective

(*i*) What can be said about the surjectivity of j^∞?

(*ii*) More generally what can be said about the image $j^\infty(\mathcal{E}^{0,n}_{[\mathcal{M}]}) \subseteq \Lambda^n_{[\mathcal{M}]}$: "How large" is the image in $\Lambda^n_{[\mathcal{M}]}$?

(*iii*) Let $a \in \Lambda^n_{[\mathcal{M}]}$ be given, does a belong to the image of j^∞?

Known results

(*i*) Classes $\mathcal{E}^{0,n}_{\{M\}}$ - T. Carleman (1920's), V. Thilliez (2008):
If $\mathcal{E}^{0,n}_{\{M\}}$ is quasianalytic and $\mathcal{O}^{0,n} \subsetneq \mathcal{E}^{0,n}_{\{M\}}$, then j^∞ is never surjective.
Proof: Carleman uses variational arguments, Thilliez uses functional analysis

(*ii*) Classes $\mathcal{E}^{0,n}_{\{\omega\}}$ and $\mathcal{E}^{0,n}_{(\omega)}$ - J. Bonet/R. Meise (2013):
If $\mathcal{E}^{0,n}_{[\omega]}$ is quasianalytic and $\mathcal{O}^{0,n} \subsetneq \mathcal{E}^{0,n}_{[\omega]}$, then j^∞ is never surjective. Proof: very much functional analysis is involved

(*iii*) H. Sfouli (2014): Proves the non-surjectivity for abstract quasianalytic local rings, but requires stability under differentiation and composition - really restrictive assumptions for the ultradifferentiable case!

Our results (2015)

(i) elementary proofs, no functional analysis is used

(ii) proof for classes $\mathcal{E}^{0,n}_{[\mathcal{M}]}$ - very general setting

(iii) show not only the non-surjectivity of j^{∞}, but a little bit more

(iv) obtain some information of sequences which are not contained in the image of j^{∞}

Important theorem from T. Bang (1953)

Theorem

Let M be quasianalytic, $f \in \mathcal{E}([0,1])$ such that

$$\forall\, j \in \mathbb{N}: \quad \sup_{t\in[0,1]} |f^{(j)}(t)| \le M_j.$$

If $f \neq 0$ and for all $j \in \mathbb{N}$ there exists $x_j \in [0,1]$ with $f^{(j)}(x_j) = 0$, then

$$\sum_{j=0}^{\infty} |x_j - x_{j+1}| = +\infty.$$

Important consequence from T. Bang (1953)

Corollary

Let M be quasianalytic and f as above. If $f^{(j)}(0) > 0$ for all $j \in \mathbb{N}$, then $f^{(j)}(t) > 0$ for all $t \in [0,1]$ and $j \in \mathbb{N}$, i.e. f is absolutely monotonic.

Proof: Apply the previous theorem and Rolle's theorem.

Non-surjectivity for j^∞ - Roumieu case $\mathcal{E}^{0,n}_{\{M\}}$

Theorem

Let M be quasianalytic and such that $\mathcal{O}^{0,n} \subsetneq \mathcal{E}^{0,n}_{\{M\}} \Leftrightarrow \sup_{k\in\mathbb{N}_{>0}}(m_k)^{1/k} = +\infty$. Then there exist elements in $\Lambda^n_{\{M\}}$ which are not contained in $j^\infty(\mathcal{E}^{0,n}_{\{N\}})$ for any quasianalytic N.

Proof: Use the previous Corollary and *Bernstein's theorem*: Absolutely monotonic functions are real analytic.
Consequence ($n=1$): Each strictly positive sequence $b=(b_p)_p \in \Lambda^n_{\{M\}}$ (i.e. $b_p > 0$ for all $p \in \mathbb{N}$) is not contained in $j^\infty(\mathcal{E}^{0,n}_{\{N\}})$ for any quasianalytic N unless b defines a real-analytic germ.

Non-surjectivity for j^∞ - Beurling case $\mathcal{E}^{0,n}_{(M)}$

Theorem

Let M be quasianalytic and such that $\mathcal{O}^{0,n} \subsetneq \mathcal{E}^{0,n}_{(M)} \Leftrightarrow \lim_{k\to\infty}(m_k)^{1/k} = +\infty$. Then there exist elements in $\Lambda^n_{(M)}$ which are not contained in $j^\infty(\mathcal{E}^{0,n}_{\{N\}})$ for any quasianalytic N (and so consequently are not contained in any $j^\infty(\mathcal{E}^{0,n}_{(N)})$).

Proof: Reduction to the Roumieu case using

Proposition

Let M be arbitrary with $\lim_{k\to\infty}(m_k)^{1/k} = +\infty$. Then

$$\Lambda^n_{(M)} = \bigcup \{\Lambda^n_{\{L\}} : L \lhd M, \lim_{k\to\infty}(l_k)^{1/k} = +\infty\}.$$

Non-surjectivity for j^∞ - Roumieu case $\mathcal{E}^{0,n}_{\{\mathcal{M}\}}$

Theorem

Let $\mathcal{M}$ be quasianalytic and such that $\mathcal{O}^{0,n} \subsetneq \mathcal{E}^{0,n}_{\{\mathcal{M}\}}$, i.e. there exists some $x_0 \in \mathcal{I}$ s.th.
$\sup_{k\in\mathbb{N}_{>0}}(m_k^{x_0})^{1/k} = +\infty \Leftrightarrow \mathcal{O}^{0,n} \subsetneq \mathcal{E}^{0,n}_{\{M^{x_0}\}}$.
Then there exist elements in $\Lambda^n_{\{\mathcal{M}\}}$ which are not contained in $j^\infty(\mathcal{E}^{0,n}_{\{\mathcal{N}\}})$ for any quasianalytic $\mathcal{N} := \{N^y : y \in \mathbb{R}_{>0}\}$.

Proof: Reduction to the $\mathcal{E}_{\{M\}}$ case.

Non-surjectivity for j^∞ - Roumieu case $\mathcal{E}^{0,n}_{\{\omega\}}$

Corollary

Let $\omega \in \mathcal{W}$ be quasianalytic, i.e. $\int_1^\infty \frac{\omega(t)}{t^2} dt = \infty$ (does not satisfy (ω_{nq})), and such that

$$\mathcal{O}^{0,n} \subsetneq \mathcal{E}^{0,n}_{\{\omega\}} \Leftrightarrow \liminf_{t\to\infty} \frac{\omega(t)}{t} = 0.$$

Then there exist elements in $\Lambda^n_{\{\omega\}}$ which are not contained in $j^\infty(\mathcal{E}^{0,n}_{\{\sigma\}})$ for any quasianalytic $\sigma \in \mathcal{W}$.

Non-surjectivity for j^∞ - Beurling case $\mathcal{E}^{0,n}_{(\mathcal{M})}$

Theorem

Let $\mathcal{M}$ be quasianalytic and such that $\mathcal{O}^{0,n} \subsetneq \mathcal{E}^{0,n}_{(\mathcal{M})}$, i.e. $(\mathcal{M}_{(\mathcal{C}^\omega)})$. Then there exist elements in $\Lambda^n_{(\mathcal{M})}$ which are not contained in $j^\infty(\mathcal{E}^{0,n}_{\{\mathcal{N}\}})$ for any quasianalytic $\mathcal{N} := \{N^y : y \in \mathbb{R}_{>0}\}$ (and so consequently not contained in any $j^\infty(\mathcal{E}^{0,n}_{(\mathcal{N})})$).

Proof: Reduction to the $\mathcal{E}_{\{\mathcal{M}\}}$ case by using

Proposition

Let $\mathcal{M}$ be $(\mathcal{M})$ with $(\mathcal{M}_{(\mathcal{C}^\omega)})$, i.e.
$\forall\, x \in \mathcal{I}:\ \lim_{k\to\infty}(m^x_k)^{1/k} = +\infty$. *Then*

$$\Lambda^n_{(\mathcal{M})} = \bigcup \{\Lambda^n_{\{L\}} : L \lhd \mathcal{M}, \lim_{k\to\infty}(l_k)^{1/k} = +\infty\}.$$

Non-surjectivity for j^∞ - Beurling case $\mathcal{E}^{0,n}_{(\omega)}$

Corollary

Let $\omega \in \mathcal{W}$ be quasianalytic, i.e. does not satisfy (ω_{nq}), and such that

$$\mathcal{O}^{0,n} \subsetneq \mathcal{E}^{0,n}_{(\omega)} \Leftrightarrow \omega(t) = o(t) \text{ as } t \to +\infty.$$

Then there exist elements in $\Lambda^n_{(\omega)}$ which are not contained in $j^\infty(\mathcal{E}^{0,n}_{\{\sigma\}})$ for any quasianalytic $\sigma \in \mathcal{W}$ (and so consequently not contained in any $j^\infty(\mathcal{E}^{0,n}_{(\sigma)})$).

(*i*) For weight matrices, their properties and associated function spaces (Sections 3 and 4) see [1] resp. [3, Sections 3-9],

(*ii*) for the characterization of the non-quasianalyticity (Section 5) see [4, Section 4],

(*iii*) for Section 6 see [2].

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