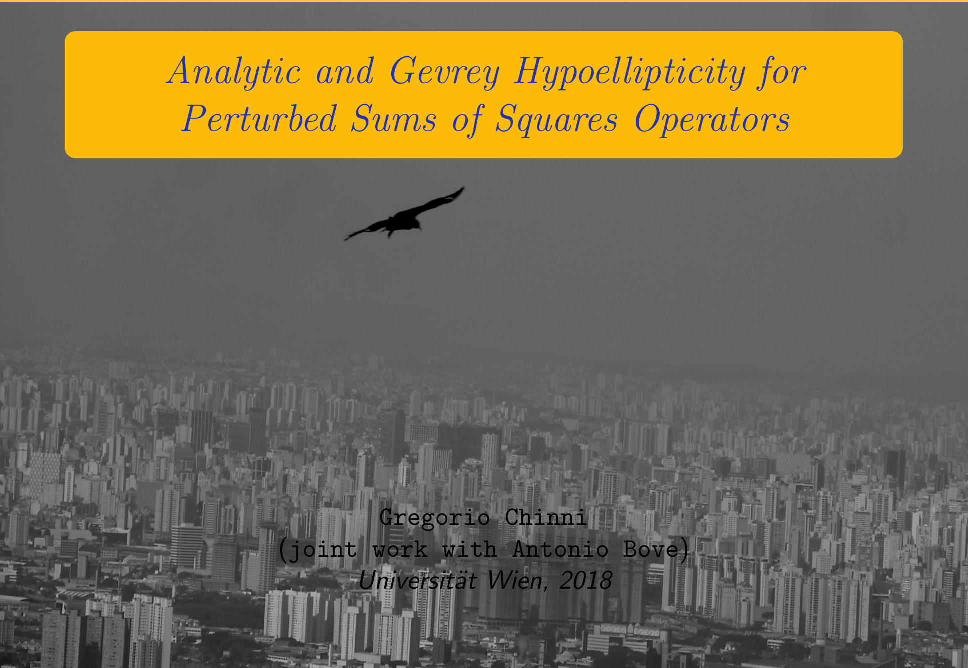
Analytic and Gevrey Hypoellipticity for Perturbed Sums of Squares Operators



Gregorio Chinni
(joint work with Antonio Bove)
Universität Wien, 2018

The Perturbation Problem

Let us consider N vector fields with real-valued real analytic (C^ω) coefficients

$$X_j(x; D), \qquad j = 1, \ldots, N, \qquad x \in U \subset \mathbb{R}^n.$$

Let P denote the corresponding ''sum of squares'' operator

$$P(x, D) = \sum_{j=1}^{N} X_j(x, D)^2. \tag{1}$$

Assumption:

(H) The fields X_j satisfy the *Hörmander condition*, i.e. the Lie algebra generated by the X_j as well as by their commutators of length up to r has dimension n.

The Perturbation Problem

Let $s \geq 1$. We denote by $G^s(\Omega)$, $\Omega \subset \mathbb{R}^n$, the space of Gevrey functions of order s. $u \in G^s(\Omega)$ if and only if $u \in C^\infty(\Omega)$ and for every compact subset K of Ω there is a constant $C_K > 0$ such that

$$\|D^\alpha u\|_{L^2(K)} \leq C_K^{|\alpha|+1}(\alpha!)^s, \qquad \forall\, \alpha \in \mathbb{Z}_+^N.$$

Definition

We say that P is G^s (C^∞)-hypoelliptic in U, $s \geq 1$, if for every $u \in \mathscr{D}'(U)$ and every open set $\Omega \subset U$, the following holds:

$$P(x, D)u \in G^s(\Omega)(C^\infty(\Omega)) \Longrightarrow u \in G^s(\Omega)(C^\infty(\Omega)).$$

When $s = 1$ we shell say that $P(x, D)$ is analytic hypoelliptic.

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The Perturbation Problem

Hörmander's therem (1967)

Theorem [H] [RS]

Let $P(x, D)$ be a ''sum of squares'' operator as above, assume that the fields $X_j(x,D)$ satisfy *Hörmander condition at the step r.*

Then $P(x, D)$ is hypoelliptic.

Furthermore the following *a priori* estimate holds:

$$\|u\|^2_{1/r} + \sum_{j=1}^{N} \|X_j u\|^2_0 \leq C \left(|\langle Pu,\ u\rangle| + \|u\|^2_0 \right).$$

(Subelliptic estimate)

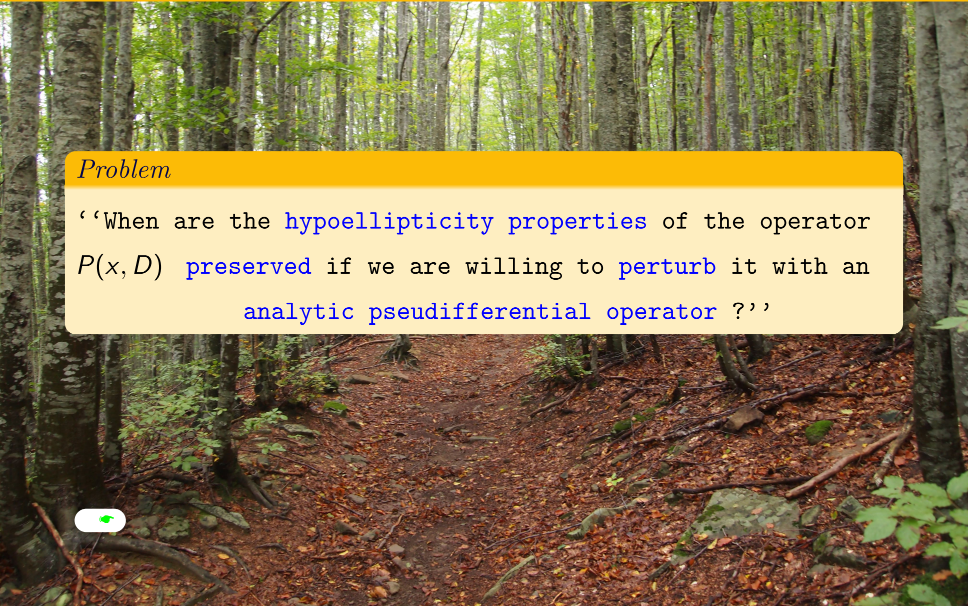
The Perturbation Problem

Derridj and Zuily therem (1973)

Theorem [DZ]

Assume that $P(x, D)$ is defined as above and that the Hörmander condition is satisfied at the step r. Assume that $Pu \in C^\omega(\Omega)$ then $u \in G^r(\Omega)$, i.e. P is G^r-hypoelliptic.

The Perturbation Problem



Problem

''When are the hypoellipticity properties of the operator $P(x, D)$ preserved if we are willing to perturb it with an analytic pseudifferential operator ?''

The Perturbation Problem

Problem

" It is true that the hypoellipticity properties of the operator $P(x, D)$ are preserved if we perturb it with an analytic pseudifferential operator of order strictly less than the subelliptic index of $P(x, D)$?"

Some Positive Results

- C. Parenti and A. Parmeggiani, ([PP]) , have studied the perturbation problem in the local setting for the C^∞-hypoelliptic case. They study the stability of the C^∞-hypoellipticity of a linear partial differential operator, which loses finitely many derivatives, after perturbation by a lower order linear partial differential operator.
- In the global setting, i.e. on the Torus, for the C^ω-hypoelliptic case the perturbation problem was studied for some classes of operators, by C. and Cordaro, ([CC]), and by Braun Rodrigues, C., Cordaro and Jahnke, ([BCCJ]) .

Statements of the results

Write $\{X_i, X_j\}$ for the Poisson bracket of the symbols of the vector fields X_i, X_j:

$$\{X_i, X_j\}(x, \xi) = \sum_{\ell=1}^{n} \left(\frac{\partial X_i}{\partial \xi_\ell} \frac{\partial X_j}{\partial x_\ell} - \frac{\partial X_j}{\partial \xi_\ell} \frac{\partial X_i}{\partial x_\ell} \right) (x, \xi).$$

Definition

Fix a point

$$(x_0, \xi_0) \in \mathrm{Char}(P) \doteq \{(x, \xi) \in T^*\mathbb{R}^n \setminus \{0\} : X_j(x, \xi) = 0 \; j = 1, \ldots, N\}.$$

Consider all the *iterated Poisson brackets* $\{X_i, X_j\}$, $\{\{X_i, X_j\}, X_k\}$ etcetera.

We define $\nu(x_0, \xi_0)$ as the *length* of the *shortest* iterated Poisson bracket of the symbols of the vector fields which is *non zero* at (x_0, ξ_0).

Statements of the results

Definition (Microlocal Gevrey Hypoellipticity)

We say that an operator P is G^s-hypoelliptic at (x_0, ξ_0) if $(x_0, \xi_0) \notin WF_s(u)$ **provided** $(x_0, \xi_0) \notin WF_s(Pu)$.

Albano, Bove and C. theorem (2009)

Theorem [ABC]

Let P be defined as above. Let $(x_0, \xi_0) \in Char(P)$ and $\nu(x_0, \xi_0)$ its length. **Then** P is $G^{\nu(x_0,\xi_0)}$- *hypoelliptic* at (x_0, ξ_0), i.e. if $(x_0, \xi_0) \notin WF_{\nu(x_0,\xi_0)}(Pu)$ **then** $(x_0, \xi_0) \notin WF_{\nu(x_0,\xi_0)}(u)$.

Statement of the Result-1

Theorem (A. Bove, G.C.)

Let $P(x, D)$ be as in (1) and denote by $Q(x, D)$ an *analytic pseudodifferential* operator defined in a conical neighborhood of the point $(x_0, \xi_0) \in \mathrm{Char}(P)$. **If**

$$\mathrm{ord}(Q) < 2/\nu(x_0, \xi_0)$$

then $P + Q$ is $G^{\nu(x_0,\xi_0)}$-hypoelliptic at (x_0, ξ_0).

SkTh1

Statement of the Result-1

Corollary (Local statement)

Let V denote a neighborhood of the point x_0 and

$$r = \sup_{x \in V, |\xi|=1} \nu(x, \xi).$$

Let moreover P be as above with G^r coefficients defined in V and $Q \in OPS_r^m(V)$ be a G^r-pseudodifferential operator of *order* $m < 2/r$.

Then $P + Q$ is G^r-hypoelliptic at x_0.

Statement of the Result-2 (Analytic Case)

Assumptions: Def

(A1) Let $U \times \Gamma$ be a conic neighborhood of (x_0, ξ_0). **There exists** a real analytic function, $h(x, \xi)$, $h\colon U \times \Gamma \to [0, +\infty[$ such that $h(x_0, \xi_0) = 0$ and $h(x, \xi) > 0$ in $U \times \Gamma \setminus \{(x_0, \xi_0)\}$.

(A2) **There exist** real analytic functions $\alpha_{jk}(x, \xi)$ defined in $U \times \Gamma$, such that

$$\{h(x,\xi), X_j(x,\xi)\} = \sum_{\ell=1}^{N} \alpha_{j\ell}(x,\xi) X_\ell(x,\xi), \quad j = 1, \ldots, N. \tag{2}$$

Albano, Bove theorem (2013)

Theorem [AB]

If P, defined as in (1), satisfies (A1) and (A2) **then** P is analytic hypoelliptic at (x_0, ξ_0).

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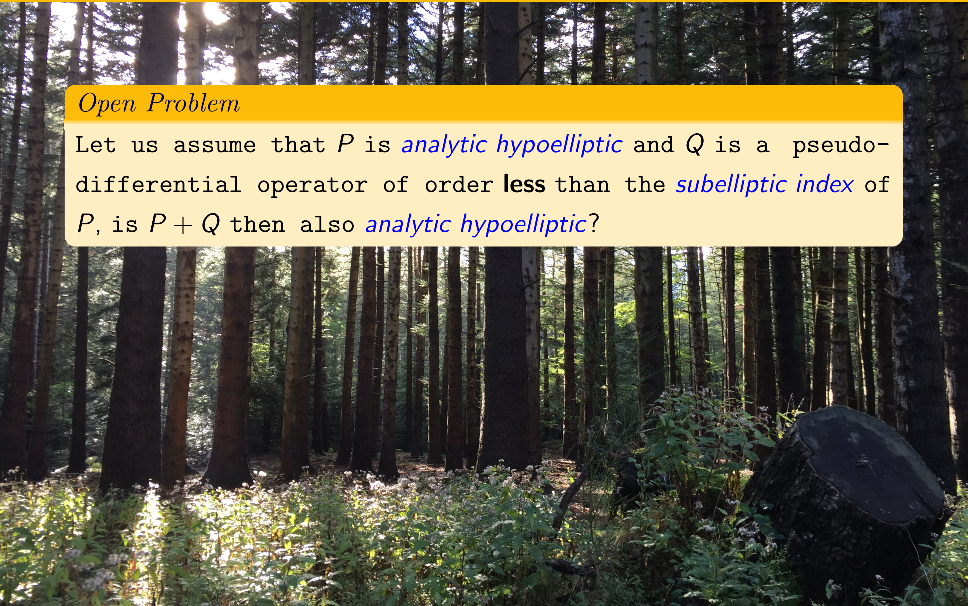
Statement of the Result-2 (Analytic Case)

Theorem (A. Bove, G.C.)

Let P be as in (1) and assume that (A1) and (A2) above are satisfied. Let Q be a real analytic pseudodifferential operator of order *strictly less* than $2/\nu(x_0, \xi_0)$, **then** $P + Q$ is *analytic hypoelliptic* at (x_0, ξ_0).

SkTh2

Open Problem



Open Problem

Let us assume that P is *analytic hypoelliptic* and Q is a pseudo-differential operator of order **less** than the *subelliptic index* of P, is $P+Q$ then also *analytic hypoelliptic*?

Fourier-Bros-Iagolnitzer (FBI) Transform

We define the FBI-*Transform* of a temperate distribution u as

$$T_\varphi u(z,\lambda) = \int_{\mathbb{R}^n} e^{i\lambda\varphi(z,y)}\, u(y)\, dy, \quad z \in \mathbb{C}^n,$$

where $\lambda \gg 1$, $\varphi(z,w)$ is an *holomorphic* function such that

- $\det \partial_z \partial_w \varphi \neq 0$
- $\operatorname{Im} \partial_w^2 \varphi > 0$.

The classical phase function is $\varphi_0(z,x) = \frac{i}{2}(z-x)^2$.

The *wight function*, ϕ, associated to the phase function φ is defined by

$$\phi(z) = \sup_{x\in\mathbb{R}^n} -\operatorname{Im}\varphi(z,x), \qquad z \in \mathbb{C}^n.$$

In the classical case we have

$$\phi_0(z) = \sup_{x\in\mathbb{R}^n} -\operatorname{Im}\varphi_0(z,x) = \frac{(\operatorname{Im} z)^2}{2}.$$

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FBI-Transform

T_φ is associated to the canonical transform

$$\mathscr{H}_{T_\varphi} : \quad \mathbb{C}^{2n}_{w,\theta} \longrightarrow \mathbb{C}^{2n}_{z,\zeta}$$
$$(w, -\partial_w \varphi(z,w)) \longmapsto (z, \partial_z \varphi(z,w)).$$

We have

$$\mathscr{H}_{T_\varphi}\left(\mathbb{R}^{2n}_{x,\xi}\right) \doteq \Lambda_\phi = \left\{\left(z, -2i\frac{\partial \phi}{\partial z}(z)\right)\right\}, \qquad z = x - i\xi.$$

In the case of classical phase function we have

$$\mathscr{H}_{T_{\varphi_0}}\left(\mathbb{R}^{2n}_{x,\xi}\right) = \{(x - i\xi, \xi)\} \doteq \Lambda_{\phi_0}.$$

Λ_{ϕ_0} is a I-Lagrangian, $\mathbb{R}$-Symplectic (Totally Real) sub-manifold of $\mathbb{C}^{2n}$. We have that

$$u \in L^2(\mathbb{R}^n) \Rightarrow T_\varphi u \in L^2(\mathbb{C}^n, e^{-2\lambda\phi(z)} L(dz)).$$

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Wave Front Set from FBI Point of View

Definition ET2

Let u a compactly supported distribution on $\mathbb{R}^n$. We say that the point $(x_0, \xi_0) \in T^*\mathbb{R}^n \setminus \{0\}$ is not in the *$s-$Gevrey wave front set* of u, $(x_0, \xi_0) \notin WF_s(u)$, where $s \geq 1$, if there exist a neighborhood Ω of $x_0 - i\xi_0$ in $\mathbb{C}^n$ and constants $\epsilon > 0$ and $C(< +\infty)$ such that

$$|T_{\varphi_0} u(z, \lambda)| \, e^{-\frac{\lambda}{2}\phi_0(z)} \leq C e^{-\epsilon \lambda^{1/s}}, \qquad \forall z \in \Omega.$$

We say that $(x_0, \xi_0) \notin WF(u)$ ($C^\infty-$ *wave front set*) if there exist Ω neighborhood of $x_0 - i\xi_0$ in $\mathbb{C}^n$ such that $\forall N \in \mathbb{N}$ there is a constant $C_N (> 0)$ for which

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Pseudodifferential Operators (Ω-realization)

Let $(z_0, \zeta_0) \in \mathbb{C}^{2n}$ and $\phi(z)$ a real valued real analytic function defined near z_0 such that

- ϕ is *strictly plurisubharmonic*;
- $\frac{2}{i}\frac{\partial \phi}{\partial z}(z_0) = \zeta_0$.

Denote $\psi(z, w)$ the holomorphic function defined near $(z_0, \bar{z}_0)$ by

$$\psi(z, \bar{z}) = \phi(z).$$

The plurisubharmonicity of ϕ implies that

$$\det \partial_z \partial_w \psi \neq 0$$

and

$$\mathrm{Re}\, \psi(z, \bar{w}) - \frac{1}{2}\left[\varphi(z) + \varphi(w)\right] \sim -|z - w|^2.$$

Pseudodifferential Operators (Ω-realization) [GS]

Denote by $q(z, \zeta, \lambda)$ an *analytic classical symbol* and by $Q(z, \tilde{D}, \lambda)$, $\tilde{D} = (\lambda i)^{-1}\partial$, the formal *classical pseudodifferential operator* associated to q. Using "Kuranishi's trick" one may represent $Q(z, \tilde{D}, \lambda)$ as

$$Qu(z, \lambda) = \left(\frac{\lambda}{2i\pi}\right)^n \int e^{2\lambda(\psi(z,\theta)-\psi(w,\theta))} \tilde{q}(z, \theta, \lambda) u(w) dw d\theta. \quad (3)$$

$\tilde{q}$ is the symbol of Q in the actual representation. To realize the above operator we need a prescription for the int. path. **Ω-realization**. This is accomplished by transforming the classical integration path (Kuranishi change of variables/Stokes theorem):

$$Q^\Omega u(z, \lambda) = \left(\frac{\lambda}{\pi}\right)^n \int_\Omega e^{2\lambda\psi(z,\bar{w})} \tilde{q}(z, \bar{w}, \lambda) u(w) e^{-2\lambda\Phi(w)} L(dw), \quad (4)$$

where $L(dw) = (2i)^{-n} dw \wedge d\bar{w}$, the *integration path* is $\theta = \bar{w}$ and Ω is a small nbhd of $(z_0, \bar{z}_0)$.

Pseudodifferential Operators (Ω-realization)

Remarks. The advantages of such a definition are:

- If the principal symbol is real, Q^Ω is formally *self adjoint* in $L^2(\Omega, e^{-2\lambda\phi})$;
- If $\tilde{q}$ is a classical symbol of order zero, Q^Ω is *uniformly bounded* as $\lambda \to +\infty$, from $H_\phi(\Omega)$ into itself.

Definition

Let Ω be an open subset of $\mathbb{C}^n$. We denote by $H_\phi(\Omega)$ the space of all holomorphic functions $u(z,\lambda)$ such that for every $\epsilon > 0$ and for every compact $K \subset\subset \Omega$ there exists a constant $C > 0$ such that

$$|u(z,\lambda)| \leq Ce^{\lambda(\phi(z)+\epsilon)},$$

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Pseudodifferential Operators (Ω-realization)

Remark: The definition (3) of a pseudodifferential operator on Ω is not the classical one. Via the Kuranishi trick it can be reduced to the classical definition. The function ψ allows us to use a weight function not explicitly related to an FBI phase.

Grigis and Sjöstrand (1985):

Proposition [GS]

Let Q_1 and Q_2 be two pseudodifferential operator of order zero. Then they can be composed and

$$Q_1^\Omega \circ Q_2^\Omega = (Q_1 \circ Q_2)^\Omega + R^\Omega.$$

R^Ω is an error term whose norm is $\mathscr{O}(1)$ as an operator from $H_{\phi+(1/C)d^2}$ to $H_{\phi-(1/C)d^2}$, $d(z) = \mathrm{dist}(z, \complement\Omega)$

A Priori Estimate

Ω-realization of P

Arguing as Grigis and Sjöstrand, 1985, the Ω-*realization* of P can be written as

$$P^{\Omega} = \sum_{j=1}^{N} (X_j^{\Omega})^2 + \mathscr{O}(\lambda^2), \tag{5}$$

$\mathscr{O}(\lambda^2)$ is **continuous** from $H_{\tilde{\phi}}$ to $H_{\phi-(1/C)d^2}$ with norm bounded by $C'\lambda^2$, $\tilde{\phi}$ given by $\phi(z) + \frac{1}{C}d^2(z)$, $d(z) = \operatorname{dist}(z, \complement\Omega)$.

A Priori Estimate

Using the theory of Fourier Integral Operators (FIO) via FBI, developed by A. Grigis, J. Sjöstrand, ([GS]) , allows us, following the ideas of P. Bolley, J. Camus, J. Nourrigat, ([BCN]) , to obtain

Theorem (Sub-elliptic Micro-local Estimate) [ABC]

Let P^Ω be the Ω-**realization** of $P(x,D)$, (1). Let $\Omega_1 \subset\subset \Omega$. **Then**

$$\lambda^{\frac{2}{r}}\|u\|_\phi^2 + \sum_{j=1}^{N}\|X_j^\Omega u\|_\phi^2 \leq C\left(\langle P^\Omega u, u\rangle_\phi + \lambda^\alpha\|u\|^2_{\phi,\Omega\setminus\Omega_1}\right). \quad (6)$$

α is a positive integer, $u \in L^2(\Omega, e^{-2\phi}L(dz))$ and $r = \nu(x_0,\xi_0)$

Albano, Bove and C. 2009

A Priori estimate

Corollary

With the same notation of the above Theorem we have

$$\lambda^{\frac{2}{r}}\|u\|_\phi^2 \leq C\left(\|P^\Omega u\|_\phi^2 + \lambda^\alpha \|u\|_{\phi,\Omega\setminus\Omega_1}^2\right). \tag{7}$$

Here we denote by

$$\|u\|_\phi^2 = \int_\Omega e^{-2\lambda\phi(z)}|u(z)|^2 L(dz).$$

The "Deformation" Argument (Λ_{ϕ_0} deformation)

We use a ''deformation'' argument (due to Sjöstrand) to obtain a canonical deformation of ϕ_0.

We consider a real analytic function $h(z, \zeta, \lambda)$ defined near the point $(x_0 - i\xi_0, \xi_0) = \mathscr{H}_{T_{\varphi_0}}(x_0, \xi_0) \in \Lambda_{\phi_0}$. We solve, for small positive t, the Hamilton-Jacobi problem

$$\begin{cases} \frac{\partial \phi}{\partial t}(t, z, \lambda) & = h\left(z, \frac{2}{i}\frac{\partial \phi}{\partial z}(t, z, \lambda), \lambda\right) \\ \phi(0, z, \lambda) & = \phi_0(z) \end{cases}.$$

Set

$$\phi_t(z, \lambda) = \phi(t, z, \lambda).$$

We have

$$\Lambda_{\phi_t} = \exp\left(itH_h\right)\Lambda_{\phi_0}.$$

The "Deformation" Argument

General Case. We choose the function h as

$$h(z,\zeta,\lambda) = \lambda^{-\frac{r-1}{r}} |z - (x_0 - i\xi_0)|^2 \quad \text{on} \quad \Lambda_{\Phi_0}.$$

Here $r = \nu(x_0, \xi_0)$.

The function ϕ_t can be expanded as a power series in the variable t:

$$\begin{aligned} \phi_t(z,\lambda) &= \phi_0(z) + \frac{t}{2}\, h(\cdot,\cdot,\lambda)\Big|_{\Lambda_{\phi_0}} + \mathcal{O}(\lambda^{-1}) \\ &= \phi_0(z) + \frac{t}{2} \lambda^{\frac{r-1}{r}} |z - (x_0 - i\xi_0)|^2 + \mathcal{O}(\lambda^{-1}). \end{aligned}$$

A Priori Estimate

Albano, Bove, C. theorem

Theorem [ABC]

There exist a neighborhood Ω_0 of $x_0 - i\xi_0$, a positive number $\delta > 0$, and a positive integer α such that, for every $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega \subset \Omega_0$, there exists a constant $C > 0$ such that, for $0 < t < \delta$, we have

$$\lambda^{\frac{2}{r}} \|u\|_{\phi_t,\Omega_1} \leq C \left(\|P^{t^\Omega} u\|_{\phi_t,\Omega_2} + \lambda^\alpha \|u\|_{\phi_t,\Omega\setminus\Omega_1} \right), \quad r = \nu(x_0,\xi_0). \tag{8}$$

P^{t^Ω} is the Ω-*realization* of P^t, the symbol of P restricted to Λ_{ϕ_t}.

Corollary

Let Pu be analytic at (x_0,ξ_0), then the point (x_0,ξ_0) does not belong to $WF_{\nu(x_0,\xi_0)}(u)$.

A Priori Estimate

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The “Deformation” Argument

Operators which satisfy the assumptions *(A1)* and *(A2)*. Show A1/2

We choose the function h of the assumptions.

h does not depend on λ.

We have

$$\phi_t(z) = \phi_0(z) + \frac{1}{2}\int_0^t h\left(z, \frac{2}{i}\partial_z \phi_s(z)\right) ds.$$

Also in this case we obtain

$$\lambda^{\frac{2}{r}} \|u\|_{\phi_t,\Omega_1} \le C\left(\|P^{t^\Omega} u\|_{\phi_t,\Omega_2} + \lambda^\alpha \|u\|_{\phi_t,\Omega\setminus\Omega_1}\right), \qquad r = \nu(x_0,\xi_0).$$

We point out that

$$h_{\big|\Lambda_{\phi_0}\cap\Omega\setminus\Omega_1} \ge a > 0 \qquad (\phi_t(z) \ge \phi_0(z) + c' t, \quad x \in \Omega\setminus\Omega_1),$$

$$\phi_t \le \phi_0 + t/(2C) \text{ in } \Omega_2,\ \Omega_2 \subset\subset \Omega_1 \text{ neighborhood of } x_0 - i\xi_0.$$

Theorem 1 Show Th1

Denote by θ the *order* of the pseudodifferential operator Q. Let $Q^{t\Omega}$ is the Ω-*realization* of Q^t, the symbol of Q restricted to Λ_{ϕ_t}. From (8) we have

$$\lambda^{\frac{2}{r}}\|u\|_{\phi_t,\Omega_1} \leq C\left(\|(P+Q)^{t\Omega}u\|_{\phi_t,\Omega_2} + \|Q^{t\Omega}u\|_{\phi_t,\Omega_2} + \lambda^{\alpha}\|u\|_{\phi_t,\Omega\setminus\Omega_1}\right).$$

We have

$$\|Q^{t\Omega}u\|_{\phi_t,\Omega_2} \leq C_1\lambda^{\theta}\|u\|_{\phi_t,\Omega_2} \leq C_1\lambda^{\theta}\left(\|u\|_{\phi_t,\Omega_1} + \|u\|_{\phi_t,\Omega\setminus\Omega_1}\right)$$

The first term of above inequality is absorbed on the left hand side($\theta < 2/r$, λ large enough.) Hence we have

$$\lambda^{\frac{2}{r}}\|u\|_{\phi_t,\Omega_1} \leq C\left(\|\left(P+Q\right)u\|_{\phi_t,\Omega_2} + \lambda^{\alpha}\|u\|_{\phi_t,\Omega\setminus\Omega_1}\right).$$

Theorem 1

We assume that $(x_0, \xi_0) \notin WF_a((P+Q)u)$ then

$$\|(P+Q)^{t^{\Omega}} u\|_{\phi_t,\Omega_2} \leq Ce^{-\lambda/C}.$$

Λ_{Φ_t} is a small perturbation of Λ_{Φ_0}.
Since

$$\phi_t(z,\lambda) = \phi_0(z) + \frac{t}{2} \underbrace{\lambda^{\frac{r-1}{r}} |z-(x_0 - i\xi_0)|^2}_{=h(\cdot,\cdot,\lambda)_{|\Lambda_{\phi_0}}} + \mathscr{O}(\lambda^{-1}),$$

we have that $\|u\|_{\phi_t,\Omega\setminus\Omega_1} \leq Ce^{-\lambda^{1/r}/C}.$

Thus we obtain that

$$\|u\|_{\phi_t,\Omega_1} \leq C_1 e^{-\lambda^{1/r}/C_1}.$$

Theorem 2

Let $\Omega_3 \subset\subset \Omega_2$. For $z \in \Omega_3$, for a fixed small positive value of t, we have

$$\phi_t(z) - \phi_0(z) \leq \frac{\lambda^{-1+1/r}}{C_2(t)}.$$

Therefore

$$\|u\|_{\phi_0,\Omega_3} \leq ce^{-\lambda^{1/r}/c} \quad ((x_0,\xi_0) \notin WF_r(u), r = \nu(x_0,\xi_0)).$$ Show DWF

An analogous strategy allows us to obtain also the second Theorem. Show Th2


Thank you for your patience!!
Vielen Dank für Ihre Geduld!!!

Example I :

- Let k be an integer, $k \geq 2$, and consider

$$P(x, D) = D_1^2 + x_1^{2(k-1)} D_2^2, \quad x \in \mathbb{R}^2.$$

Let $Q(x, D)$ the *analytic pseudodifferential* operator given by

$$\lambda |D_2|^{2/k}.$$

λ is a constant that we shall choose later.

Q is microlocally elliptic near points in

$$\mathrm{Char}(P) = \{(x_1, x_2, \xi_1, \xi_2) \in T^*\mathbb{R}^2 \setminus \{0\} \;:\; x_1 = 0 = \xi_1,\ \xi_2 \neq 0\}.$$

Remark: $P(x, D)$ is analytic hypoelliptic. $\frac{2}{k}$ is the *subelliptic index* of $P(x, D)$.

Example I :

Performing a Fourier transform w.r.t. x_2, and the dilation

$$x_1 \to |\xi_2|^{-1/k} x_1,$$

$P + Q$ becomes

$$D_1^2 + x_1^{2(k-1)} + \lambda.$$

(modulo a microlocally elliptic factor)

Let λ be the opposite of an eigenvalue.

Let $\phi_\lambda(x_1)$ be such that

$$-\phi_\lambda'' + x_1^{2(k-1)} \phi_\lambda + \lambda \phi_\lambda = 0.$$

Example I :

Consider

$$u(x) = \int_0^{+\infty} e^{ix_2\rho}\ \phi_\lambda(x_1\rho^{1/k})\,(1+\rho^4)^{-1}d\rho.$$

$(P+Q)u = 0$ **and** $u \notin C^\infty$.

We have:

$$\text{If } \phi_\lambda(0) \neq 0 \Longrightarrow u(0,x_2) = \phi_\lambda(0)\int_0^{+\infty} e^{ix_2\rho}(1+\rho^4)^{-1}d\rho;$$

$$\text{If } \phi_\lambda(0) = 0\,(\phi'_\lambda(0) \neq 0!) \Rightarrow \frac{\partial u}{\partial x_1}(0,x_2) = \phi'_\lambda(0)\int_0^{+\infty} e^{ix_2\rho}(1+\rho^4)^{-1}\rho^{1/k}d\rho.$$

In both cases we do not have a C^∞ function!

Example I :

Taking

$$Q = \lambda|D_2|^{2/k} + \mu(x_2)|D_2|^{\epsilon}, \text{ with } \epsilon < 2/k,$$

then

$$P + Q \quad \text{can be} \quad \begin{cases} C^\omega - \textit{hypoelliptic}, \\ G^s - \textit{hypoelliptic} \text{ for some } s, \\ \textbf{not even } C^\infty - \textit{hypoelliptic}. \end{cases}$$

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The Stein example (converse statement):

Consider Kohn's Laplacian,

$$\Box_b = -Z\,\bar{Z} \quad \text{where} \quad Z = \frac{\partial}{\partial z} + i\bar{z}\frac{\partial}{\partial t}, \quad (z,t) \in \mathbb{C} \times \mathbb{R},$$

which is **neither** C^∞ **nor** C^ω*-hypoelliptic*.

Stein 1982, ([S]) :

Perturbing it with a non zero complex number,

$$\Box_b + \alpha, \quad \alpha \in \mathbb{C} \setminus \{0\},$$

we obtain an operator being both C^∞ and C^ω*-hypoelliptic*.

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Consider Kohn's Laplacian,

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