On standard models of CR–submanifolds

Jan Gregorovič

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Introduction

Real submanifolds $M \subset \mathbb{C}^N$.
$\mathcal{D} := TM \cap iTM$ complex tangent space, I complex structure on $\mathcal{D}$ induced by i
If $\mathcal{D}$ is a distribution, then M is called CR–submanifold of complex dimension $dim_{\mathbb{C}}\mathcal{D}$.
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CR–automorphisms on $\mathbb{C}^N$... holomorphic maps preserving $M \subset \mathbb{C}^N$
Infinitesimal CR–automorphism on $\mathbb{C}^N$ ... holomorphic vector fields such that their flows preserve M
CR–automorphisms of $(M, \mathcal{D}, I)$ ... diffeomorphisms on M preserving $\mathcal{D}$ and I
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Introduction – Example

Hypersurface Q in $\mathbb{C}^{n+1}$ given by $Im(w) = h(z, \bar{z})$ for a (non–degenerate) Hermitian form h on $\mathbb{C}^n$.

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Q is homogeneous w.r.t. the action of CR–automorphisms.
Lie algebra of infinitesimal CR–automorphisms of Q is $\mathfrak{su}(p+1, q+1) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$:

$$\mathfrak{g}_{-2} = \{2Re(c\frac{\partial}{\partial w})\},\ c \in \mathbb{R}$$

$$\mathfrak{g}_{-1} = \{2Re(d\frac{\partial}{\partial z} + 2ih(z, \bar{d})\frac{\partial}{\partial w})\},\ d \in \mathbb{C}^n$$

$$\mathfrak{g}_0 = \{2Re(\lambda z\frac{\partial}{\partial z} + \rho w\frac{\partial}{\partial w})\}, 2Re(h(\lambda z, \bar{z})) = \rho h(z, \bar{z})$$

$$\mathfrak{g}_1 = \{2Re((aw + 2ih(z, \bar{a})z)\frac{\partial}{\partial z} + 2ih(z, \bar{a})w\frac{\partial}{\partial w})\},\ a \in \mathbb{C}^n$$

$$\mathfrak{g}_2 = \{2Re(rwz\frac{\partial}{\partial z} + rw^2\frac{\partial}{\partial w})\},\ r \in \mathbb{R}$$

Introduction – Quadrics of higher codimension

Submanifolds in $\mathbb{C}^{n+k}$ given by $Im(w_i) = h_i(z, \bar{z}), i = 1 \dots k$ for a (non–degenerate) linearly independent Hermitian forms h_i on $\mathbb{C}^n$.

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Lie algebras of infinitesimal CR–automorphisms are finite dimensional, consist of polynomial vector fields of weighted degree ≤ 2

$$\mathfrak{g}_{-2} = \{2Re(q\frac{\partial}{\partial w})\},\ q \in \mathbb{R}^k$$
$$\mathfrak{g}_{-1} = \{2Re(p\frac{\partial}{\partial z} + 2ih(z,\bar{p})\frac{\partial}{\partial w})\},\ p \in \mathbb{C}^n$$

Generically $\mathfrak{g}_2 = \mathfrak{g}_1 = 0$. The cases (as the hyperquadric Q) with $\mathfrak{g}_1 \neq 0$ are exceptional.

Introduction – Higher order polynomials

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It is conjectured that $\mathfrak{g}_k = \cdots = \mathfrak{g}_1 = 0$, when $k > 2$. (Proved in specific cases)

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1. Fundamental CR–algebras
2. Standard models of CR–submanifolds – abstract setting
3. Infinitesimal CR–automorphisms of standard models
4. Standard models of CR–submanifolds – embedding
5. Explicit example

Fundamental CR–algebras

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Definition

We say that a Lie algebra $\mathfrak{m} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ satisfying that

1. $\mathfrak{m}$ is generated by $\mathfrak{g}_{-1}$, [fundamental Lie algebra]
2. $[\mathfrak{g}_a, \mathfrak{g}_b] = \mathfrak{g}_{a+b}$ ($\mathfrak{g}_l = 0$ if $l < -k$), [graded Lie algebra]
3. there is complex structure I (i.e. $I^2 = \epsilon^2 \mathrm{id}$) on $\mathfrak{g}_{-1}$ such that for all $X, Y \in \mathfrak{g}_{-1}$

$$[I(X), I(Y)] = -\epsilon^2 [X, Y]$$

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Lemma

If $\mathfrak{m}$ is a complex Lie algebra, then $\mathfrak{m}$ is a fundamental CR–algebra if and only if $\mathfrak{m}$ is a fundamental para CR–algebra.

Standard models of CR–submanifolds – abstract setting

Definition

An almost CR–structure on smooth manifold M is a tuple $(\mathcal{D}, I)$ of a distribution $\mathcal{D}$ on M and complex structure I on $\mathcal{D}$ (i.e. $I^2 = \epsilon^2 \mathrm{id}_{\mathcal{D}}$) such that the tensorial map

$$\mathcal{L} : \mathcal{D} \otimes \mathcal{D} \to \mathcal{D}_2 := [\mathcal{D}, \mathcal{D}] \mod \mathcal{D}$$

provided by the bracket of vector fields satisfy for all $x \in M$ and $X, Y \in \mathcal{D}_x$

$$\mathcal{L}_x(I(X), I(Y)) = -\epsilon^2 \mathcal{L}_x(X, Y)$$

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Definition

Standard model of a fundamental CR–algebra $(\mathfrak{m}, I)$ is an (almost) CR–structure on the Lie group $M = \exp(\mathfrak{m})$ provided by the left–invariant distribution $\mathcal{D}$ given by $\mathfrak{g}_{-1}$ and the complex structure on $\mathcal{D}$ given by I.

Symbol algebras

Inductively, we can extend the tensorial map $\mathcal{L}$ using the bracket of vector fields

$$\mathcal{L} : \mathcal{D} \otimes \mathcal{D}_{i-1} \to \mathcal{D}_i := [\mathcal{D}, \mathcal{D}_{i-1}] \mod (\mathcal{D} + \cdots + \mathcal{D}_{i-1}).$$

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For each x, the map $\mathcal{L}_x$ is a Lie bracket on
$\mathfrak{m}(x) = \mathfrak{g}_{-k}(x) \oplus \cdots \oplus \mathfrak{g}_{-1}(x)$, $\mathfrak{g}_{-1}(x) := \mathcal{D}(x)$, $\mathfrak{g}_{-i}(x) := \mathcal{D}_i(x)$.
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Theorem (Bloom, Graham)

Dimension of the Lie algebra of infinitesimal CR–automorphisms at x is bounded by the dimension of the Lie algebra of infinitesimal CR–automorphisms of standard model $M = \exp(\mathfrak{m}(x))$ of the symbol $(\mathfrak{m}(x), I_x)$.

Infinitesimal CR–automorphisms of the standard model

Definition

Tanaka prolongation of the fundamental CR–algebra $(\mathfrak{m} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}, I)$ is the maximal graded Lie algebra $\mathfrak{g} = \mathfrak{m} \oplus \oplus_{i \geq 0} \mathfrak{g}_i$ such that

1. $\mathfrak{g}_0$ consists of grading preserving derivations of $\mathfrak{m}$ commuting with I,
2. for all $X \in \oplus_{i \geq 0} \mathfrak{g}_i$ the condition $[X, \mathfrak{g}_{-1}] = 0$ implies $X = 0$.

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The ith prolongation $\mathfrak{g}_i$ can be algebraically computed as

$$\mathfrak{g}_i := \{f \in \oplus_{j<0} \mathfrak{g}_j^* \otimes \mathfrak{g}_{j+i} : \ f([X, Y]) = [f(X), Y] + [X, f(Y)], \ X, Y \in \mathfrak{m}\}.$$

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Theorem (Tanaka)

Suppose for all $X \in \mathfrak{g}_{-1}$ the condition $[X, \mathfrak{g}_{-1}] = 0$ implies $X = 0$. Then $\mathfrak{g}_l = 0$ for all l large enough and $\mathfrak{g}$ is finite dimensional Lie algebra.

Parabolic geometries

$\mathfrak{g}$ ...complex simple Lie algebra
Each $|k|$–grading $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$ of $\mathfrak{g}$ is equivalent (up to conjugation) to a subset Σ of the set of positive simple roots.
Root spaces of $\mathfrak{g}$ belong to $\mathfrak{g}_i$ according to sum of coefficients by roots in Σ.
$\mathfrak{m} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ ... fundamental graded nilpotent Lie algebra
$\mathfrak{p} = \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k$ ... parabolic subalgebra of $\mathfrak{g}$

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Theorem

There is I on $\mathfrak{g}_{-1} \subset \mathfrak{m}$ such that $\mathfrak{g}$ is a Tanaka prolongation of the fundamental CR–algebra $(\mathfrak{m}, I)$ in the following cases

1. $\mathfrak{g} = \mathfrak{sl}(n+1, \mathbb{C})$, $k = |\Sigma| > 1$
2. $\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$, $\Sigma = \{\alpha_{i_1}, \dots, \alpha_{i_{l-1}}, \alpha_{i_{l-1}+1}\}, k \geq 2|\Sigma| - 1$
3. $\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C})$, $\Sigma = \{\alpha_{i_1}, \dots, \alpha_{i_{l-1}}, \alpha_{i_n}\}, k = 2|\Sigma| - 1$
4. ...

We want to embed $M = \exp(\mathfrak{m})$ into $\mathbb{C}^N$ in a way that $\mathcal{D}$, becomes the maximal complex subspace of $TM \subset T\mathbb{C}^N$ and I the restriction of the complex structure on $T\mathbb{C}^N$ to $\mathcal{D}$.

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Obstruction for (local) embedability of real analytic almost CR–structure $(\mathcal{D}, I)$ on M is the Nijenhuis tensor $\mathcal{N} : \wedge^2\mathcal{D} \to \mathcal{D}$ defined for all $X, Y \in \mathcal{D}_x$

$$\mathcal{N}_x(X, Y) := [X, Y] + \epsilon^2[I(X), I(Y)] - \epsilon^2 I([X, I(Y)] + [I(X), Y])$$

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$\mathcal{N} \equiv 0$ on standard model $M = \exp(\mathfrak{m})$ of fundamental CR–algebra $(\mathfrak{m}, I)$ and the embedding exists globally.

Explicit construction of emending ϕ of $M = \exp(\mathfrak{m})$ into $\mathbb{C}^N$

$\mathfrak{m} \oplus i\mathfrak{m}$... complexification of $\mathfrak{m}$
$\mathfrak{g}_{-1}^{\pm} := \{X \pm i\epsilon^2 I(X) : X \in \mathfrak{g}_{-1}\}$ are the $\pm i$–eigenspaces of I
Decompose $\mathfrak{m} \oplus i\mathfrak{m} = (\mathfrak{g}_{-1}^{-}) \oplus (\mathfrak{g}_{-1}^{+} \oplus \mathfrak{g}_{-2} \oplus i\mathfrak{g}_{-2} \dots)$ to complex abelian subalgebra $\mathfrak{g}_{-1}^{-}$ and complex ideal $\mathfrak{n} = (\mathfrak{g}_{-1}^{+} \oplus \mathfrak{g}_{-2} \oplus i\mathfrak{g}_{-2} \dots)$

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For $X \in \mathfrak{m}$, we can uniquely decompose $\exp(X)$ to $\exp(\mathfrak{n})\exp(\mathfrak{g}_{-1}^{-})$
Thus $\exp(X) = \exp(\phi(X))\exp(\frac{1}{2}(X_{-1} - i\epsilon^2 I(X_{-1})))$ for
$\phi : \mathfrak{m} \to \mathfrak{n} = \mathbb{C}^N$ and $X = X_{-k} + \dots + X_{-1} \in \mathfrak{g}_{-k} \oplus \dots \oplus \mathfrak{g}_{-1}$

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Theorem (Naruki)

The $\exp^{-1} : M \to \mathfrak{m}$ is a global chart and in this chart the map $\phi : \mathfrak{m} \to \mathfrak{n}$ defined as $\phi(X) := \exp^{-1}(\exp(X)\exp(-\frac{1}{2}(X_{-1} - i\epsilon^2 I(X_{-1}))))$ is embeding of M into $\mathbb{C}^N$.

Baker–Campbell-Hausdorff formula

$$\exp^{-1}(\exp(X)\exp(Y)) = X + Y + \sum_{n=1}^{\infty}\frac{(-1)^n}{n+1}\sum_{r_i+s_i>0} f(r_1, s_1, \dots, r_n, s_n)$$

$$f(r_1, \dots, s_n) := \frac{ad(X)^{r_1}ad(Y)^{s_1}\dots ad(X)^{r_n}ad(Y)^{s_n}(X)}{(1+\sum_{i=1}^{n})(r_i+s_i)\prod_{i=1}^{n} r_i!s_i!}$$

$$\exp^{-1}(\exp(X)\exp(Y)) =$$
$$X + Y + \tfrac{1}{2}[X, Y] + \tfrac{1}{12}([X, [X, Y]] + [Y, [Y, X]]) + \dots$$

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The gradings of $\mathfrak{m}$, $\mathfrak{n}$ provides weights for variables and the embeding $\phi(X)$ decomposes to homogeneous weighted polynomials given by Baker–Campbell-Hausdorff formula.

The infinitesimal automorphisms in coordinates

For $X \in \mathfrak{g}$ and $Y \in \mathfrak{m}$ given by $\frac{d}{dt}|_{t=0}\phi \circ \exp^{-1} \circ p(\exp(tX)\exp(Y)))$, where $p : \exp(\mathfrak{g}) \to \exp(\mathfrak{m})$ is the projection along the stabilizer of a point.

For $X \in \mathfrak{g}_j, j > -1$,

$$\frac{d}{dt}|_{t=0}\phi \circ \exp^{-1} \circ p(\exp(tX)\exp(Y))$$

$$= \frac{d}{dt}|_{t=0}\phi(\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1} \sum_{s_i>0, \sum_{i=1}^n s_i > j} f(0, s_1, \dots, 0, s_n))$$

$$f(0, s_1, \dots, 0, s_n) := \frac{ad(Y)^{s_1+\dots+s_n}(tX)}{(1+\sum_{i=1}^n s_i)\prod_{i=1}^n s_i!}$$

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Theorem

For $X \in \mathfrak{g}_j$, the corresponding infinitesimal CR–automorphism is a polynomial vector fields of weighted degree j.

Expressing standard submanifolds by equations

$z = \phi(X)_{-1} = \frac{1}{2}X_{-1}$
$Re(\phi(X)_{-2}) = X_{-2},\ Im(\phi(X)_{-2}) = \frac{1}{4}\epsilon^2[X_{-1}, I(X_{-1})]$
=> $Im(w_{-2}) = \epsilon^2[z, I(z)]$ ... $[z, I(z)]$ Hermitian forms

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$=> Im(w_{-2}) = \epsilon^2[z, I(z)]$ ... $[z, I(z)]$ Hermitian forms

$Re(\phi(X)_{-3}) = X_{-3} - \frac{1}{4}[X_{-2}, X_{-1}] + \epsilon^2\frac{1}{48}[I(X_{-1}), [I(X_{-1}), X_{-1}]]$
$Im(\phi(X)_{-3}) = \frac{1}{4}\epsilon^2[X_{-2}, I(X_{-1})] + \epsilon^2\frac{1}{24}[X_{-1}, [X_{-1}, I(X_{-1})]]$
$=> Im(w_{-3}) = \frac{1}{2}\epsilon^2[Re(w_{-2}), I(z)] + \epsilon^2\frac{1}{3}[z, [z, I(z)]]$

Expressing standard submanifolds by equations

$z = \phi(X)_{-1} = \frac{1}{2}X_{-1}$
$Re(\phi(X)_{-2}) = X_{-2},\ Im(\phi(X)_{-2}) = \frac{1}{4}\epsilon^2[X_{-1}, I(X_{-1})]$
$=> Im(w_{-2}) = \epsilon^2[z, I(z)]$ ... $[z, I(z)]$ Hermitian forms

$Re(\phi(X)_{-3}) = X_{-3} - \frac{1}{4}[X_{-2}, X_{-1}] + \epsilon^2\frac{1}{48}[I(X_{-1}), [I(X_{-1}), X_{-1}]]$
$Im(\phi(X)_{-3}) = \frac{1}{4}\epsilon^2[X_{-2}, I(X_{-1})] + \epsilon^2\frac{1}{24}[X_{-1}, [X_{-1}, I(X_{-1})]]$
$=> Im(w_{-3}) = \frac{1}{2}\epsilon^2[Re(w_{-2}), I(z)] + \epsilon^2\frac{1}{3}[z, [z, I(z)]]$

...

Explicit example – |3|–grading of $\mathfrak{sp}(4,\mathbb{C})$

$$\mathfrak{m} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ x_{-1,1} & -x_{-1,2} & 0 & 0 & 0 & 0 & 0 & 0 \\ x_{-1,2} & x_{-1,1} & 0 & 0 & 0 & 0 & 0 & 0 \\ x_{-2,5} & -x_{-2,6} & x_{-1,3} & -x_{-1,4} & 0 & 0 & 0 & 0 \\ x_{-2,6} & x_{-2,5} & x_{-1,4} & x_{-1,3} & 0 & 0 & 0 & 0 \\ x_{-3,7} & -x_{-3,8} & x_{-2,5} & -x_{-2,6} & -x_{-1,1} & x_{-1,2} & 0 & 0 \\ x_{-3,8} & x_{-3,7} & x_{-2,6} & x_{-2,5} & -x_{-1,2} & -x_{-1,1} & 0 & 0 \end{pmatrix},$$

where all $x_{a,b} \in \mathfrak{g}_a$ are real.

Explicit example – |3|–grading of $\mathfrak{sp}(4, \mathbb{C})$

$$\mathfrak{n} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ y_{-1,1} & iy_{-1,1} & 0 & 0 & 0 & 0 & 0 & 0 \\ -iy_{-1,1} & y_{-1,1} & 0 & 0 & 0 & 0 & 0 & 0 \\ y_{-2,3} & -y_{-2,4} & -iy_{-1,2} & -y_{-1,2} & 0 & 0 & 0 & 0 \\ y_{-2,4} & y_{-2,3} & y_{-1,2} & -iy_{-1,2} & 0 & 0 & 0 & 0 \\ y_{-3,5} & -y_{-3,6} & y_{-2,3} & -y_{-2,4} & -y_{-1,1} & -iy_{-1,1} & 0 & 0 \\ y_{-3,6} & y_{-3,5} & y_{-2,4} & y_{-2,3} & iy_{-1,1} & -y_{-1,1} & 0 & 0 \end{pmatrix},$$

where all $y_{a,b} \in \mathfrak{g}_{a,\mathbb{C}}$ are complex.

Explicit example – |3|–grading of $\mathfrak{sp}(4, \mathbb{C})$

$$y_{-1,1} = \frac{x_{-1,1} + ix_{-1,2}}{2}$$

$$y_{-1,2} = \frac{x_{-1,4} + ix_{-1,3}}{2}$$

$$y_{-2,3} = x_{-2,5} + \frac{i}{2}(x_{-1,1}x_{-1,4} + x_{-1,2}x_{-1,3})$$

$$y_{-2,4} = x_{-2,6} + \frac{i}{2}(x_{-1,2}x_{-1,4} - x_{-1,1}x_{-1,3})$$

$$y_{-3,5} = x_{-3,7} + \frac{1}{12}(6(x_{-1,2} + ix_{-1,1})(ix_{-2,5} + x_{-2,6})$$

$$+ (3ix^2_{-1,2} - 2x_{-1,1}x_{-1,2} - 3ix^2_{-1,1})x_{-1,4}$$

$$+ (-x^2_{-1,2} - 6ix_{-1,1}x_{-1,2} + x^2_{-1,1})x_{-1,3})$$

$$y_{-3,6} = x_{-3,8} + \frac{1}{12}(6(x_{-1,2} + ix_{-1,1})(-x_{-2,5} + ix_{-2,6})$$

$$+ (-3ix^2_{-1,2} + 2x_{-1,1}x_{-1,2} + 3ix^2_{-1,1})x_{-1,3} + (-x^2_{-1,2} - 6ix_{-1,1}x_{-1,2} + x^2_{-1,1})x$$

Explicit example – |3|–grading of $\mathfrak{sp}(4,\mathbb{C})$

$$Im(w_{-2,1}) = z_1\bar{z}_2 + \bar{z}_1 z_2$$

$$h_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$Im(w_{-2,2}) = -iz_1\bar{z}_2 + i\bar{z}_1 z_2$$

$$h_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$w = \frac{1}{2}(Re(w_{-2,2}) + iRe(w_{-2,1}))$$

$$Im(w_{-3,1}) = -z_1^2\bar{z}_2 - \bar{z}_1{}^2 z_2 + w\bar{z}_1 + \bar{w}z_1$$

$$Im(w_{-3,2}) = iz_1^2\bar{z}_2 - i\bar{z}_1{}^2 z_2 + iw\bar{z}_1 - i\bar{w}z_1$$