Real Paley-Wiener theorems in spaces of ultradifferentiable functions

David Jornet

Universitat Politècnica de València, Spain

Complex Analysis Seminar
Fakultät für Mathematik, Universität Wien

Vienna, 26 February 2019

Work in collaboration with C. Boiti (Università di Ferrara) and A. Oliaro (Università di Torino)

Notations

For $f \in L^1(\mathbb{R}^d)$ we define the Fourier transform as

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} dx, \quad \xi \in \mathbb{R}^d.$$

Notations

For $f \in L^1(\mathbb{R}^d)$ we define the Fourier transform as

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} dx, \quad \xi \in \mathbb{R}^d.$$

Given a *window function* $\psi \in L^2(\mathbb{R}^d)$, the short-time Fourier transform (Gabor transform) is defined as

$$V_\psi f(x,\xi) := \int_{\mathbb{R}^d} f(y)\overline{\psi(y-x)} e^{-iy\cdot\xi}\, dy, \quad x, \xi \in \mathbb{R}^d,$$

for $f \in L^2(\mathbb{R}^d)$.

Notations

For $f \in L^1(\mathbb{R}^d)$ we define the Fourier transform as

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} dx, \quad \xi \in \mathbb{R}^d.$$

Given a *window function* $\psi \in L^2(\mathbb{R}^d)$, the short-time Fourier transform (Gabor transform) is defined as

$$V_\psi f(x,\xi) := \int_{\mathbb{R}^d} f(y)\overline{\psi(y-x)} e^{-iy\cdot\xi}\, dy, \quad x,\xi \in \mathbb{R}^d,$$

for $f \in L^2(\mathbb{R}^d)$. Moreover, the Wigner transform of $f \in L^2(\mathbb{R}^d)$ is

$$\mathrm{Wig}\, f(x,\xi) = \int_{\mathbb{R}^d} f\left(x+\frac{t}{2}\right)\overline{f\left(x-\frac{t}{2}\right)} e^{-it\cdot\xi}\, dt, \quad x,\xi \in \mathbb{R}^d.$$

Paley-Wiener Theorems

Classical Paley-Wiener Theorem

A function f satisfies $\hat{f} \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} \hat{f} \subseteq [-R, R]$ if and only if f is an entire function such that for every $k \in \mathbb{N}_0$ there exists $C_k > 0$ such that

$$|f(z)| \leq C_k(1 + |z|)^{-k} e^{R|\mathrm{Im}\, z|},$$

for every $z \in \mathbb{C}$.

Paley-Wiener Theorems

Classical Paley-Wiener Theorem

A function f satisfies $\hat{f} \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} \hat{f} \subseteq [-R, R]$ if and only if f is an entire function such that for every $k \in \mathbb{N}_0$ there exists $C_k > 0$ such that

$$|f(z)| \leq C_k(1 + |z|)^{-k} e^{R|\operatorname{Im} z|},$$

for every $z \in \mathbb{C}$.

Real Paley-Wiener Theorem (Bang, 1990, Proc. AMS)

Let $1 \leq p \leq +\infty$ and $f \in C^\infty(\mathbb{R})$ such that $f^{(n)} \in L^p(\mathbb{R})$ for every $n \in \mathbb{N}_0$. Then the following limit exists

$$\lim_{n\to\infty} \|f^{(n)}\|_p^{1/n} = R,$$

where $R = \sup\{|\xi| : \xi \in \operatorname{supp} \hat{f}(\xi)\}$.

Proof (simple case)

$$\lim_{n\to\infty} \|f^{(n)}\|_p^{1/n} = R, \qquad R = \sup\{|\xi| : \xi \in \operatorname{supp} \hat{f}(\xi)\}$$

Case p=2

Proof (simple case)

$$\lim_{n\to\infty} \|f^{(n)}\|_p^{1/n} = R, \qquad R = \sup\{|\xi| : \xi \in \operatorname{supp} \hat{f}(\xi)\}$$

Case p=2

$$\|f^{(n)}\|_2^{1/n} \leq \|\widehat{f^{(n)}}\|_2^{1/n}$$

Proof (simple case)

$$\lim_{n\to\infty} \|f^{(n)}\|_p^{1/n} = R, \qquad R = \sup\{|\xi| : \xi \in \operatorname{supp} \hat{f}(\xi)\}$$

Case p=2

$$\|f^{(n)}\|_2^{1/n} \leq \|\widehat{f^{(n)}}\|_2^{1/n} = \|(-i\xi)^n \hat{f}(\xi)\|_2^{1/n}$$

Proof (simple case)

$$\lim_{n\to\infty} \|f^{(n)}\|_p^{1/n} = R, \qquad R = \sup\{|\xi| : \xi \in \operatorname{supp} \hat{f}(\xi)\}$$

Case p=2

$$\|f^{(n)}\|_2^{1/n} \leq \|\widehat{f^{(n)}}\|_2^{1/n} = \|(-i\xi)^n \hat{f}(\xi)\|_2^{1/n} \leq \left(R^n \|\hat{f}\|_2\right)^{1/n} \to R, \quad n \to \infty$$

Proof (simple case)

$$\lim_{n\to\infty} \|f^{(n)}\|_p^{1/n} = R, \qquad R = \sup\{|\xi| : \xi \in \operatorname{supp} \hat{f}(\xi)\}$$

Case p=2

$$\|f^{(n)}\|_2^{1/n} \leq \|\widehat{f^{(n)}}\|_2^{1/n} = \|(-i\xi)^n \hat{f}(\xi)\|_2^{1/n} \leq \left(R^n \|\hat{f}\|_2\right)^{1/n} \to R, \quad n \to \infty$$

$$\Rightarrow \quad \limsup_{n\to\infty} \|f^{(n)}\|_2^{1/n} \leq R.$$

Proof (simple case)

$$\lim_{n\to\infty} \|f^{(n)}\|_p^{1/n} = R, \qquad R = \sup\{|\xi| : \xi \in \operatorname{supp} \hat{f}(\xi)\}$$

Case p=2

$$\|f^{(n)}\|_2^{1/n} \leq \|\widehat{f^{(n)}}\|_2^{1/n} = \|(-i\xi)^n \hat{f}(\xi)\|_2^{1/n} \leq \left(R^n \|\hat{f}\|_2\right)^{1/n} \to R, \quad n \to \infty$$

$$\Rightarrow \quad \limsup_{n\to\infty} \|f^{(n)}\|_2^{1/n} \leq R.$$

Fix $\varepsilon > 0$. We have

$$\|f^{(n)}\|_2^{1/n} = \|(-i\xi)^n \hat{f}(\xi)\|_2^{1/n}$$

Proof (simple case)

$$\lim_{n\to\infty} \|f^{(n)}\|_p^{1/n} = R, \qquad R = \sup\{|\xi| : \xi \in \operatorname{supp} \hat{f}(\xi)\}$$

Case p=2

$$\|f^{(n)}\|_2^{1/n} \leq \|\widehat{f^{(n)}}\|_2^{1/n} = \|(-i\xi)^n \hat{f}(\xi)\|_2^{1/n} \leq \left(R^n \|\hat{f}\|_2\right)^{1/n} \to R, \quad n \to \infty$$

$$\Rightarrow \quad \limsup_{n\to\infty} \|f^{(n)}\|_2^{1/n} \leq R.$$

Fix $\varepsilon > 0$. We have

$$\|f^{(n)}\|_2^{1/n} = \|(-i\xi)^n \hat{f}(\xi)\|_2^{1/n} \geq \left(\int_{R-\varepsilon\leq|\xi|\leq R} |\xi^n \hat{f}(\xi)|^2 \, d\xi\right)^{1/2n}$$

Proof (simple case)

$$\lim_{n\to\infty} \|f^{(n)}\|_p^{1/n} = R, \qquad R = \sup\{|\xi| : \xi \in \operatorname{supp} \hat{f}(\xi)\}$$

Case p=2

$$\|f^{(n)}\|_2^{1/n} \le \|\widehat{f^{(n)}}\|_2^{1/n} = \|(-i\xi)^n \hat{f}(\xi)\|_2^{1/n} \le \left(R^n \|\hat{f}\|_2\right)^{1/n} \to R, \quad n \to \infty$$

$$\Rightarrow \quad \limsup_{n\to\infty} \|f^{(n)}\|_2^{1/n} \le R.$$

Fix $\varepsilon > 0$. We have

$$\|f^{(n)}\|_2^{1/n} = \|(-i\xi)^n \hat{f}(\xi)\|_2^{1/n} \ge \left(\int_{R-\varepsilon \le |\xi| \le R} |\xi^n \hat{f}(\xi)|^2 \, d\xi \right)^{1/2n}$$

$$\ge (R-\varepsilon) \underbrace{\left(\int_{R-\varepsilon \le |\xi| \le R} |\hat{f}(\xi)|^2 \, d\xi \right)}_{:= C_\varepsilon > 0}^{1/2n}$$

Proof (simple case)

$$\lim_{n\to\infty} \|f^{(n)}\|_p^{1/n} = R, \qquad R = \sup\{|\xi| : \xi \in \operatorname{supp} \hat{f}(\xi)\}$$

Case p=2

$$\|f^{(n)}\|_2^{1/n} \le \|\widehat{f^{(n)}}\|_2^{1/n} = \|(-i\xi)^n \hat{f}(\xi)\|_2^{1/n} \le \left(R^n \|\hat{f}\|_2\right)^{1/n} \to R, \quad n \to \infty$$

$$\Rightarrow \quad \limsup_{n\to\infty} \|f^{(n)}\|_2^{1/n} \le R.$$

Fix $\varepsilon > 0$. We have

$$\|f^{(n)}\|_2^{1/n} = \|(-i\xi)^n \hat{f}(\xi)\|_2^{1/n} \ge \left(\int_{R-\varepsilon\le|\xi|\le R} |\xi^n \hat{f}(\xi)|^2 \, d\xi\right)^{1/2n}$$

$$\ge (R-\varepsilon)\Bigg(\underbrace{\int_{R-\varepsilon\le|\xi|\le R} |\hat{f}(\xi)|^2 \, d\xi}_{:=C_\varepsilon>0}\Bigg)^{1/2n} \to R-\varepsilon, \quad n \to \infty$$

Proof (simple case)

$$\lim_{n\to\infty} \|f^{(n)}\|_p^{1/n} = R, \qquad R = \sup\{|\xi| : \xi \in \operatorname{supp} \hat{f}(\xi)\}$$

Case p=2

$$\|f^{(n)}\|_2^{1/n} \le \|\widehat{f^{(n)}}\|_2^{1/n} = \|(-i\xi)^n \hat{f}(\xi)\|_2^{1/n} \le \left(R^n \|\hat{f}\|_2\right)^{1/n} \to R, \quad n \to \infty$$

$$\Rightarrow \quad \limsup_{n\to\infty} \|f^{(n)}\|_2^{1/n} \le R.$$

Fix $\varepsilon > 0$. We have

$$\|f^{(n)}\|_2^{1/n} = \|(-i\xi)^n \hat{f}(\xi)\|_2^{1/n} \ge \left(\int_{R-\varepsilon\le|\xi|\le R} |\xi^n \hat{f}(\xi)|^2 \, d\xi\right)^{1/2n}$$

$$\ge (R-\varepsilon)\Bigg(\underbrace{\int_{R-\varepsilon\le|\xi|\le R} |\hat{f}(\xi)|^2 \, d\xi}_{:=C_\varepsilon>0}\Bigg)^{1/2n} \to R-\varepsilon, \quad n \to \infty$$

$$\Rightarrow \quad \liminf_{n\to\infty} \|f^{(n)}\|_2^{1/n} \ge R-\varepsilon, \quad \forall \varepsilon > 0.$$

Case $p \neq 2$

The proof in the case $p \neq 2$ is more complicated.

Case $p \neq 2$

The proof in the case $p \neq 2$ is more complicated.

Theorem (Andersen, 2004, Bull. London Math. Soc.)

Define, for $R > 0$,

$$PW_R(\mathbb{R}) := \{f \in C^\infty(\mathbb{R}) : \text{for all } N \in \mathbb{N}_0,$$

$$\sup_{x\in\mathbb{R},\ n\in\mathbb{N}_0} R^{-n} n^{-N} (1+|x|)^N |f^{(n)}(x)| < \infty\}.$$

Case $p \neq 2$

The proof in the case $p \neq 2$ is more complicated.

Theorem (Andersen, 2004, Bull. London Math. Soc.)

Define, for $R > 0$,

$$PW_R(\mathbb{R}) := \{f \in C^\infty(\mathbb{R}) : \text{for all } N \in \mathbb{N}_0,$$

$$\sup_{x \in \mathbb{R},\ n \in \mathbb{N}_0} R^{-n} n^{-N} (1+|x|)^N |f^{(n)}(x)| < \infty\}.$$

Then

1) $PW_R(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$.

Case $p \neq 2$

The proof in the case $p \neq 2$ is more complicated.

Theorem (Andersen, 2004, Bull. London Math. Soc.)

Define, for $R > 0$,

$$PW_R(\mathbb{R}) := \{f \in C^\infty(\mathbb{R}) : \text{for all } N \in \mathbb{N}_0,$$

$$\sup_{x\in\mathbb{R},\ n\in\mathbb{N}_0} R^{-n} n^{-N} (1+|x|)^N |f^{(n)}(x)| < \infty\}.$$

Then

1) $PW_R(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$.
2) The Fourier transform $\mathscr{F}$ is a bijection from $PW_R(\mathbb{R})$ onto

$$\{g \in C_c^\infty(\mathbb{R}) : \operatorname{supp} g \subseteq [-R, R]\}.$$

Case $p \neq 2$

The proof in the case $p \neq 2$ is more complicated.

Theorem (Andersen, 2004, Bull. London Math. Soc.)

Define, for $R > 0$,

$$PW_R(\mathbb{R}) := \{f \in C^\infty(\mathbb{R}) : \text{for all } N \in \mathbb{N}_0,$$

$$\sup_{x\in\mathbb{R},\ n\in\mathbb{N}_0} R^{-n} n^{-N} (1+|x|)^N |f^{(n)}(x)| < \infty\}.$$

Then

1) $PW_R(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$.
2) The Fourier transform $\mathscr{F}$ is a bijection from $PW_R(\mathbb{R})$ onto

$$\{g \in C_c^\infty(\mathbb{R}) : \operatorname{supp} g \subseteq [-R, R]\}.$$

As a consequence of this theorem, Andersen gives an alternative proof of the result of Bang, simpler than the original one.

Aims of this work

1. Define the Paley-Wiener space PW_R in the ultradifferentiable setting and give corresponding real Paley-Wiener theorems in the lines of Bang and Andersen.

Aims of this work

1. Define the Paley-Wiener space PW_R in the ultradifferentiable setting and give corresponding real Paley-Wiener theorems in the lines of Bang and Andersen.

2. Analyze the relations between the size of the support of the Fourier transform of a function f and time-frequency representations, and give new real Paley-Wiener theorems involving Gabor and Wigner transform.

Ultradifferentiable setting

Definition

A non-quasianalytic weight function is a continuous increasing function $\omega : [0, +\infty) \to [0, +\infty)$ satisfying:

(α) There exists $L > 0$ such that $\omega(2t) \leq L(\omega(t) + 1), \quad \forall t \geq 0$;

(β) $\int_1^{+\infty} \frac{\omega(t)}{t^2} dt < +\infty$;

(γ) $\exists a \in \mathbb{R}, b > 0$ s.t. $\quad \omega(t) \geq a + b \log(1 + t), \quad \forall t \geq 0$;

(δ) $\varphi_\omega : \; t \mapsto \omega(e^t)$ is convex.

Define $\omega(\zeta) := \omega(|\zeta|)$ for $\zeta \in \mathbb{C}^d$.

Ultradifferentiable setting

Definition

A non-quasianalytic weight function is a continuous increasing function $\omega : [0, +\infty) \to [0, +\infty)$ satisfying:

(α) There exists $L > 0$ such that $\omega(2t) \leq L(\omega(t) + 1), \quad \forall t \geq 0$;

(β) $\int_1^{+\infty} \frac{\omega(t)}{t^2} dt < +\infty$;

(γ) $\exists a \in \mathbb{R}, b > 0$ s.t. $\quad \omega(t) \geq a + b \log(1 + t), \quad \forall t \geq 0$;

(δ) $\varphi_\omega : \ t \mapsto \omega(e^t)$ is convex.

Define $\omega(\zeta) := \omega(|\zeta|)$ for $\zeta \in \mathbb{C}^d$.

Young Conjugate: $\varphi^*_\omega(s) := \sup_{t \geq 0}\{ts - \varphi_\omega(t)\}$.

Ultradifferentiable setting

Definition

A non-quasianalytic weight function is a continuous increasing function $\omega : [0, +\infty) \to [0, +\infty)$ satisfying:

(α) There exists $L > 0$ such that $\omega(2t) \leq L(\omega(t) + 1), \quad \forall t \geq 0$;

(β) $\int_1^{+\infty} \frac{\omega(t)}{t^2} dt < +\infty$;

(γ) $\exists a \in \mathbb{R}, b > 0$ s.t. $\quad \omega(t) \geq a + b \log(1 + t), \quad \forall t \geq 0$;

(δ) $\varphi_\omega : \ t \mapsto \omega(e^t)$ is convex.

Define $\omega(\zeta) := \omega(|\zeta|)$ for $\zeta \in \mathbb{C}^d$.

Young Conjugate: $\varphi^*_\omega(s) := \sup_{t \geq 0}\{ts - \varphi_\omega(t)\}$.

Definition (Björck, 1966)

$\mathcal{S}_\omega(\mathbb{R}^d)$ is the set of all $u \in L^1(\mathbb{R}^d)$ such that $u, \hat{u} \in C^\infty(\mathbb{R}^d)$ and

(i) $\forall \lambda > 0, \alpha \in \mathbb{N}_0^d : \quad \sup_{x \in \mathbb{R}^d} e^{\lambda \omega(x)} |D^\alpha u(x)| < +\infty$;

(ii) $\forall \lambda > 0, \alpha \in \mathbb{N}_0^d : \quad \sup_{\xi \in \mathbb{R}^d} e^{\lambda \omega(\xi)} |D^\alpha \hat{u}(\xi)| < +\infty$.

Characterization of $\mathcal{S}_\omega(\mathbb{R}^d)$

Theorem

For $u \in \mathcal{S}(\mathbb{R}^d)$, $u \in \mathcal{S}_\omega$ iff one of the following equivalent conditions is satisfied:

1. $\forall \lambda > 0, \alpha \in \mathbb{N}_0^d : \ \|e^{\lambda\omega(x)} D^\alpha u(x)\|_{L^p} < +\infty, \qquad p \in [1,\infty]$
   $\forall \lambda > 0, \alpha \in \mathbb{N}_0^d : \ \|e^{\lambda\omega(\xi)} D^\alpha \hat{u}(\xi)\|_{L^q} < +\infty, \qquad q \in [1,\infty]$
2. $\forall \lambda > 0, \alpha \in \mathbb{N}_0^d : \ \|e^{\lambda\omega(x)} x^\alpha u(x)\|_{L^p} < +\infty$
   $\forall \lambda > 0, \alpha \in \mathbb{N}_0^d : \ \|e^{\lambda\omega(\xi)} \xi^\alpha \hat{u}(\xi)\|_{L^q} < +\infty$
3. $\forall \lambda > 0 : \ \|e^{\lambda\omega(x)} u(x)\|_{L^p} < +\infty$
   $\forall \lambda > 0 : \ \|e^{\lambda\omega(\xi)} \hat{u}(\xi)\|_{L^q} < +\infty$
4. $\forall \beta \in \mathbb{N}_0^d, \lambda > 0 \ \exists C_{\beta,\lambda} > 0 : \|x^\beta D^\alpha u(x)\|_{L^p} e^{-\lambda\varphi^*_\omega\left(\frac{|\alpha|}{\lambda}\right)} \le C_{\beta,\lambda} \ \forall \alpha \in \mathbb{N}_0^d$
   $\forall \alpha \in \mathbb{N}_0^d, \mu > 0 \ \exists C_{\alpha,\mu} > 0 : \|x^\beta D^\alpha u(x)\|_{L^q} e^{-\mu\varphi^*_\omega\left(\frac{|\beta|}{\mu}\right)} \le C_{\alpha,\mu} \ \forall \beta \in \mathbb{N}_0^d$
5. $\forall \mu, \lambda > 0, \exists C_{\mu,\lambda} > 0 : \|x^\beta D^\alpha u(x)\|_{L^p} e^{-\lambda\varphi^*_\omega\left(\frac{|\alpha|}{\lambda}\right)} e^{-\mu\varphi^*_\omega\left(\frac{|\beta|}{\mu}\right)} \le C_{\mu,\lambda} \ \forall \alpha, \beta$
6. $\forall \lambda > 0, \exists\, C_\lambda > 0 : \ \|x^\beta D^\alpha u(x)\|_{L^p} e^{-\lambda\varphi^*_\omega\left(\frac{|\alpha+\beta|}{\lambda}\right)} \le C_\lambda \ \forall \alpha, \beta \in \mathbb{N}_0^d$
7. $\forall \mu, \lambda > 0, \exists C_{\mu,\lambda} > 0 : \ \|e^{\mu\omega(x)} D^\alpha u(x)\|_{L^p} e^{-\lambda\varphi^*_\omega\left(\frac{|\alpha|}{\lambda}\right)} \le C_{\mu,\lambda} \ \forall \alpha \in \mathbb{N}_0^d$
8. $\varphi \in \mathcal{S}_\omega(\mathbb{R}^d) \setminus \{0\}$; $\forall \lambda > 0 : \ \|V_\varphi u(z) e^{\lambda\omega(z)}\|_{L^{p,q}} < +\infty$

Paley-Wiener Theorem in ultradifferentiable classes

Let $K \subset \mathbb{R}^d$ a **compact convex** set and let $f \in L^1(\mathbb{R}^d)$.

Let $K \subset \mathbb{R}^d$ a **compact convex** set and let $f \in L^1(\mathbb{R}^d)$.

Supporting function of K: $$H_K(x) := \sup_{y \in K} \langle x, y \rangle.$$

Paley-Wiener Theorem in ultradifferentiable classes

Let $K \subset \mathbb{R}^d$ a **compact convex** set and let $f \in L^1(\mathbb{R}^d)$.

Supporting function of K: $$H_K(x) := \sup_{y \in K} \langle x, y \rangle.$$

Theorem (Braun, Meise, Taylor, 1990)

The function $f \in \mathcal{S}_\omega(\mathbb{R}^d)$ satisfies

$$\operatorname{supp} \hat{f} \subset K$$

if and only if f is an entire function and for all $\ell \in \mathbb{N}_0$ there exists $C_\ell > 0$ such that

$$|f(z)| \le C_\ell e^{H_K(\operatorname{Im} z) - \ell\omega(z)}$$

for all $z \in \mathbb{C}^d$.

Real Paley-Wiener Theorem in ultradifferentiable classes

Let ω be a non-quasianalytic weight function and $R > 0$. We define $\mathrm{PW}^{\omega}_{R}(\mathbb{R}^d)$ as the set of all functions $f \in C^{\infty}(\mathbb{R}^d)$ such that for every $\lambda > 0$,

$$\sup_{\alpha \in \mathbb{N}_0^d} \sup_{x \in \mathbb{R}^d} R^{-|\alpha|} e^{\lambda\omega\left(\frac{x}{|\alpha|+1}\right)} |f^{(\alpha)}(x)| < +\infty.$$

Real Paley-Wiener Theorem in ultradifferentiable classes

Let ω be a non-quasianalytic weight function and $R > 0$. We define $\mathrm{PW}_R^{\omega}(\mathbb{R}^d)$ as the set of all functions $f \in C^{\infty}(\mathbb{R}^d)$ such that for every $\lambda > 0$,

$$\sup_{\alpha \in \mathbb{N}_0^d} \sup_{x \in \mathbb{R}^d} R^{-|\alpha|} e^{\lambda \omega\left(\frac{x}{|\alpha|+1}\right)} |f^{(\alpha)}(x)| < +\infty.$$

Theorem

1. $\mathrm{PW}_R^{\omega}(\mathbb{R}^d) \subseteq \mathcal{S}_{\omega}(\mathbb{R}^d)$.

Real Paley-Wiener Theorem in ultradifferentiable classes

Let ω be a non-quasianalytic weight function and $R > 0$. We define $\mathrm{PW}_R^\omega(\mathbb{R}^d)$ as the set of all functions $f \in C^\infty(\mathbb{R}^d)$ such that for every $\lambda > 0$,

$$\sup_{\alpha \in \mathbb{N}_0^d} \sup_{x \in \mathbb{R}^d} R^{-|\alpha|} e^{\lambda\omega\left(\frac{x}{|\alpha|+1}\right)} |f^{(\alpha)}(x)| < +\infty.$$

Theorem

1. $\mathrm{PW}_R^\omega(\mathbb{R}^d) \subseteq \mathcal{S}_\omega(\mathbb{R}^d)$.
2. A function $f \in \mathcal{S}_\omega(\mathbb{R}^d)$ satisfies

$$\sup\{|\xi|_\infty : \ \xi \in \operatorname{supp} \hat{f}\} = R < +\infty$$

if and only if $f \in \mathrm{PW}_R^\omega(\mathbb{R}^d)$.

Real Paley-Wiener Theorem in ultradifferentiable classes

Let ω be a non-quasianalytic weight function and $R > 0$. We define $\mathrm{PW}_R^\omega(\mathbb{R}^d)$ as the set of all functions $f \in C^\infty(\mathbb{R}^d)$ such that for every $\lambda > 0$,

$$\sup_{\alpha \in \mathbb{N}_0^d} \sup_{x \in \mathbb{R}^d} R^{-|\alpha|} e^{\lambda \omega\left(\frac{x}{|\alpha|+1}\right)} |f^{(\alpha)}(x)| < +\infty.$$

Theorem

1. $\mathrm{PW}_R^\omega(\mathbb{R}^d) \subseteq \mathcal{S}_\omega(\mathbb{R}^d)$.
2. A function $f \in \mathcal{S}_\omega(\mathbb{R}^d)$ satisfies

$$\sup\{|\xi|_\infty : \ \xi \in \operatorname{supp} \hat{f}\} = R < +\infty$$

if and only if $f \in \mathrm{PW}_R^\omega(\mathbb{R}^d)$.

Notation

We denote

$$R_{\hat{f}} := \sup\{|\xi|_\infty : \ \xi \in \operatorname{supp} \hat{f}\}.$$

In the following it may happen that $R = +\infty$.

Proof of (2) of the Theorem, sufficiency

Assume $f \in \mathcal{S}_\omega(\mathbb{R}^d)$ satisfies $\sup\{|\xi|_\infty : \ \xi \in \operatorname{supp} \hat{f}\} = R < +\infty$.

Proof of (2) of the Theorem, sufficiency

Assume $f \in \mathcal{S}_\omega(\mathbb{R}^d)$ satisfies $\sup\{|\xi|_\infty : \xi \in \operatorname{supp} \hat{f}\} = R < +\infty$. By Fourier inversion formula for $x \neq 0$ and $N \in \mathbb{N}_0$:

Proof of (2) of the Theorem, sufficiency

Assume $f \in \mathcal{S}_\omega(\mathbb{R}^d)$ satisfies $\sup\{|\xi|_\infty : \xi \in \operatorname{supp} \hat{f}\} = R < +\infty$. By Fourier inversion formula for $x \neq 0$ and $N \in \mathbb{N}_0$:

$$|D^\alpha f(x)| = \frac{1}{(2\pi)^d} \left| \int_{\mathbb{R}^d} \mathcal{F}(D^\alpha f)(\xi) e^{i\langle x,\xi\rangle} d\xi \right| \leq \frac{1}{|x|^{2N}} \int_{\mathbb{R}^d} |\Delta_\xi^N \xi^\alpha \hat{f}(\xi)| d\xi$$

Proof of (2) of the Theorem, sufficiency

Assume $f \in \mathcal{S}_\omega(\mathbb{R}^d)$ satisfies $\sup\{|\xi|_\infty : \xi \in \operatorname{supp} \hat{f}\} = R < +\infty$. By Fourier inversion formula for $x \neq 0$ and $N \in \mathbb{N}_0$:

$$|D^\alpha f(x)| = \frac{1}{(2\pi)^d} \left| \int_{\mathbb{R}^d} \mathcal{F}(D^\alpha f)(\xi) e^{i\langle x, \xi \rangle} d\xi \right| \leq \frac{1}{|x|^{2N}} \int_{\mathbb{R}^d} |\Delta_\xi^N \xi^\alpha \hat{f}(\xi)| d\xi$$

$$\leq \frac{1}{|x|^{2N}} \int_{\mathbb{R}^d} \sum_{|\nu|=N} \frac{N!}{\nu!} \left| D_{\xi_1}^{2\nu_1} \cdots D_{\xi_d}^{2\nu_d} \left(\xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d} \hat{f}(\xi) \right) \right| d\xi$$

Proof of (2) of the Theorem, sufficiency

Assume $f \in \mathcal{S}_\omega(\mathbb{R}^d)$ satisfies $\sup\{|\xi|_\infty : \xi \in \operatorname{supp} \hat{f}\} = R < +\infty$. By Fourier inversion formula for $x \neq 0$ and $N \in \mathbb{N}_0$:

$$|D^\alpha f(x)| = \frac{1}{(2\pi)^d} \left| \int_{\mathbb{R}^d} \mathcal{F}(D^\alpha f)(\xi) e^{i\langle x,\xi\rangle} d\xi \right| \leq \frac{1}{|x|^{2N}} \int_{\mathbb{R}^d} |\Delta_\xi^N \xi^\alpha \hat{f}(\xi)| d\xi$$
$$\leq \frac{1}{|x|^{2N}} \int_{\mathbb{R}^d} \sum_{|\nu|=N} \frac{N!}{\nu!} \left| D_{\xi_1}^{2\nu_1} \cdots D_{\xi_d}^{2\nu_d} \left(\xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d} \hat{f}(\xi) \right) \right| d\xi$$

Now, we use that $f \in \mathcal{S}_\omega(\mathbb{R}^d)$ and the properties of the weight function ω to obtain:

Proof of (2) of the Theorem, sufficiency

Assume $f \in \mathcal{S}_\omega(\mathbb{R}^d)$ satisfies $\sup\{|\xi|_\infty : \xi \in \operatorname{supp} \hat{f}\} = R < +\infty$. By Fourier inversion formula for $x \neq 0$ and $N \in \mathbb{N}_0$:

$$|D^\alpha f(x)| = \frac{1}{(2\pi)^d} \left| \int_{\mathbb{R}^d} \mathcal{F}(D^\alpha f)(\xi) e^{i\langle x,\xi\rangle} d\xi \right| \leq \frac{1}{|x|^{2N}} \int_{\mathbb{R}^d} |\Delta_\xi^N \xi^\alpha \hat{f}(\xi)| d\xi$$

$$\leq \frac{1}{|x|^{2N}} \int_{\mathbb{R}^d} \sum_{|\nu|=N} \frac{N!}{\nu!} \left| D_{\xi_1}^{2\nu_1} \cdots D_{\xi_d}^{2\nu_d} \left(\xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d} \hat{f}(\xi) \right) \right| d\xi$$

Now, we use that $f \in \mathcal{S}_\omega(\mathbb{R}^d)$ and the properties of the weight function ω to obtain:

$$|D^\alpha f(x)| \leq C_{\lambda'} e^{-\lambda' \omega\left(\frac{x}{|\alpha|+1}\right)} R^{|\alpha|}.$$

Proof of (2) of the Theorem, necessity

Let $f \in \mathrm{PW}^{\omega}_{R}(\mathbb{R}^d)$.

Proof of (2) of the Theorem, necessity

Let $f \in \mathrm{PW}_R^\omega(\mathbb{R}^d)$. We integrate by parts,

$$|\hat{f}(\xi)| \leq \frac{1}{\xi_1^{2N} + \cdots + \xi_d^{2N}} \sum_{j=1}^{d} \int_{\mathbb{R}^d} |D_{x_j}^{2N} f(x)|\, dx.$$

Proof of (2) of the Theorem, necessity

Let $f \in \mathrm{PW}_R^\omega(\mathbb{R}^d)$. We integrate by parts,

$$|\hat{f}(\xi)| \leq \frac{1}{\xi_1^{2N} + \cdots + \xi_d^{2N}} \sum_{j=1}^{d} \int_{\mathbb{R}^d} |D_{x_j}^{2N} f(x)|\, dx.$$

By hypothesis, we have that for every $\lambda > 0$ there exists C_λ (independent of N) such that

$$|\hat{f}(\xi)| \leq C_\lambda \frac{dR^{2N}(2N+1)^d}{\xi_1^{2N} + \cdots + \xi_d^{2N}}.$$

Proof of (2) of the Theorem, necessity

Let $f \in \mathrm{PW}_R^\omega(\mathbb{R}^d)$. We integrate by parts,

$$|\hat{f}(\xi)| \leq \frac{1}{\xi_1^{2N} + \cdots + \xi_d^{2N}} \sum_{j=1}^{d} \int_{\mathbb{R}^d} |D_{x_j}^{2N} f(x)| \, dx.$$

By hypothesis, we have that for every $\lambda > 0$ there exists C_λ (independent of N) such that

$$|\hat{f}(\xi)| \leq C_\lambda \frac{dR^{2N}(2N+1)^d}{\xi_1^{2N} + \cdots + \xi_d^{2N}}.$$

We conclude since $|\xi|_\infty > R$ implies $\xi_1^{2N} + \cdots + \xi_d^{2N} > R^{2N}$.

Real Paley-Wiener Theorem in ultradifferentiable classes

Theorem

Let $1 \leq p \leq +\infty$ and $f \in C^\infty(\mathbb{R}^d)$. We have:

1. If $f^{(\alpha)}(x) \in L^p(\mathbb{R}^d)$ for all $\alpha \in \mathbb{N}_0^d$, we have

$$\lim_{n \to +\infty} \left(\max_{|\alpha|=n} \left\| f^{(\alpha)}(x) \right\|_{L^p} \right)^{1/n} = R_{\hat{f}}.$$

Real Paley-Wiener Theorem in ultradifferentiable classes

Theorem

Let $1 \leq p \leq +\infty$ and $f \in C^\infty(\mathbb{R}^d)$. We have:

1. If $f^{(\alpha)}(x) \in L^p(\mathbb{R}^d)$ for all $\alpha \in \mathbb{N}_0^d$, we have

$$\lim_{n \to +\infty} \left(\max_{|\alpha|=n} \left\| f^{(\alpha)}(x) \right\|_{L^p} \right)^{1/n} = R_{\hat{f}}.$$

2. Assume that $e^{\lambda\omega\left(\frac{x}{|\alpha|+1}\right)} f^{(\alpha)}(x) \in L^p(\mathbb{R}^d)$ for all $\alpha \in \mathbb{N}_0^d$ and for some $\lambda > 0$, and that the weight function ω satisfies some mild conditions. Then

$$\lim_{n \to +\infty} \left(\max_{|\alpha|=n} \left\| e^{\mu\omega\left(\frac{x}{|\alpha|+1}\right)} f^{(\alpha)}(x) \right\|_{L^p} \right)^{1/n} = R_{\hat{f}}, \text{ for all } \ 0 \leq \mu \leq \lambda.$$

Real Paley-Wiener Theorem in ultradifferentiable classes

Theorem

Let $1 \leq p \leq +\infty$ and $f \in C^\infty(\mathbb{R}^d)$. We have:

1. If $f^{(\alpha)}(x) \in L^p(\mathbb{R}^d)$ for all $\alpha \in \mathbb{N}_0^d$, we have

$$\lim_{n \to +\infty} \left(\max_{|\alpha|=n} \left\| f^{(\alpha)}(x) \right\|_{L^p} \right)^{1/n} = R_{\hat{f}}.$$

2. Assume that $e^{\lambda\omega\left(\frac{x}{|\alpha|+1}\right)} f^{(\alpha)}(x) \in L^p(\mathbb{R}^d)$ for all $\alpha \in \mathbb{N}_0^d$ and for some $\lambda > 0$, and that the weight function ω satisfies some mild conditions. Then

$$\lim_{n \to +\infty} \left(\max_{|\alpha|=n} \left\| e^{\mu\omega\left(\frac{x}{|\alpha|+1}\right)} f^{(\alpha)}(x) \right\|_{L^p} \right)^{1/n} = R_{\hat{f}}, \text{ for all } \ 0 \leq \mu \leq \lambda.$$

- The first part is an extension to several variables of previous results of Bang and Andersen.

Real Paley-Wiener Theorem in ultradifferentiable classes

Theorem

Let $1 \leq p \leq +\infty$ and $f \in C^\infty(\mathbb{R}^d)$. We have:

1. If $f^{(\alpha)}(x) \in L^p(\mathbb{R}^d)$ for all $\alpha \in \mathbb{N}_0^d$, we have
$$\lim_{n\to+\infty}\left(\max_{|\alpha|=n}\left\|f^{(\alpha)}(x)\right\|_{L^p}\right)^{1/n} = R_{\hat{f}}.$$
2. Assume that $e^{\lambda\omega\left(\frac{x}{|\alpha|+1}\right)}f^{(\alpha)}(x) \in L^p(\mathbb{R}^d)$ for all $\alpha \in \mathbb{N}_0^d$ and for some $\lambda > 0$, and that the weight function ω satisfies some mild conditions. Then
$$\lim_{n\to+\infty}\left(\max_{|\alpha|=n}\left\|e^{\mu\omega\left(\frac{x}{|\alpha|+1}\right)}f^{(\alpha)}(x)\right\|_{L^p}\right)^{1/n} = R_{\hat{f}}, \text{ for all } \ 0 \leq \mu \leq \lambda.$$

- The first part is an extension to several variables of previous results of Bang and Andersen.
- The second part is satisfied for example when ω is sub-additive.

Proof of (2) of the Theorem, $\leq$

Assume $p < \infty$.

Proof of (2) of the Theorem, $\leq$

Assume $p < \infty$. Take $\phi \in \mathcal{S}_\omega(\mathbb{R}^d)$ such that $\hat{\phi}$ has compact support.

Proof of (2) of the Theorem, $\leq$

Assume $p < \infty$. Take $\phi \in \mathcal{S}_\omega(\mathbb{R}^d)$ such that $\hat{\phi}$ has compact support. We have $\phi \in \mathrm{PW}^{\omega}_{R_{\hat{\phi}}}(\mathbb{R}^d)$ and it is easy to see that

$$\limsup_{n\to+\infty} \left(\max_{|\alpha|=n} \left\| e^{\lambda\omega\left(\frac{x}{|\alpha|+1}\right)} \phi^{(\alpha)}(x) \right\|_{L^p}^{1/n} \right) \leq R_{\hat{\phi}}\,, \qquad \lambda > 0.$$

Proof of (2) of the Theorem, $\leq$

Assume $p < \infty$. Take $\phi \in \mathcal{S}_\omega(\mathbb{R}^d)$ such that $\hat{\phi}$ has compact support. We have $\phi \in \mathrm{PW}^\omega_{R_{\hat{\phi}}}(\mathbb{R}^d)$ and it is easy to see that

$$\limsup_{n\to+\infty} \left(\max_{|\alpha|=n} \left\| e^{\lambda\omega\left(\frac{x}{|\alpha|+1}\right)} \phi^{(\alpha)}(x) \right\|_{L^p}^{1/n} \right) \leq R_{\hat{\phi}}\,, \qquad \lambda > 0.$$

Now, we fix $\varepsilon > 0$ and choose $\phi \in \mathcal{S}_\omega(\mathbb{R}^d)$ such that $\hat{\phi} \equiv 1$ in a neighborhood of $[-R_{\hat{f}}, R_{\hat{f}}]^d$ and $\hat{\phi} \equiv 0$ outside $[-R_{\hat{f}} - \varepsilon, R_{\hat{f}} + \varepsilon]^d$.

Proof of (2) of the Theorem, $\leq$

Assume $p < \infty$. Take $\phi \in \mathcal{S}_\omega(\mathbb{R}^d)$ such that $\hat{\phi}$ has compact support. We have $\phi \in \mathrm{PW}^{\omega}_{R_{\hat{\phi}}}(\mathbb{R}^d)$ and it is easy to see that

$$\limsup_{n\to+\infty} \left(\max_{|\alpha|=n} \left\| e^{\lambda\omega\left(\frac{x}{|\alpha|+1}\right)} \phi^{(\alpha)}(x) \right\|_{L^p}^{1/n} \right) \leq R_{\hat{\phi}}\,, \qquad \lambda > 0.$$

Now, we fix $\varepsilon > 0$ and choose $\phi \in \mathcal{S}_\omega(\mathbb{R}^d)$ such that $\hat{\phi} \equiv 1$ in a neighborhood of $[-R_{\hat{f}}, R_{\hat{f}}]^d$ and $\hat{\phi} \equiv 0$ outside $[-R_{\hat{f}} - \varepsilon, R_{\hat{f}} + \varepsilon]^d$. Hence, $f = f * \phi$.

Proof of (2) of the Theorem, $\leq$

Assume $p < \infty$. Take $\phi \in \mathcal{S}_\omega(\mathbb{R}^d)$ such that $\hat{\phi}$ has compact support. We have $\phi \in \mathrm{PW}^\omega_{R_{\hat{\phi}}}(\mathbb{R}^d)$ and it is easy to see that

$$\limsup_{n\to+\infty}\left(\max_{|\alpha|=n}\left\|e^{\lambda\omega\left(\frac{x}{|\alpha|+1}\right)}\phi^{(\alpha)}(x)\right\|_{L^p}^{1/n}\right) \leq R_{\hat{\phi}}\,, \qquad \lambda > 0.$$

Now, we fix $\varepsilon > 0$ and choose $\phi \in \mathcal{S}_\omega(\mathbb{R}^d)$ such that $\hat{\phi} \equiv 1$ in a neighborhood of $[-R_{\hat{f}}, R_{\hat{f}}]^d$ and $\hat{\phi} \equiv 0$ outside $[-R_{\hat{f}} - \varepsilon, R_{\hat{f}} + \varepsilon]^d$. Hence, $f = f * \phi$. So,

$$\limsup_{n\to+\infty}\left(\max_{|\alpha|=n}\left\|e^{\lambda\omega\left(\frac{x}{|\alpha|+1}\right)}f^{(\alpha)}(x)\right\|_{L^p}^{1/n}\right) = \limsup_{n\to+\infty}\left(\max_{|\alpha|=n}\left\|e^{\lambda\omega\left(\frac{x}{n+1}\right)}f * \phi^{(\alpha)}(x)\right\|_{L^p}^{1/n}\right)$$

Proof of (2) of the Theorem, $\leq$

Assume $p < \infty$. Take $\phi \in \mathcal{S}_\omega(\mathbb{R}^d)$ such that $\hat{\phi}$ has compact support. We have $\phi \in \mathrm{PW}^{\omega}_{R_{\hat{\phi}}}(\mathbb{R}^d)$ and it is easy to see that

$$\limsup_{n\to+\infty} \left(\max_{|\alpha|=n} \left\| e^{\lambda\omega\left(\frac{x}{|\alpha|+1}\right)} \phi^{(\alpha)}(x) \right\|_{L^p}^{1/n} \right) \leq R_{\hat{\phi}} , \qquad \lambda > 0.$$

Now, we fix $\varepsilon > 0$ and choose $\phi \in \mathcal{S}_\omega(\mathbb{R}^d)$ such that $\hat{\phi} \equiv 1$ in a neighborhood of $[-R_{\hat{f}}, R_{\hat{f}}]^d$ and $\hat{\phi} \equiv 0$ outside $[-R_{\hat{f}} - \varepsilon, R_{\hat{f}} + \varepsilon]^d$. Hence, $f = f * \phi$. So,

$$\limsup_{n\to+\infty} \left(\max_{|\alpha|=n} \left\| e^{\lambda\omega\left(\frac{x}{|\alpha|+1}\right)} f^{(\alpha)}(x) \right\|_{L^p}^{1/n} \right) = \limsup_{n\to+\infty} \left(\max_{|\alpha|=n} \left\| e^{\lambda\omega\left(\frac{x}{n+1}\right)} f * \phi^{(\alpha)}(x) \right\|_{L^p}^{1/n} \right)$$

$$\leq \limsup_{n\to+\infty} \left(\max_{|\alpha|=n} \left\| e^{\lambda L\omega\left(\frac{x}{n+1}\right)} \phi^{(\alpha)}(x) \right\|_{L^1}^{1/n} \right) \left\| e^{\lambda\omega(x)} f(x) \right\|_{L^p}^{1/n} \leq R_{\hat{\phi}} \leq R_{\hat{f}} + \varepsilon.$$

Proof of (2) of the Theorem, $\leq$

Assume $p < \infty$. Take $\phi \in \mathcal{S}_\omega(\mathbb{R}^d)$ such that $\hat{\phi}$ has compact support. We have $\phi \in \mathrm{PW}^{\omega}_{R_{\hat{\phi}}}(\mathbb{R}^d)$ and it is easy to see that

$$\limsup_{n\to+\infty} \left(\max_{|\alpha|=n} \left\| e^{\lambda\omega\left(\frac{x}{|\alpha|+1}\right)} \phi^{(\alpha)}(x) \right\|_{L^p}^{1/n} \right) \leq R_{\hat{\phi}}\,, \qquad \lambda > 0.$$

Now, we fix $\varepsilon > 0$ and choose $\phi \in \mathcal{S}_\omega(\mathbb{R}^d)$ such that $\hat{\phi} \equiv 1$ in a neighborhood of $[-R_{\hat{f}}, R_{\hat{f}}]^d$ and $\hat{\phi} \equiv 0$ outside $[-R_{\hat{f}} - \varepsilon, R_{\hat{f}} + \varepsilon]^d$. Hence, $f = f * \phi$. So,

$$\limsup_{n\to+\infty} \left(\max_{|\alpha|=n} \left\| e^{\lambda\omega\left(\frac{x}{|\alpha|+1}\right)} f^{(\alpha)}(x) \right\|_{L^p}^{1/n} \right) = \limsup_{n\to+\infty} \left(\max_{|\alpha|=n} \left\| e^{\lambda\omega\left(\frac{x}{n+1}\right)} f * \phi^{(\alpha)}(x) \right\|_{L^p}^{1/n} \right)$$

$$\leq \limsup_{n\to+\infty} \left(\max_{|\alpha|=n} \left\| e^{\lambda L\omega\left(\frac{x}{n+1}\right)} \phi^{(\alpha)}(x) \right\|_{L^1}^{1/n} \right) \left\| e^{\lambda\omega(x)} f(x) \right\|_{L^p}^{1/n} \leq R_{\hat{\phi}} \leq R_{\hat{f}} + \varepsilon.$$

Therefore

$$\boxed{\limsup_{n\to+\infty} \left(\max_{|\alpha|=n} \left\| e^{\lambda\omega\left(\frac{x}{n+1}\right)} f^{(\alpha)}(x) \right\|_{L^p}^{1/n} \right) \leq R_{\hat{f}}.}$$

Proof of (2) of the Theorem, $\geq$

Proof of (2) of the Theorem, $\geq$

Take now $0 \neq \xi^0 \in \operatorname{supp} \hat{f}$, and assume w.l.o.g. that $0 < \varepsilon < |\xi_1^0| = |\xi^0|_\infty$, where $\xi^0 = (\xi_1^0, \ldots, \xi_d^0) \in \mathbb{R}^d$.

Proof of (2) of the Theorem, $\geq$

Take now $0 \neq \xi^0 \in \operatorname{supp} \hat{f}$, and assume w.l.o.g. that $0 < \varepsilon < |\xi_1^0| = |\xi^0|_\infty$, where $\xi^0 = (\xi_1^0, \dots, \xi_d^0) \in \mathbb{R}^d$.
We take $\psi \in \mathcal{S}_\omega(\mathbb{R}^d)$ a suitable function with compact support s.t. $\langle \hat{f}, \psi \rangle \neq 0$ with $\Pi_1 \operatorname{supp} \psi \subseteq \left[\xi_1^0 - \frac{\varepsilon}{2}, \xi_1^0 + \frac{\varepsilon}{2}\right]$.

Proof of (2) of the Theorem, $\geq$

Take now $0 \neq \xi^0 \in \operatorname{supp} \hat{f}$, and assume w.l.o.g. that $0 < \varepsilon < |\xi_1^0| = |\xi^0|_\infty$, where $\xi^0 = (\xi_1^0, \ldots, \xi_d^0) \in \mathbb{R}^d$.
We take $\psi \in \mathcal{S}_\omega(\mathbb{R}^d)$ a suitable function with compact support s.t. $\langle \hat{f}, \psi \rangle \neq 0$ with $\Pi_1 \operatorname{supp} \psi \subseteq \left[\xi_1^0 - \frac{\varepsilon}{2}, \xi_1^0 + \frac{\varepsilon}{2}\right]$.
Then, for $\xi \in \mathbb{R}^d$ with $\xi_1 \neq 0$, $\lambda > 0$ and $1 \leq p < +\infty$ we have

$$(|\xi_1^0| - \varepsilon)|\langle \hat{f}(\xi), \psi(\xi) \rangle|^{1/n} = (|\xi_1^0| - \varepsilon)|\langle \xi_1^n \hat{f}(\xi), \xi_1^{-n} \psi(\xi) \rangle|^{1/n}$$

Proof of (2) of the Theorem, $\geq$

Take now $0 \neq \xi^0 \in \operatorname{supp} \hat{f}$, and assume w.l.o.g. that $0 < \varepsilon < |\xi_1^0| = |\xi^0|_\infty$, where $\xi^0 = (\xi_1^0, \ldots, \xi_d^0) \in \mathbb{R}^d$.

We take $\psi \in \mathcal{S}_\omega(\mathbb{R}^d)$ a suitable function with compact support s.t. $\langle \hat{f}, \psi \rangle \neq 0$ with $\Pi_1 \operatorname{supp} \psi \subseteq \left[\xi_1^0 - \frac{\varepsilon}{2}, \xi_1^0 + \frac{\varepsilon}{2}\right]$.

Then, for $\xi \in \mathbb{R}^d$ with $\xi_1 \neq 0$, $\lambda > 0$ and $1 \leq p < +\infty$ we have

$$\begin{aligned}
(|\xi_1^0| - \varepsilon)|\langle \hat{f}(\xi), \psi(\xi)\rangle|^{1/n} &= (|\xi_1^0| - \varepsilon)|\langle \xi_1^n \hat{f}(\xi), \xi_1^{-n}\psi(\xi)\rangle|^{1/n} \\
&= (|\xi_1^0| - \varepsilon)|\langle \widehat{D_1^n f}(\xi), \xi_1^{-n}\psi(\xi)\rangle|^{1/n} \\
&= (|\xi_1^0| - \varepsilon)|\langle D_1^n f(x), \mathcal{F}^{-1}(\xi_1^{-n}\psi(\xi))(x)\rangle|^{1/n}
\end{aligned}$$

Proof of (2) of the Theorem, $\geq$

Take now $0 \neq \xi^0 \in \operatorname{supp} \hat{f}$, and assume w.l.o.g. that $0 < \varepsilon < |\xi_1^0| = |\xi^0|_\infty$, where $\xi^0 = (\xi_1^0, \dots, \xi_d^0) \in \mathbb{R}^d$.
We take $\psi \in \mathcal{S}_\omega(\mathbb{R}^d)$ a suitable function with compact support s.t. $\langle \hat{f}, \psi \rangle \neq 0$ with $\Pi_1 \operatorname{supp} \psi \subseteq \left[\xi_1^0 - \frac{\varepsilon}{2}, \xi_1^0 + \frac{\varepsilon}{2}\right]$.
Then, for $\xi \in \mathbb{R}^d$ with $\xi_1 \neq 0$, $\lambda > 0$ and $1 \leq p < +\infty$ we have

$$\begin{aligned}
(|\xi_1^0| - \varepsilon)|\langle \hat{f}(\xi), \psi(\xi) \rangle|^{1/n} &= (|\xi_1^0| - \varepsilon)|\langle \xi_1^n \hat{f}(\xi), \xi_1^{-n} \psi(\xi) \rangle|^{1/n} \\
&= (|\xi_1^0| - \varepsilon)|\langle \widehat{D_1^n f}(\xi), \xi_1^{-n} \psi(\xi) \rangle|^{1/n} \\
&= (|\xi_1^0| - \varepsilon)|\langle D_1^n f(x), \mathcal{F}^{-1}(\xi_1^{-n} \psi(\xi))(x) \rangle|^{1/n} \\
&\leq (|\xi_1^0| - \varepsilon)\|D_1^n f\|_{L^p}^{1/n} \|\mathcal{F}^{-1}(\xi_1^{-n} \psi(\xi))\|_{L^{p'}}^{1/n}
\end{aligned}$$

Proof of (2) of the Theorem, $\geq$

Take now $0 \neq \xi^0 \in \operatorname{supp} \hat{f}$, and assume w.l.o.g. that $0 < \varepsilon < |\xi_1^0| = |\xi^0|_\infty$, where $\xi^0 = (\xi_1^0, \ldots, \xi_d^0) \in \mathbb{R}^d$.
We take $\psi \in \mathcal{S}_\omega(\mathbb{R}^d)$ a suitable function with compact support s.t. $\langle \hat{f}, \psi \rangle \neq 0$ with $\Pi_1 \operatorname{supp} \psi \subseteq \left[\xi_1^0 - \frac{\varepsilon}{2}, \xi_1^0 + \frac{\varepsilon}{2}\right]$.
Then, for $\xi \in \mathbb{R}^d$ with $\xi_1 \neq 0$, $\lambda > 0$ and $1 \leq p < +\infty$ we have

$$\begin{aligned}
(|\xi_1^0| - \varepsilon)|\langle \hat{f}(\xi), \psi(\xi)\rangle|^{1/n} &= (|\xi_1^0| - \varepsilon)|\langle \xi_1^n \hat{f}(\xi), \xi_1^{-n}\psi(\xi)\rangle|^{1/n} \\
&= (|\xi_1^0| - \varepsilon)|\langle \widehat{D_1^n f}(\xi), \xi_1^{-n}\psi(\xi)\rangle|^{1/n} \\
&= (|\xi_1^0| - \varepsilon)|\langle D_1^n f(x), \mathcal{F}^{-1}(\xi_1^{-n}\psi(\xi))(x)\rangle|^{1/n} \\
&\leq (|\xi_1^0| - \varepsilon)\|D_1^n f\|_{L^p}^{1/n} \|\mathcal{F}^{-1}(\xi_1^{-n}\psi(\xi))\|_{L^{p'}}^{1/n} \\
&\leq \frac{|\xi_1^0| - \varepsilon}{|\xi_1^0| - \varepsilon/2} \|e^{\lambda\omega\left(\frac{x}{n+1}\right)} D_1^n f(x)\|_{L^p}^{1/n} (n + 2d)^{2d/n} C(\psi)^{1/n}.
\end{aligned}$$

Proof of (2) of the Theorem, $\geq$

Take now $0 \neq \xi^0 \in \operatorname{supp} \hat{f}$, and assume w.l.o.g. that $0 < \varepsilon < |\xi_1^0| = |\xi^0|_\infty$, where $\xi^0 = (\xi_1^0, \ldots, \xi_d^0) \in \mathbb{R}^d$.
We take $\psi \in \mathcal{S}_\omega(\mathbb{R}^d)$ a suitable function with compact support s.t. $\langle \hat{f}, \psi \rangle \neq 0$ with $\Pi_1 \operatorname{supp} \psi \subseteq \left[\xi_1^0 - \frac{\varepsilon}{2}, \xi_1^0 + \frac{\varepsilon}{2}\right]$.
Then, for $\xi \in \mathbb{R}^d$ with $\xi_1 \neq 0$, $\lambda > 0$ and $1 \leq p < +\infty$ we have

$$\begin{aligned}
(|\xi_1^0| - \varepsilon)|\langle \hat{f}(\xi), \psi(\xi)\rangle|^{1/n} &= (|\xi_1^0| - \varepsilon)|\langle \xi_1^n \hat{f}(\xi), \xi_1^{-n}\psi(\xi)\rangle|^{1/n} \\
&= (|\xi_1^0| - \varepsilon)|\langle \widehat{D_1^n f}(\xi), \xi_1^{-n}\psi(\xi)\rangle|^{1/n} \\
&= (|\xi_1^0| - \varepsilon)|\langle D_1^n f(x), \mathcal{F}^{-1}(\xi_1^{-n}\psi(\xi))(x)\rangle|^{1/n} \\
&\leq (|\xi_1^0| - \varepsilon)\|D_1^n f\|_{L^p}^{1/n} \|\mathcal{F}^{-1}(\xi_1^{-n}\psi(\xi))\|_{L^{p'}}^{1/n} \\
&\leq \frac{|\xi_1^0| - \varepsilon}{|\xi_1^0| - \varepsilon/2} \|e^{\lambda\omega\left(\frac{x}{n+1}\right)} D_1^n f(x)\|_{L^p}^{1/n} (n+2d)^{2d/n} C(\psi)^{1/n}.
\end{aligned}$$

By the arbitrariness of $\varepsilon > 0$ and then of $\xi^0 \in \operatorname{supp} \hat{f}$:

$$R_{\hat{f}} \leq \liminf_{n\to+\infty} \left(\max_{|\alpha|=n} \left\| e^{\lambda\omega\left(\frac{x}{n+1}\right)} f^{(\alpha)}(x) \right\|_{L^p}^{1/n} \right).$$

Paley-Wiener Theorems and Time-Frequency analysis

Is it possible to describe the 'size' of the support of the Fourier transform of f with a limit involving time-frequency representations of f instead of the derivatives $f^{(n)}(x)$?

Is it possible to describe the 'size' of the support of the Fourier transform of f with a limit involving time-frequency representations of f instead of the derivatives $f^{(n)}(x)$?

Gabor transform

$$V_\psi f(x,\xi) := \int_{\mathbb{R}^d} f(y)\overline{\psi(y-x)}e^{-iy\xi}\,dy, \qquad (x,\xi)\in\mathbb{R}^{2d}$$

Is it possible to describe the 'size' of the support of the Fourier transform of f with a limit involving time-frequency representations of f instead of the derivatives $f^{(n)}(x)$?

Gabor transform

$$V_\psi f(x,\xi) := \int_{\mathbb{R}^d} f(y)\overline{\psi(y-x)}e^{-iy\xi}\,dy, \qquad (x,\xi)\in\mathbb{R}^{2d}$$

Wigner transform

$$\mathrm{Wig}\, f(x,\xi) := \int_{\mathbb{R}^d} f\left(x+\frac{t}{2}\right)\overline{f\left(x-\frac{t}{2}\right)}e^{-i\xi t}\,dt, \qquad (x,\xi)\in\mathbb{R}^{2d}$$

$$\mathrm{PW}_R^{\omega}(\mathbb{R}^d) = \{f \in C^{\infty}(\mathbb{R}^d) \,:\, \forall \lambda > 0, \sup_{\alpha \in \mathbb{N}_0^d} \sup_{x \in \mathbb{R}^d} R^{-|\alpha|} e^{\lambda \omega\left(\frac{x}{|\alpha|+1}\right)} |f^{(\alpha)}(x)| < +\infty\}.$$

Paley-Wiener Theorems and Time-Frequency analysis

$$\mathrm{PW}_R^\omega(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) \,:\, \forall \lambda > 0, \sup_{\alpha \in \mathbb{N}_0^d} \sup_{x \in \mathbb{R}^d} R^{-|\alpha|} e^{\lambda \omega\left(\frac{x}{|\alpha|+1}\right)} |f^{(\alpha)}(x)| < +\infty\}.$$

Let $T, R > 0$ and $\psi \in \mathrm{PW}_T^\omega(\mathbb{R}^d)$. We define

$$\mathrm{PWG}_R^{\omega,\psi}(\mathbb{R}^d) := \{f \in C^\infty(\mathbb{R}^d) \cap \mathcal{S}'_\omega(\mathbb{R}^d) : \ \forall \lambda, \mu > 0$$

$$\sup_{n \in \mathbb{N}_0} \sup_{x, \xi \in \mathbb{R}^d} (R+T)^{-n} \frac{1}{\sqrt{n+1}} e^{\lambda \omega\left(\frac{x}{n+1}\right) + \mu \omega(\xi)} |\xi|_\infty^n |V_\psi f(x,\xi)| < +\infty\}.$$

$$\mathrm{PW}^{\omega}_{R}(\mathbb{R}^d) = \{f \in C^{\infty}(\mathbb{R}^d) \,:\, \forall\lambda > 0, \sup_{\alpha\in\mathbb{N}_0^d} \sup_{x\in\mathbb{R}^d} R^{-|\alpha|} e^{\lambda\omega\left(\frac{x}{|\alpha|+1}\right)} |f^{(\alpha)}(x)| < +\infty\}.$$

Let $T, R > 0$ and $\psi \in \mathrm{PW}^{\omega}_{T}(\mathbb{R}^d)$. We define

$$\mathrm{PWG}^{\omega,\psi}_{R}(\mathbb{R}^d) := \{f \in C^{\infty}(\mathbb{R}^d) \cap \mathcal{S}'_{\omega}(\mathbb{R}^d) : \ \forall\lambda, \mu > 0$$

$$\sup_{n\in\mathbb{N}_0} \sup_{x,\xi\in\mathbb{R}^d} (R+T)^{-n} \frac{1}{\sqrt{n+1}} e^{\lambda\omega\left(\frac{x}{n+1}\right)+\mu\omega(\xi)} |\xi|_{\infty}^{n} |V_{\psi}f(x,\xi)| < +\infty\}.$$

Proposition

1. Let $\psi \in \mathrm{PW}^{\omega}_{T}(\mathbb{R}^d)$. Then $\mathrm{PW}^{\omega}_{R}(\mathbb{R}^d) \subseteq \mathrm{PWG}^{\omega,\psi}_{R}(\mathbb{R}^d)$.

Paley-Wiener Theorems and Time-Frequency analysis

$$\mathrm{PW}_R^\omega(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) \,:\, \forall\lambda > 0, \sup_{\alpha\in\mathbb{N}_0^d} \sup_{x\in\mathbb{R}^d} R^{-|\alpha|} e^{\lambda\omega\left(\frac{x}{|\alpha|+1}\right)} |f^{(\alpha)}(x)| < +\infty\}.$$

Let $T, R > 0$ and $\psi \in \mathrm{PW}_T^\omega(\mathbb{R}^d)$. We define

$$\mathrm{PWG}_R^{\omega,\psi}(\mathbb{R}^d) := \{f \in C^\infty(\mathbb{R}^d) \cap \mathcal{S}'_\omega(\mathbb{R}^d) : \ \forall\lambda, \mu > 0$$

$$\sup_{n\in\mathbb{N}_0} \sup_{x,\xi\in\mathbb{R}^d} (R+T)^{-n} \frac{1}{\sqrt{n+1}} e^{\lambda\omega\left(\frac{x}{n+1}\right)+\mu\omega(\xi)} |\xi|_\infty^n |V_\psi f(x,\xi)| < +\infty\}.$$

Proposition

1. Let $\psi \in \mathrm{PW}_T^\omega(\mathbb{R}^d)$. Then $\mathrm{PW}_R^\omega(\mathbb{R}^d) \subseteq \mathrm{PWG}_R^{\omega,\psi}(\mathbb{R}^d)$.
2. Let $f, \psi \in \mathcal{S}_\omega(\mathbb{R}^d)$ and $p, q \in [1, +\infty]$. Then, for every $\lambda, \mu \geq 0$,

$$\limsup_{n\to+\infty} \left\| e^{\lambda\omega\left(\frac{x}{n+1}\right)+\mu\omega(\xi)} |\xi|_\infty^n V_\psi f(x,\xi) \right\|_{L^{p,q}}^{1/n} \leq R_{\hat{f}} + R_{\hat{\psi}},$$

where $R_g = \sup\{|y|_\infty : y \in \operatorname{supp} g\}$.

Paley-Wiener Theorems and Time-Frequency analysis

$$\mathrm{PW}_R^\omega(\mathbb{R}^d) = \{f \in C^\infty(\mathbb{R}^d) \,:\, \forall\lambda > 0, \sup_{\alpha\in\mathbb{N}_0^d} \sup_{x\in\mathbb{R}^d} R^{-|\alpha|} e^{\lambda\omega\left(\frac{x}{|\alpha|+1}\right)} |f^{(\alpha)}(x)| < +\infty\}.$$

Let $T, R > 0$ and $\psi \in \mathrm{PW}_T^\omega(\mathbb{R}^d)$. We define

$$\mathrm{PWG}_R^{\omega,\psi}(\mathbb{R}^d) := \{f \in C^\infty(\mathbb{R}^d) \cap \mathcal{S}'_\omega(\mathbb{R}^d) : \ \forall\lambda,\mu > 0$$

$$\sup_{n\in\mathbb{N}_0} \sup_{x,\xi\in\mathbb{R}^d} (R+T)^{-n} \frac{1}{\sqrt{n+1}} e^{\lambda\omega\left(\frac{x}{n+1}\right)+\mu\omega(\xi)} |\xi|_\infty^n |V_\psi f(x,\xi)| < +\infty\}.$$

Proposition

1. Let $\psi \in \mathrm{PW}_T^\omega(\mathbb{R}^d)$. Then $\mathrm{PW}_R^\omega(\mathbb{R}^d) \subseteq \mathrm{PWG}_R^{\omega,\psi}(\mathbb{R}^d)$.
2. Let $f, \psi \in \mathcal{S}_\omega(\mathbb{R}^d)$ and $p, q \in [1,+\infty]$. Then, for every $\lambda, \mu \geq 0$,

$$\limsup_{n\to+\infty} \left\| e^{\lambda\omega\left(\frac{x}{n+1}\right)+\mu\omega(\xi)} |\xi|_\infty^n V_\psi f(x,\xi) \right\|_{L^{p,q}}^{1/n} \leq R_{\hat{f}} + R_{\hat{\psi}},$$

where $R_g = \sup\{|y|_\infty : y \in \operatorname{supp} g\}$. The inequality above can be strict.

Paley-Wiener Theorems and Time-Frequency analysis

$$\limsup_{n\to+\infty} \left\| e^{\lambda\omega\left(\frac{x}{n+1}\right)+\mu\omega(\xi)} |\xi|_\infty^n V_\psi f(x,\xi) \right\|_{L^{p,q}}^{1/n} \leq R_{\hat{f}} + R_{\hat{\psi}},$$

$$\limsup_{n\to+\infty} \left\| e^{\lambda\omega\left(\frac{x}{n+1}\right)+\mu\omega(\xi)} |\xi|_\infty^n V_\psi f(x,\xi) \right\|_{L^{p,q}}^{1/n} \leq R_{\hat{f}} + R_{\hat{\psi}},$$

Example

Let $f \in \mathcal{S}_\omega(\mathbb{R})$ with $\operatorname{supp} \hat{f} \subseteq [R_{\hat{f}} - \mu, R_{\hat{f}}]$ for some $0 < \mu < R_{\hat{f}} < +\infty$. Then

$$\| |\xi|_\infty^n V_f f(x,\xi) \|_{L^{p,q}}^{1/n} \leq \mu \| V_f f(x,\xi) \|_{L^{p,q}}^{1/n} \tag{1}$$

since $\Pi_\xi \operatorname{supp} V_f f(x,\xi) = [-\mu, \mu]$.

Paley-Wiener Theorems and Time-Frequency analysis

$$\limsup_{n\to+\infty} \left\| e^{\lambda\omega\left(\frac{x}{n+1}\right)+\mu\omega(\xi)} |\xi|_\infty^n V_\psi f(x,\xi) \right\|_{L^{p,q}}^{1/n} \leq R_{\hat{f}} + R_{\hat{\psi}},$$

Example

Let $f \in \mathcal{S}_\omega(\mathbb{R})$ with $\operatorname{supp} \hat{f} \subseteq [R_{\hat{f}} - \mu, R_{\hat{f}}]$ for some $0 < \mu < R_{\hat{f}} < +\infty$. Then

$$\| |\xi|_\infty^n V_f f(x,\xi) \|_{L^{p,q}}^{1/n} \leq \mu \| V_f f(x,\xi) \|_{L^{p,q}}^{1/n} \tag{1}$$

since $\Pi_\xi \operatorname{supp} V_f f(x,\xi) = [-\mu, \mu]$.

So, we get

$$\limsup_{n\to+\infty} \| |\xi|_\infty^n V_f f(x,\xi) \|_{L^{p,q}}^{1/n} \leq \mu < R_{\hat{f}} < 2R_{\hat{f}}.$$

Paley-Wiener Theorems and Time-Frequency analysis

$$\limsup_{n\to+\infty} \left\| e^{\lambda\omega\left(\frac{x}{n+1}\right)+\mu\omega(\xi)} |\xi|^n V_\psi f(x,\xi) \right\|_{L^{p,q}}^{1/n} \leq R_{\hat{f}} + R_{\hat{\psi}}$$

$$\limsup_{n\to+\infty} \left\| e^{\lambda\omega\left(\frac{x}{n+1}\right)+\mu\omega(\xi)} |\xi|^n V_\psi f(x,\xi) \right\|_{L^{p,q}}^{1/n} \leq R_{\hat{f}} + R_{\hat{\psi}}$$

Concerning the Wigner transform, we recall that

$$\mathrm{Wig}\, f(x,\xi) = 2e^{2ix\xi} V_{\tilde{f}} f(2x, 2\xi), \quad \text{where } \tilde{f}(y) = f(-y).$$

Paley-Wiener Theorems and Time-Frequency analysis

$$\limsup_{n\to+\infty} \left\| e^{\lambda\omega\left(\frac{x}{n+1}\right)+\mu\omega(\xi)} |\xi|^n V_\psi f(x,\xi) \right\|_{L^{p,q}}^{1/n} \leq R_{\hat{f}} + R_{\hat{\psi}}$$

Concerning the Wigner transform, we recall that

$$\mathrm{Wig}\, f(x,\xi) = 2e^{2ix\xi} V_{\tilde{f}} f(2x, 2\xi), \quad \text{where } \tilde{f}(y) = f(-y).$$

Proposition

Let $f \in \mathcal{S}_\omega(\mathbb{R}^d)$ and $p, q \in [1, +\infty]$. Then, for all $\lambda, \mu \geq 0$:

$$\lim_{n\to+\infty} \left\| e^{\lambda\omega\left(\frac{x}{n+1}\right)+\mu\omega(\xi)} |\xi|_\infty^n \,\mathrm{Wig}\, f(x,\xi) \right\|_{L^{p,q}}^{1/n} = R_{\hat{f}},$$

where $R_{\hat{f}} = \sup\{|\eta|_\infty : \eta \in \mathrm{supp}\, \hat{f}\}$.

Paley-Wiener Theorems and Time-Frequency analysis

$$\limsup_{n\to+\infty} \left\| e^{\lambda\omega\left(\frac{x}{n+1}\right)+\mu\omega(\xi)} |\xi|^n V_\psi f(x,\xi) \right\|_{L^{p,q}}^{1/n} \leq R_{\hat{f}} + R_{\hat{\psi}}$$

Concerning the Wigner transform, we recall that

$$\mathrm{Wig}\, f(x,\xi) = 2e^{2ix\xi} V_{\tilde{f}} f(2x, 2\xi), \quad \text{where } \tilde{f}(y) = f(-y).$$

Proposition

Let $f \in \mathcal{S}_\omega(\mathbb{R}^d)$ and $p, q \in [1, +\infty]$. Then, for all $\lambda, \mu \geq 0$:

$$\lim_{n\to+\infty} \left\| e^{\lambda\omega\left(\frac{x}{n+1}\right)+\mu\omega(\xi)} |\xi|_\infty^n \,\mathrm{Wig}\, f(x,\xi) \right\|_{L^{p,q}}^{1/n} = R_{\hat{f}},$$

where $R_{\hat{f}} = \sup\{|\eta|_\infty : \eta \in \mathrm{supp}\, \hat{f}\}$.

We have

$$\lim_{n\to+\infty} \||\xi|^n \,\mathrm{Wig}\, f(x,\xi)\|_{L^{p,q}}^{1/n} = R_{\hat{f}},$$

$$\limsup_{n\to+\infty} \left\| e^{\lambda\omega\left(\frac{x}{n+1}\right)+\mu\omega(\xi)} |\xi|^n V_\psi f(x,\xi) \right\|_{L^{p,q}}^{1/n} \leq R_{\hat{f}} + R_{\hat{\psi}}$$

Concerning the Wigner transform, we recall that

$$\mathrm{Wig}\, f(x,\xi) = 2e^{2ix\xi} V_{\tilde{f}} f(2x, 2\xi), \quad \text{where } \tilde{f}(y) = f(-y).$$

Proposition

Let $f \in \mathcal{S}_\omega(\mathbb{R}^d)$ and $p, q \in [1, +\infty]$. Then, for all $\lambda, \mu \geq 0$:

$$\lim_{n\to+\infty} \left\| e^{\lambda\omega\left(\frac{x}{n+1}\right)+\mu\omega(\xi)} |\xi|_\infty^n \,\mathrm{Wig}\, f(x,\xi) \right\|_{L^{p,q}}^{1/n} = R_{\hat{f}},$$

where $R_{\hat{f}} = \sup\{|\eta|_\infty : \eta \in \mathrm{supp}\, \hat{f}\}$.

We have

$$\lim_{n\to+\infty} \||\xi|^n \,\mathrm{Wig}\, f(x,\xi)\|_{L^{p,q}}^{1/n} = R_{\hat{f}}, \quad \lim_{n\to+\infty} \||x|^n \,\mathrm{Wig}\, f(x,\xi)\|_{L^{p,q}}^{1/n} = R_f.$$

Theorem (main result)

$$Q_R := \{\xi \in \mathbb{R}^d : |\xi|_\infty \leq R\}, \qquad H_R(x) := \sup_{y \in Q_R} \langle x, y \rangle.$$

Theorem (main result)

$$Q_R := \{\xi \in \mathbb{R}^d : |\xi|_\infty \le R\}, \qquad H_R(x) := \sup_{y \in Q_R} \langle x, y \rangle.$$

Let $1 \le p, q \le +\infty$ and $R > 0$. Then the following conditions are equivalent:

(a) f is an entire function in $\mathbb{C}^d$ and for all $k \in \mathbb{N}_0$ there exists $C_k > 0$ such that

$$|f(z)| \le C_k e^{H_R(\mathrm{Im}\, z) - k\omega(z)}, \qquad z \in \mathbb{C}^d.$$

Theorem (main result)

$$Q_R := \{\xi \in \mathbb{R}^d : |\xi|_\infty \leq R\}, \qquad H_R(x) := \sup_{y \in Q_R} \langle x, y \rangle.$$

Let $1 \leq p, q \leq +\infty$ and $R > 0$. Then the following conditions are equivalent:

(a) f is an entire function in $\mathbb{C}^d$ and for all $k \in \mathbb{N}_0$ there exists $C_k > 0$ such that

$$|f(z)| \leq C_k e^{H_R(\operatorname{Im} z) - k\omega(z)}, \qquad z \in \mathbb{C}^d.$$

(b) $f \in \mathrm{PW}^\omega_R(\mathbb{R}^d)$.

Theorem (main result)

$$Q_R := \{\xi \in \mathbb{R}^d : |\xi|_\infty \le R\}, \qquad H_R(x) := \sup_{y \in Q_R} \langle x, y \rangle.$$

Let $1 \le p, q \le +\infty$ and $R > 0$. Then the following conditions are equivalent:

(a) f is an entire function in $\mathbb{C}^d$ and for all $k \in \mathbb{N}_0$ there exists $C_k > 0$ such that

$$|f(z)| \le C_k e^{H_R(\mathrm{Im}\, z) - k\omega(z)}, \qquad z \in \mathbb{C}^d.$$

(b) $f \in \mathrm{PW}_R^\omega(\mathbb{R}^d)$.

(c) $f \in C^\infty(\mathbb{R}^d)$, $e^{\lambda\omega(x)} f^{(\alpha)}(x) \in L^p(\mathbb{R}^d)$ for all $\alpha \in \mathbb{N}_0^d$ and $\lambda \ge 0$ and

$$\lim_{n \to +\infty} \left(\max_{|\alpha| = n} \left\| e^{\lambda\omega\left(\frac{x}{n+1}\right)} f^{(\alpha)}(x) \right\|_{L^p} \right)^{1/n} = R_{\hat{f}} \le R. \tag{2}$$

Theorem (main result)

$$Q_R := \{\xi \in \mathbb{R}^d : |\xi|_\infty \leq R\}, \qquad H_R(x) := \sup_{y \in Q_R} \langle x, y \rangle.$$

Let $1 \leq p, q \leq +\infty$ and $R > 0$. Then the following conditions are equivalent:

(a) f is an entire function in $\mathbb{C}^d$ and for all $k \in \mathbb{N}_0$ there exists $C_k > 0$ such that

$$|f(z)| \leq C_k e^{H_R(\operatorname{Im} z) - k\omega(z)}, \qquad z \in \mathbb{C}^d.$$

(b) $f \in \mathrm{PW}_R^\omega(\mathbb{R}^d)$.

(c) $f \in C^\infty(\mathbb{R}^d)$, $e^{\lambda\omega(x)} f^{(\alpha)}(x) \in L^p(\mathbb{R}^d)$ for all $\alpha \in \mathbb{N}_0^d$ and $\lambda \geq 0$ and

$$\lim_{n \to +\infty} \left(\max_{|\alpha| = n} \left\| e^{\lambda\omega\left(\frac{x}{n+1}\right)} f^{(\alpha)}(x) \right\|_{L^p} \right)^{1/n} = R_{\hat{f}} \leq R. \qquad (2)$$

(d) $f \in \mathcal{S}_\omega(\mathbb{R}^d)$ and $\operatorname{supp} \hat{f} \subseteq Q_R$.

Theorem (main result)

$$Q_R := \{\xi \in \mathbb{R}^d : |\xi|_\infty \leq R\}, \qquad H_R(x) := \sup_{y \in Q_R} \langle x, y \rangle.$$

Let $1 \leq p, q \leq +\infty$ and $R > 0$. Then the following conditions are equivalent:

(a) f is an entire function in $\mathbb{C}^d$ and for all $k \in \mathbb{N}_0$ there exists $C_k > 0$ such that

$$|f(z)| \leq C_k e^{H_R(\mathrm{Im}\, z) - k\omega(z)}, \qquad z \in \mathbb{C}^d.$$

(b) $f \in \mathrm{PW}_R^\omega(\mathbb{R}^d)$.

(c) $f \in C^\infty(\mathbb{R}^d)$, $e^{\lambda\omega(x)} f^{(\alpha)}(x) \in L^p(\mathbb{R}^d)$ for all $\alpha \in \mathbb{N}_0^d$ and $\lambda \geq 0$ and

$$\lim_{n \to +\infty} \left(\max_{|\alpha|=n} \left\| e^{\lambda\omega\left(\frac{x}{n+1}\right)} f^{(\alpha)}(x) \right\|_{L^p} \right)^{1/n} = R_{\hat{f}} \leq R. \tag{2}$$

(d) $f \in \mathcal{S}_\omega(\mathbb{R}^d)$ and $\mathrm{supp}\, \hat{f} \subseteq Q_R$.

(e) $f \in \mathcal{S}'_\omega(\mathbb{R}^d)$, $e^{\lambda\omega(x) + \mu\omega(\xi)} \,\mathrm{Wig}\, f(x, \xi) \in L^{p,q}(\mathbb{R}^{2d})$ for all $\lambda, \mu \geq 0$ and

$$\lim_{n \to +\infty} \left\| e^{\lambda\omega\left(\frac{x}{n+1}\right) + \mu\omega(\xi)} |\xi|_\infty^n \,\mathrm{Wig}\, f(x, \xi) \right\|_{L^{p,q}}^{1/n} = R_{\hat{f}} \leq R. \tag{3}$$

Corollary (C^∞ case)

Case $\omega(t) = \log(1+t)$

Let $1 \leq p, q \leq +\infty$ and $R > 0$. Then the following conditions are equivalent:

(a) f is an entire function in $\mathbb{C}^d$ and for all $k \in \mathbb{N}_0$ there exists $C_k > 0$ such that

$$|f(z)| \leq C_k(1+|z|)^{-k} e^{H_R(\operatorname{Im} z)}, \quad z \in \mathbb{C}^d.$$

Corollary (C^∞ case)

Case $\omega(t) = \log(1+t)$

Let $1 \leq p, q \leq +\infty$ and $R > 0$. Then the following conditions are equivalent:

(a) f is an entire function in $\mathbb{C}^d$ and for all $k \in \mathbb{N}_0$ there exists $C_k > 0$ such that

$$|f(z)| \leq C_k (1+|z|)^{-k} e^{H_R(\mathrm{Im}\, z)}, \quad z \in \mathbb{C}^d.$$

(b) $f \in \mathcal{S}(\mathbb{R}^d)$ and for all $\lambda > 0$ there exists $C_\lambda > 0$ such that

$$|f^{(\alpha)}(x)| \leq C_\lambda R^{|\alpha|} (|\alpha|+1)^\lambda (1+|x|)^{-\lambda} \qquad x \in \mathbb{R}^d,\ \alpha \in \mathbb{N}_0^d.$$

Corollary (C^∞ case)

Case $\omega(t) = \log(1+t)$

Let $1 \le p, q \le +\infty$ and $R > 0$. Then the following conditions are equivalent:

(a) f is an entire function in $\mathbb{C}^d$ and for all $k \in \mathbb{N}_0$ there exists $C_k > 0$ such that

$$|f(z)| \le C_k (1+|z|)^{-k} e^{H_R(\mathrm{Im}\, z)}, \quad z \in \mathbb{C}^d.$$

(b) $f \in \mathcal{S}(\mathbb{R}^d)$ and for all $\lambda > 0$ there exists $C_\lambda > 0$ such that

$$|f^{(\alpha)}(x)| \le C_\lambda R^{|\alpha|} (|\alpha|+1)^\lambda (1+|x|)^{-\lambda} \qquad x \in \mathbb{R}^d,\ \alpha \in \mathbb{N}_0^d.$$

(c) $f \in \mathcal{S}(\mathbb{R}^d)$ and

$$\lim_{n \to +\infty} \left(\max_{|\alpha|=n} \left\| f^{(\alpha)}(x) \right\|_{L^p} \right)^{1/n} = R_{\hat{f}} \le R.$$

Corollary (C^∞ case)

Case $\omega(t) = \log(1+t)$

Let $1 \leq p, q \leq +\infty$ and $R > 0$. Then the following conditions are equivalent:

(a) f is an entire function in $\mathbb{C}^d$ and for all $k \in \mathbb{N}_0$ there exists $C_k > 0$ such that

$$|f(z)| \leq C_k(1+|z|)^{-k} e^{H_R(\mathrm{Im}\, z)}, \quad z \in \mathbb{C}^d.$$

(b) $f \in \mathcal{S}(\mathbb{R}^d)$ and for all $\lambda > 0$ there exists $C_\lambda > 0$ such that

$$|f^{(\alpha)}(x)| \leq C_\lambda R^{|\alpha|}(|\alpha|+1)^\lambda (1+|x|)^{-\lambda} \qquad x \in \mathbb{R}^d,\ \alpha \in \mathbb{N}_0^d.$$

(c) $f \in \mathcal{S}(\mathbb{R}^d)$ and

$$\lim_{n\to+\infty} \left(\max_{|\alpha|=n} \left\| f^{(\alpha)}(x) \right\|_{L^p} \right)^{1/n} = R_{\hat{f}} \leq R.$$

(d) $f \in \mathcal{S}(\mathbb{R}^d)$ and $\mathrm{supp}\, \hat{f} \subseteq Q_R$.

Corollary (C^∞ case)

Case $\omega(t) = \log(1+t)$

Let $1 \le p, q \le +\infty$ and $R > 0$. Then the following conditions are equivalent:

(a) f is an entire function in $\mathbb{C}^d$ and for all $k \in \mathbb{N}_0$ there exists $C_k > 0$ such that

$$|f(z)| \le C_k (1+|z|)^{-k} e^{H_R(\mathrm{Im}\, z)}, \quad z \in \mathbb{C}^d.$$

(b) $f \in \mathcal{S}(\mathbb{R}^d)$ and for all $\lambda > 0$ there exists $C_\lambda > 0$ such that

$$|f^{(\alpha)}(x)| \le C_\lambda R^{|\alpha|} (|\alpha|+1)^\lambda (1+|x|)^{-\lambda} \qquad x \in \mathbb{R}^d,\ \alpha \in \mathbb{N}_0^d.$$

(c) $f \in \mathcal{S}(\mathbb{R}^d)$ and

$$\lim_{n\to+\infty} \left(\max_{|\alpha|=n} \left\| f^{(\alpha)}(x) \right\|_{L^p} \right)^{1/n} = R_{\hat{f}} \le R.$$

(d) $f \in \mathcal{S}(\mathbb{R}^d)$ and $\mathrm{supp}\, \hat{f} \subseteq Q_R$.

(e) $f \in \mathcal{S}(\mathbb{R}^d)$ and

$$\lim_{n\to+\infty} \| |\xi|_\infty^n \,\mathrm{Wig}\, f(x,\xi) \|_{L^{p,q}}^{1/n} = R_{\hat{f}} \le R.$$

An example

For $k \in \mathbb{N}_0$, let e_k be the Hermite function on $\mathbb{R}$ defined by

$$e_k(x) = \frac{1}{(2^k k! \sqrt{\pi})^{1/2}} e^{-x^2/2} H_k(x), \qquad H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2}.$$

An example

For $k \in \mathbb{N}_0$, let e_k be the Hermite function on $\mathbb{R}$ defined by

$$e_k(x) = \frac{1}{(2^k k! \sqrt{\pi})^{1/2}} e^{-x^2/2} H_k(x), \qquad H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2}.$$

1. $e_k \in \mathcal{S}_\omega(\mathbb{R})$.

An example

For $k \in \mathbb{N}_0$, let e_k be the Hermite function on $\mathbb{R}$ defined by

$$e_k(x) = \frac{1}{(2^k k! \sqrt{\pi})^{1/2}} e^{-x^2/2} H_k(x), \qquad H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2}.$$

1. $e_k \in \mathcal{S}_\omega(\mathbb{R})$.
2. Defining $e_{j,k}(y,t) := \mathscr{F}^{-1}\big(\mathrm{Wig}(e_j, e_k)\big)$ we have

$$\begin{aligned} L e_{j,k}(y,t) &= (2k+1) e_{j,k}(y,t), \qquad & L &= (D_y - t/2)^2 + (D_t + y/2)^2\,; \\ \tilde{L} \hat{e}_{j,k}(x,\xi) &= (2k+1) \hat{e}_{j,k}(x,\xi), \qquad & \tilde{L} &= (D_\xi/2 + x)^2 + (D_x/2 - \xi)^2\,. \end{aligned}$$

An example

For $k \in \mathbb{N}_0$, let e_k be the Hermite function on $\mathbb{R}$ defined by

$$e_k(x) = \frac{1}{(2^k k! \sqrt{\pi})^{1/2}} e^{-x^2/2} H_k(x), \qquad H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2}.$$

1. $e_k \in \mathcal{S}_\omega(\mathbb{R})$.
2. Defining $e_{j,k}(y,t) := \mathscr{F}^{-1}\big(\mathrm{Wig}(e_j, e_k)\big)$ we have

$$L e_{j,k}(y,t) = (2k+1) e_{j,k}(y,t), \qquad L = (D_y - t/2)^2 + (D_t + y/2)^2;$$
$$\tilde{L} \hat{e}_{j,k}(x,\xi) = (2k+1) \hat{e}_{j,k}(x,\xi), \qquad \tilde{L} = (D_\xi/2 + x)^2 + (D_x/2 - \xi)^2.$$

3. $\hat{e}_k(\xi) = \lambda e_k(\xi)$, $0 \neq \lambda \in \mathbb{C}$. Then, $R_{\hat{e}_k} = \sup\{|\xi| : \xi \in \operatorname{supp} \hat{e}_k\} = +\infty$.

An example

For $k \in \mathbb{N}_0$, let e_k be the Hermite function on $\mathbb{R}$ defined by

$$e_k(x) = \frac{1}{(2^k k! \sqrt{\pi})^{1/2}} e^{-x^2/2} H_k(x), \qquad H_k(x) = (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2}.$$

1. $e_k \in \mathcal{S}_\omega(\mathbb{R})$.
2. Defining $e_{j,k}(y,t) := \mathscr{F}^{-1}\big(\mathrm{Wig}(e_j, e_k)\big)$ we have

$$L e_{j,k}(y,t) = (2k+1) e_{j,k}(y,t), \qquad L = (D_y - t/2)^2 + (D_t + y/2)^2\,;$$
$$\tilde{L} \hat{e}_{j,k}(x,\xi) = (2k+1) \hat{e}_{j,k}(x,\xi), \qquad \tilde{L} = (D_\xi/2 + x)^2 + (D_x/2 - \xi)^2\,.$$

3. $\hat{e}_k(\xi) = \lambda e_k(\xi)$, $0 \neq \lambda \in \mathbb{C}$. Then, $R_{\hat{e}_k} = \sup\{|\xi| : \xi \in \mathrm{supp}\,\hat{e}_k\} = +\infty$.
4. We then have that for every $\mu, \lambda \geq 0$, $p, q \in [1, +\infty]$,

$$\lim_{n \to +\infty} \left\| e^{\lambda \omega\left(\frac{x}{n+1}\right) + \mu \omega(\xi)} |\xi|^n \hat{e}_{k,k}(x,\xi) \right\|_{L^{p,q}}^{1/n} = +\infty;$$
$$\lim_{n \to +\infty} \left\| e^{\lambda \omega\left(\frac{x}{n+1}\right)} \frac{d^n}{dx^n} e_k(x) \right\|_{L^p}^{1/n} = +\infty.$$

Arbitrary support

Arbitrary support

$P \in \mathbb{C}[\xi_1, \ldots, \xi_d]$ polynomial

Arbitrary support

$P \in \mathbb{C}[\xi_1, \ldots, \xi_d]$ polynomial
$P(D)$ the corresponding linear partial differential operator with symbol P, where we use the standard notation $D_j := -i\partial_j$.

Arbitrary support

$P \in \mathbb{C}[\xi_1, \ldots, \xi_d]$ polynomial
$P(D)$ the corresponding linear partial differential operator with symbol P, where we use the standard notation $D_j := -i\partial_j$.
For an ultradistribution T on $\mathbb{R}^d$, we denote

$$R(P, T) := \sup\{|P(\xi)| : \ \xi \in \operatorname{supp} T\},$$

with the convention that $R(P, T) = 0$ if $T \equiv 0$.

Arbitrary support

$P \in \mathbb{C}[\xi_1, \dots, \xi_d]$ polynomial
$P(D)$ the corresponding linear partial differential operator with symbol P, where we use the standard notation $D_j := -i\partial_j$.
For an ultradistribution T on $\mathbb{R}^d$, we denote

$$R(P, T) := \sup\{|P(\xi)| : \xi \in \operatorname{supp} T\},$$

with the convention that $R(P, T) = 0$ if $T \equiv 0$.

Theorem

Let $P \in \mathbb{C}[x_1, \dots, x_d]$ a polynomial of degree $m \geq 1$. Let $f \in \mathcal{S}_\omega(\mathbb{R}^d)$. Then the following conditions are equivalent:

(a) $\forall \lambda > 0 \ \exists C_\lambda > 0$ such that $\forall n \in \mathbb{N}_0$, $x \in \mathbb{R}^d$

$$|P(D)^n f(x)| \leq C_\lambda R^n e^{-\lambda\omega\left(\left|\frac{x}{n+1}\right|^{1/m}\right)};$$

Arbitrary support

$P \in \mathbb{C}[\xi_1, \ldots, \xi_d]$ polynomial
$P(D)$ the corresponding linear partial differential operator with symbol P, where we use the standard notation $D_j := -i\partial_j$.
For an ultradistribution T on $\mathbb{R}^d$, we denote

$$R(P, T) := \sup\{|P(\xi)| : \ \xi \in \operatorname{supp} T\},$$

with the convention that $R(P, T) = 0$ if $T \equiv 0$.

Theorem

Let $P \in \mathbb{C}[x_1, \ldots, x_d]$ a polynomial of degree $m \geq 1$. Let $f \in \mathcal{S}_\omega(\mathbb{R}^d)$. Then the following conditions are equivalent:

(a) $\forall \lambda > 0 \ \exists C_\lambda > 0$ such that $\forall n \in \mathbb{N}_0$, $x \in \mathbb{R}^d$

$$|P(D)^n f(x)| \leq C_\lambda R^n e^{-\lambda\omega\left(\left|\frac{x}{n+1}\right|^{1/m}\right)};$$

(b) $R(P, \hat{f}) \leq R$.

Arbitrary support

Theorem

Let $P \in \mathbb{C}[x_1, \ldots, x_d]$ a polynomial of degree $m \geq 1$. Let $f \in \mathcal{S}_\omega(\mathbb{R}^d)$. Then the following conditions are equivalent:

(a) $\forall \lambda > 0 \; \exists C_\lambda > 0$ such that $\forall n \in \mathbb{N}_0$, $x \in \mathbb{R}^d$

$$|P(D)^n f(x)| \leq C_\lambda R^n e^{-\lambda \omega\left(\left|\frac{x}{n+1}\right|^{1/m}\right)};$$

(b) $R(P, \hat{f}) \leq R$.

Arbitrary support

Theorem

Let $P \in \mathbb{C}[x_1, \ldots, x_d]$ a polynomial of degree $m \geq 1$. Let $f \in \mathcal{S}_\omega(\mathbb{R}^d)$. Then the following conditions are equivalent:

(a) $\forall \lambda > 0\ \exists C_\lambda > 0$ such that $\forall n \in \mathbb{N}_0$, $x \in \mathbb{R}^d$

$$|P(D)^n f(x)| \leq C_\lambda R^n e^{-\lambda\omega\left(\left|\frac{x}{n+1}\right|^{1/m}\right)};$$

(b) $R(P, \hat{f}) \leq R$.

Corollary

If $P \in \mathbb{C}[x_1, \ldots, x_d]$ is a polynomial of degree $m \geq 1$, $f \in \mathcal{S}_\omega(\mathbb{R}^d)$ and $1 \leq p \leq \infty$, we have, for all $\lambda \geq 0$,

$$\lim_{n \to +\infty} \left\| e^{\lambda\omega\left(\left|\frac{x}{n+1}\right|^{1/m}\right)} P(D)^n f(x) \right\|_{L^p}^{1/n} = R(P, \hat{f}).$$

Proof of the Corollary

Proof of the Corollary

From a result of Andersen-De Jeu (2010) we have

$$\liminf_{n\to+\infty} \left\| e^{\lambda\omega\left(\left|\frac{x}{n+1}\right|^{1/m}\right)} P(D)^n f(x) \right\|_{L^p}^{1/n} \geq R(P, \hat{f}),$$

for all $\lambda \geq 0$.

Proof of the Corollary

From a result of Andersen-De Jeu (2010) we have

$$\liminf_{n\to+\infty} \left\| e^{\lambda\omega\left(\left|\frac{x}{n+1}\right|^{1/m}\right)} P(D)^n f(x) \right\|_{L^p}^{1/n} \geq R(P, \hat{f}),$$

for all $\lambda \geq 0$.
Fix $\lambda \geq 0$ and consider $\mu > 0$ such that $\left\| e^{-\mu\omega\left(|x|^{1/m}\right)} \right\|_{L^p} < +\infty$.

Proof of the Corollary

From a result of Andersen-De Jeu (2010) we have

$$\liminf_{n\to+\infty} \left\| e^{\lambda\omega\left(\left|\frac{x}{n+1}\right|^{1/m}\right)} P(D)^n f(x) \right\|_{L^p}^{1/n} \geq R(P,\hat{f}),$$

for all $\lambda \geq 0$.

Fix $\lambda \geq 0$ and consider $\mu > 0$ such that $\left\| e^{-\mu\omega\left(|x|^{1/m}\right)} \right\|_{L^p} < +\infty$.

Assume $R(P,\hat{f}) < +\infty$.

Proof of the Corollary

From a result of Andersen-De Jeu (2010) we have

$$\liminf_{n\to+\infty} \left\| e^{\lambda\omega\left(\left|\frac{x}{n+1}\right|^{1/m}\right)} P(D)^n f(x) \right\|_{L^p}^{1/n} \geq R(P,\hat{f}),$$

for all $\lambda \geq 0$.

Fix $\lambda \geq 0$ and consider $\mu > 0$ such that $\left\| e^{-\mu\omega\left(|x|^{1/m}\right)} \right\|_{L^p} < +\infty$.

Assume $R(P,\hat{f}) < +\infty$. By the Theorem, for every $R \geq R(P,\hat{f})$ and every $n \in \mathbb{N}$, we have

$$\left\| e^{\lambda\omega\left(\left|\frac{x}{n+1}\right|^{1/m}\right)} P(D)^n f(x) \right\|_{L^p} \leq (n+1)^{d/p} C_{\lambda+\mu} \left\| e^{-\mu\omega\left(|x|^{1/m}\right)} \right\|_{L^p} R^n.$$

Proof of the Corollary

From a result of Andersen-De Jeu (2010) we have

$$\liminf_{n\to+\infty} \left\| e^{\lambda\omega\left(\left|\frac{x}{n+1}\right|^{1/m}\right)} P(D)^n f(x) \right\|_{L^p}^{1/n} \geq R(P,\hat{f}),$$

for all $\lambda \geq 0$.

Fix $\lambda \geq 0$ and consider $\mu > 0$ such that $\left\| e^{-\mu\omega\left(|x|^{1/m}\right)} \right\|_{L^p} < +\infty$.

Assume $R(P,\hat{f}) < +\infty$. By the Theorem, for every $R \geq R(P,\hat{f})$ and every $n \in \mathbb{N}$, we have

$$\left\| e^{\lambda\omega\left(\left|\frac{x}{n+1}\right|^{1/m}\right)} P(D)^n f(x) \right\|_{L^p} \leq (n+1)^{d/p} C_{\lambda+\mu} \left\| e^{-\mu\omega\left(|x|^{1/m}\right)} \right\|_{L^p} R^n.$$

We deduce that

$$\limsup_{n\to+\infty} \left\| e^{\lambda\omega\left(\left|\frac{x}{n+1}\right|^{1/m}\right)} P(D)^n f(x) \right\|_{L^p}^{1/n} \leq R,$$

for each $R \geq R(P,\hat{f})$.

Examples

Example 1

Let $P \in \mathbb{C}[\xi_1, \ldots, \xi_d]$ be a polynomial of degree $m \geq 1$. If P is hypoelliptic, then

$$V_R := \{\xi \in \mathbb{R}^d : \ |P(\xi)| \leq R\}$$

is compact.

Examples

Example 1

Let $P \in \mathbb{C}[\xi_1, \ldots, \xi_d]$ be a polynomial of degree $m \geq 1$. If P is hypoelliptic, then

$$V_R := \{\xi \in \mathbb{R}^d : \ |P(\xi)| \leq R\}$$

is compact.

Example 2

On the contrary, the fact that V_R is compact does not imply that P is hypoelliptic. Take, for instance,

$$P(z) = z_1^2 - z_2^2 + iz_2, \qquad z_1, z_2 \in \mathbb{C}.$$

In this case

$$V_R = \{\xi \in \mathbb{R}^2 : \ |\xi_1^2 - \xi_2^2 + i\xi_2| \leq R\}$$

is compact.

$P(D)$ is not hypoelliptic

$P(D)$ is not hypoelliptic

A polynomial $P(\xi)$ is hypoelliptic if and only if

$$\lim_{\substack{\zeta \in V \\ |\zeta| \to +\infty}} |\operatorname{Im} \zeta| = +\infty.$$

$P(D)$ is not hypoelliptic

A polynomial $P(\xi)$ is hypoelliptic if and only if

$$\lim_{\substack{\zeta \in V \\ |\zeta| \to +\infty}} |\operatorname{Im} \zeta| = +\infty.$$

Now,

$$\begin{aligned} V :=& \ \{z \in \mathbb{C}^2 : \ P(z) = 0\} \\ =& \ \left\{z \in \mathbb{C}^2 : \ z_2 = \frac{i \pm \sqrt{-1 + 4z_1^2}}{2}\right\}, \end{aligned}$$

where $\pm\sqrt{-1+4z_1^2}$ denote the two complex roots of $4z_1^2 - 1$.

$P(D)$ is not hypoelliptic

A polynomial $P(\xi)$ is hypoelliptic if and only if

$$\lim_{\substack{\zeta \in V \\ |\zeta| \to +\infty}} |\operatorname{Im} \zeta| = +\infty.$$

Now,

$$\begin{aligned} V :=& \ \{z \in \mathbb{C}^2 : \ P(z) = 0\} \\ =& \ \left\{z \in \mathbb{C}^2 : \ z_2 = \frac{i \pm \sqrt{-1 + 4z_1^2}}{2}\right\}, \end{aligned}$$

where $\pm\sqrt{-1 + 4z_1^2}$ denote the two complex roots of $4z_1^2 - 1$. Take, for instance,

$$\xi = \left(\xi_1, \frac{i + \sqrt{4\xi_1^2 - 1}}{2}\right) \in V, \qquad \text{for } \xi_1 \in \mathbb{R},$$

we have that $|\xi| \to +\infty$ for $|\xi_1| \to +\infty$, but

$$|\operatorname{Im} \xi| = \left|\left(0, \frac{1}{2}\right)\right| = \frac{1}{2}.$$