The problem of multisummability in higher dimensions

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Asymptotic Analysis and Borel summability in one variable

We have at our disposal a powerful summability theory useful in the study of formal solutions of analytic problems, e.g. ODEs at irregular singular points, families of PDEs, difference equations, conjugacy of diffeomorphisms of $(\mathbb{C}, 0)$, normal forms for vector fields, singular perturbation problems, normal forms of real-analytic hypersurfaces...

- Asymptotic expansions, Gevrey asymptotic expansions, $k-$summability.
- Borel and Laplace transformations. Tauberian theorems.
- Ecalle's accelerator operators, Multisummability.

Asymptotics in several variables

For several variables there are different approaches. In this framework we can mention:

- Strong Asymptotic Expansions, (Majima, 1984).
- Composite Asymptotic Expansions (Fruchard-Schäfke, 2013).
- Asymptotic Expansions in a monomial or in an analytic function (Mozo-Schäfke, 2007, 2017).

We will focus in the item and pose the problem of multisummability for those methods.

The scope of applications

- (1990 Ramis, Sibuya, Braaskma) Multisummability of non-linear equations

$$x^{p+1}\frac{d\boldsymbol{y}}{dx} = \boldsymbol{F}(x,\boldsymbol{y}).$$

When $\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{y}}(0,\mathbf{0})$ is invertible the unique formal power series solution is $p-$summable.

- (2003 Luo, Chen, Zhang) Summability in the variable x of solutions of PDEs of the form

$$t\partial_t u = F(t,x,u,\partial_x u), \qquad u(0,x)=0,$$

under certain conditions on F.

- (2007 Costin, Tanveer) Existence, uniqueness and asymptotic in several variables of solutions of PDEs of the form

$$\boldsymbol{u}_t + \mathcal{P}(\partial_{\boldsymbol{x}}^j)\boldsymbol{u} + \boldsymbol{g}(\boldsymbol{x},t,\{\partial_{\boldsymbol{x}}^j \boldsymbol{u}\}) = 0, \qquad \boldsymbol{u}(\boldsymbol{x},0) = u_{\boldsymbol{I}}(\boldsymbol{x}),$$

where the principal part of the constant coefficient $n-$th order differential operator $\mathcal{P}$ is subject to a cone condition.

- (2007 Canalis-Duran, Mozo, Schäfke) $1 - x^p\varepsilon^q-$summability of the unique formal power series solution of the doubly singular equation

$$\varepsilon^q x^{p+1}\frac{\partial \boldsymbol{y}}{\partial x} = \boldsymbol{F}(x,\varepsilon,\boldsymbol{y}),$$

when $\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{y}}(0,0,\mathbf{0})$ is invertible.

- (2018 -) $1 - \boldsymbol{x}^{\boldsymbol{\alpha}}\boldsymbol{\varepsilon}^{\boldsymbol{\alpha}'}-$summability of the unique formal power series solution of the singularly perturbed PDE

$$\boldsymbol{x}^{\boldsymbol{\alpha}}\boldsymbol{\varepsilon}^{\boldsymbol{\alpha'}}\left(\lambda_1 x_1\frac{\partial \boldsymbol{y}}{\partial x_1} + \cdots + \lambda_n x_n\frac{\partial \boldsymbol{y}}{\partial x_n}\right) = \boldsymbol{F}(\boldsymbol{x},\boldsymbol{\varepsilon},\boldsymbol{y}),$$

where $\boldsymbol{x}\in\mathbb{C}^n, \boldsymbol{\varepsilon}\in\mathbb{C}^m$, $\boldsymbol{\alpha}\in(\mathbb{N}^+)^n$, $\boldsymbol{\alpha'}\in(\mathbb{N}^+)^m$, $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_n)\in(\mathbb{R}^+)^n$ and $\boldsymbol{F}$ analytic at the origin and $\frac{\partial \boldsymbol{F}}{\partial y}(0,0,\mathbf{0})$ is invertible.

The theory in one variable

Example: Euler's equation

Consider Euler's equation:

$$x^2y' + y = x.$$

We can solve it for $x > 0$ to get

$$y(x) = ce^{1/x} + \int_0^{+\infty} \frac{e^{-\xi/x}}{1+\xi} d\xi.$$

But it also has the formal power series solution

$$\hat{y}(x) = \sum_{n=0}^{\infty} (-1)^n n! x^{n+1}.$$

The notion of asymptotic expansion

Let us fix a complex Banach space $(E, \|\cdot\|)$.

We work in sectors at the origin

$$S = S(a,b;r) = \{x \in \mathbb{C} \mid 0 < |x| < r, a < \arg(x) < b\}.$$

Definition

We say $f \in \mathcal{O}(S,E)$ has $\hat{f} = \sum_{n=0}^{\infty} a_n x^n \in E[[x]]$ *as asymptotic expansion on* S ($f \sim \hat{f}$ on S.) if for every subsector $S' \subset S$ and $N \in \mathbb{N}$ we can find $C_N(S') > 0$ such that

$$\left\| f(x) - \sum_{n=0}^{N-1} a_n x^n \right\| \leq C_N(S')|x|^N, \quad x \in S'.$$

Basic properties

Assume $f \sim \hat{f} = \sum_{n=0}^{\infty} a_n x^n$ and $g \sim \hat{g}$ on S. The following properties hold:

1. $a_n = \lim_{\substack{x \to 0 \\ x \in S'}} \frac{f^{(n)}(x)}{n!}$ for any subsector S'.
2. $f + g \sim \hat{f} + \hat{g}$, $fg \sim \hat{f}\hat{g}$, $\frac{df}{dx} \sim \frac{d\hat{f}}{dx}$ on S.
3. (Borel-Ritt) Given any $\hat{f} \in E[[t]]$ and S there is $f \in \mathcal{O}(S, E)$ such that $f \sim \hat{f}$ on S.

Gevrey type asymptotic expansions

If $f \sim \hat{f}$ on S and we can choose $C_N(S') = CA^N N!^{1/k}$, then we say that the asymptotic expansion is of type $1/k-$Gevrey ($f \sim_{1/k} \hat{f}$ on S). Then

$$\hat{f} \in E[[x]]_{1/k}, \quad \text{i.e.} \quad \|a_n\| \leq CA^n n!^{1/k},$$

the space of $1/k-$*Gevrey series in* x.

- $f \sim_{1/k} 0$ on S if and only if for every $S' \subset S$, we can find $K, M > 0$

$$\|f(x)\| \leq K \exp(-M/|x|^k).$$

- (Borel-Ritt-Gevrey) If $b - a < \pi/k$ given any $\hat{f} \in E[[x]]_{1/k}$ and $S(a, b, r)$ there is $f \in \mathcal{O}(S, E)$ such that $f \sim_{1/k} \hat{f}$ on S.
- (Watson's Lemma) If $b - a > \pi/k$ and $f \sim_{1/k} 0$ on $S(a, b, r)$ then $f \equiv 0$.

$k-$Sumability

Definition

Let $\hat{f} \in E[[x]]_{1/k}$ and $\theta \in \mathbb{R}$ a direction.

- $\hat{f}$ is $k-$summable in a direction θ if we can find $f \in \mathcal{O}(S, E)$, $S = S(\theta - \frac{\pi}{2k} - \varepsilon, \theta + \frac{\pi}{2k} + \varepsilon, r)$ such that $f \sim_{1/k} \hat{f}$.
- $\hat{f}$ is $k-$summable if it is $k-$summable in all directions, up to a finite number of them, mod. 2π.

We will use the notation $E\{x\}_{1/k,\theta}$ and $E\{x\}_{1/k}$ for the corresponding sets.

Borel-Laplace method

The series $\hat{f} = \sum_{n=0}^{\infty} a_n x^n \in E[[x]]_{1/k}$ is called $k-$*Borel-summable in direction* θ if

$$\widehat{\mathcal{B}}_k(\hat{f} - \sum_{n \leq k} a_n x_n) := \sum_{n>k} \frac{a_n}{\Gamma(n/k)} \xi^{n-k},$$

can be analytically continued, say as φ, and

$$\|\varphi(\xi)\| \leq C \exp(M|\xi|^k), \qquad \text{for some } C, M > 0.$$

Its Borel sum is defined by

$$\begin{aligned} f(x) &= \sum_{n \leq k} a_n x^n + \mathcal{L}_k(\varphi)(x) \\ &= \sum_{n \leq k} a_n x^n + \int_0^{e^{i\theta}\infty} \varphi(\xi) e^{-(\xi/x)^k} d(\xi^k). \end{aligned}$$

The Borel-Laplace analysis exploits the isomorphism between the following structures

$$\left(E[[x]]_{1/k}, +, \ \times \ , x^{k+1}\frac{d}{dx}\right) \xrightarrow{\widehat{\mathcal{B}}_k} \left(\xi^{-k}E\{\xi\}, +, *_k, k\xi^k(\cdot)\right),$$

where $\times$ denotes the usual product and $*_k$ stand for the convolution product

$$(f *_k g)(\xi) = \xi^k \int_0^1 f(\xi\tau^{1/k})g(\xi(1-\tau)^{1/k})d\tau.$$

Euler's equation II

Applying the $1-$Borel transformation to Euler's example:

$$\hat{y}(x) = \sum_{n=0}^{\infty} (-1)^n n! x^{n+1} \xrightarrow{\widehat{\mathcal{B}}_1} Y(\xi) = \sum_{n=0}^{\infty} (-1)^n \xi^n = \frac{1}{1+\xi},$$

$$x^2 y' + y = x \xrightarrow{\widehat{\mathcal{B}}_1} \xi Y + Y = 1.$$

Using the Laplace transform we get the solution

$$y(x) = \int_0^{+\infty} \frac{e^{-\xi/x}}{1+\xi} d\xi, \ \text{Re}(x) > 0.$$

Example of non-linear ODEs

Consider the differential equation

$$x^{p+1}\frac{d\boldsymbol{y}}{dx} = \boldsymbol{F}(x,\boldsymbol{y}) = b(x) + A(x)\boldsymbol{y} + \sum_{|\boldsymbol{I}|\geq 2} A_I(x)\boldsymbol{y}^I,$$

where $p \in \mathbb{N}^+$, $\boldsymbol{y} \in \mathbb{C}^N$, $\boldsymbol{F}$ is analytic in a neighborhood of $(0,\mathbf{0}) \in \mathbb{C}\times\mathbb{C}^N$ and $\boldsymbol{F}(0,\mathbf{0}) = \mathbf{0}$.
Using $\hat{\mathcal{B}} = \hat{\mathcal{B}}_p$, $* = *_p$ we obtain the convolution equation

$$\begin{aligned}(p\xi^p I_N - A_0)\boldsymbol{Y} =& \mathcal{B}(b) + \mathcal{B}(A - A_0) * \boldsymbol{Y} + \sum_{|\boldsymbol{I}|\geq 2} \mathcal{B}(A_{\boldsymbol{I}} - A_{\boldsymbol{I}}(0)) * \boldsymbol{Y}^{*\boldsymbol{I}} \\ &+ \sum_{|\boldsymbol{I}|\geq 2} A_{\boldsymbol{I}}(0)\boldsymbol{Y}^{*\boldsymbol{I}}.\end{aligned}$$

We ask for $p\xi^p I_N - A_0$ to be invertible, therefore we work on domains inside

$$\Omega := \{\xi \in \mathbb{C} \mid p\xi^p \neq \lambda_j \text{ for all } j = 1,\ldots,N\},$$

where λ_j are the eigenvalues of A_0.

Theorem

If $A_0 = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{y}}(0, \mathbf{0})$ *is invertible then the previous ODE has a unique formal power series solution* $\hat{\boldsymbol{y}} \in \mathbb{C}[[x]]^N$. *Furthermore* $\hat{\boldsymbol{y}}$ *is* $p-$*summable.*

For $\mu > 0$ consider

$$\mathcal{A}_\mu^N(S) := \{\boldsymbol{f} \in \mathcal{O}(S, \mathbb{C}^N) \mid \boldsymbol{f}(0) = \mathbf{0},\ \|\boldsymbol{f}\|_{N,\mu} := \max_{1 \leq j \leq N} \|f_j\|_\mu < +\infty\},$$

$$\|f\|_\mu := M_0 \sup_{\xi \in S} |f(\xi)|(1 + |\xi|^{2p}) e^{-\mu|\xi|^p}, \qquad f \in \mathcal{O}(S).$$

$$S := S_R = S(\theta, 2\epsilon) \cup D_R \subset \Omega,$$

$$M_0 = \sup_{s>0} s(1+s^2) I(s) \approx 3.76, \quad I(s) := \int_0^1 \frac{d\tau}{(1+s^2\tau^2)(1+s^2(1-\tau)^2)}.$$

For general linear systems it is enough to consider equations of type

$$x\frac{d\boldsymbol{y}}{dx} = b(x) + A(x)\boldsymbol{y},$$

$$A(x) = \bigoplus_{h=0}^{r} x^{-k_h} A_h + \left(\bigoplus_{h=0}^{r} x^{1-k_h} I_h\right) A_+(x),$$

where $0 = k_0 < k_1 < \cdots < k_r$, $k_h \in \mathbb{N}^+$, $N = n_0 + \cdots + n_r$, A_h is a $n_h \times n_h$ invertible matrix , I_h is the identity matrix of size n_h, A_+ and g analytic at the origin.

Theorem (Braaksma-Balser-Ramis-Sibuya)

If the system posses a formal solution $\hat{\boldsymbol{y}}$ then it is $\boldsymbol{k}-$multisummable, where $\boldsymbol{k} = (k_1, \ldots, k_n)$.

Tauberian properties in one variable

Theorem (Martinet-Ramis)

The followings statements are true for $0 < k < k'$ and $0 < k_0, k_1, \ldots, k_n$:

1. *If $\hat{f} \in E\{t\}_{1/k}$ has no singular directions then it is convergent.*
2. $E[[x]]_{1/k'} \cap E\{t\}_{1/k} = E\{t\}_{1/k'} \cap E\{t\}_{1/k} = E\{t\}$.
3. *Consider $\hat{f}_j \in E\{t\}_{1/k_j} \setminus E\{t\}$ for $j = 1, ..., n$. Then $\hat{f}_0 = \hat{f}_1 + \cdots + \hat{f}_n \in E\{t\}_{1/k_0}$ if and only if $k_0 = k_j$ for all $j = 1, ..., n$.*

Multisummability

For two levels $0 < k_2 < k_1$ we can compose both k_j-summability methods:

$$\mathcal{L}_{k_1} \circ (\mathcal{B}_{k_1} \circ \mathcal{L}_{k_2}) \circ \hat{\mathcal{B}}_{k_2}.$$

We can work with the central term

$$\mathcal{B}_{k_1} \circ \mathcal{L}_{k_2}(\varphi)(x) = \frac{1}{x^{k_1}} \int_0^{e^{i\theta}\infty} \varphi(\xi) C_\alpha((\xi/x)^{k_2}) d\xi^{k_2} := \mathfrak{A}_{k_1,k_2}(\varphi)(x),$$

Here $\alpha = k_1/k_2 > 1$ and

$$C_\alpha(t) = \frac{1}{2\pi i} \int_\gamma \exp(u - tu^{1/\alpha}) du = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma\left(\frac{-n}{\alpha}\right)} t^n,$$

it is called a *Ecalle's acceleration kernel*. It is entire and $|C_\alpha(t)| \leq c_1 \exp(-c_2|t|^\beta)$, $1/\alpha + 1/\beta = 1$, $|\text{arg}(t)| \leq \pi/2\beta - \epsilon$.

Definition

Consider $\boldsymbol{k} = (k_1, k_2, ..., k_q)$ with $0 < k_q < \cdots < k_2 < k_1$. A (multi-)direction $\boldsymbol{\theta} = (\theta_1, ..., \theta_q)$ is $\boldsymbol{k}$*−admissible* if

$$|\theta_j - \theta_{j-1}| \leq \frac{\pi}{2\kappa_j}, \quad \text{with } \frac{1}{\kappa_j} = \frac{1}{k_j} - \frac{1}{k_{j-1}}, \quad 2 \leq j \leq q.$$

This is equivalent to say that the intervals $I_j = \left[\theta_j - \frac{\pi}{2k_j}, \theta_j + \frac{\pi}{2k_j}\right]$ satisfy

$$I_1 \subset I_2 \subset \cdots \subset I_q.$$

Definition

A formal power series $\hat{f} \in E[[x]]_{1/k_q}$ is $\boldsymbol{k}-$*multisumable* in direction $\boldsymbol{\theta}$ if:

1. $f_q = \hat{\mathcal{B}}_{k_q}(\hat{f})$ can be analitically extended to a sector of infinite radius bisected by θ_d and with exponential growing at most κ_q. We can then calculate $\mathfrak{A}_{k_{q-1},k_q}(f_q)$.
2. For every $1 \leq j \leq q-1$, consider $f_j = \mathfrak{A}_{k_j,k_{j+1}}(f_{j+1})$. We ask f_j to be analitically extended to a sector of infinite radius bisected by θ_j and with exponential growing at most κ_j (k_1 for $j=1$).

Then $f = \mathcal{L}_{k_1,\theta_1}(f_1)$ is well-defined in a small sector bisected by θ_1 and opening larger than π/k_1. It is called the $\boldsymbol{k}-$*multisumm* of $\hat{f}$ in direction $\boldsymbol{\theta}$.

The functions f_j satisfy

$$f_j \sim_{1/\widetilde{k}_j} \hat{f}_j = \hat{\mathcal{B}}_{k_j}(\hat{f}), \quad 1/\widetilde{k_j} = 1/k_q - 1/k_j.$$

Let $E\{x\}_{\boldsymbol{k},\boldsymbol{\theta}}$ be the space of $\boldsymbol{k}-$multisummable series in direction $\boldsymbol{\theta}$.

Decomposition theorem

Theorem (Decomposition theorem - W. Balser)

Consider $\boldsymbol{k} = (k_1, ..., k_q) \in (\mathbb{R}^+)^q$ with $\frac{1}{k_j} - \frac{1}{k_{j-1}} < 2$, $1 \leq j \leq q$ and a $\boldsymbol{k}-$admissible direction $\boldsymbol{\theta} = (\theta_1, ..., \theta_q)$.
Then for every $\hat{f} \in E\{x\}_{\boldsymbol{k},\boldsymbol{\theta}}$ we can find $\hat{f}_j \in E\{x\}_{1/k_j,\theta_j}$ such that

$$\hat{f} = \hat{f}_1 + \cdots + \hat{f}_q,$$

and the $\boldsymbol{k}-$sum correspond to the sum of the k_j-sums if the respective series.

Asymptotic and Summabiliy in an analytic function

The formal framework

We work in $(\mathbb{C}^d, \mathbf{0})$ with coordinates $\boldsymbol{x} = (x_1, \ldots, x_d)$. Consider $\hat{f} = \sum_{\boldsymbol{\beta}\in\mathbb{N}^d} f_{\boldsymbol{\beta}} \boldsymbol{x}^{\boldsymbol{\beta}} \in E[[\boldsymbol{x}]]$.

Given $\boldsymbol{\alpha} \in \mathbb{N}^d \setminus \{\mathbf{0}\}$, we can write uniquely

$$\hat{f} = \sum_{n=0}^{\infty} \hat{f}_{\boldsymbol{\alpha},n}(\boldsymbol{x})\boldsymbol{x}^{n\boldsymbol{\alpha}}, \quad \hat{f}_{\boldsymbol{\alpha},n}(\boldsymbol{x}) = \sum_{\boldsymbol{\alpha}\not\leq\boldsymbol{\beta}} f_{n\boldsymbol{\alpha}+\boldsymbol{\beta}} \boldsymbol{x}^{\boldsymbol{\beta}}.$$

Given $\boldsymbol{P} = \sum_{\boldsymbol{\beta}\in\mathbb{N}^d} P_{\boldsymbol{\beta}} \boldsymbol{x}^{\boldsymbol{\beta}} \in \mathbb{C}\{\boldsymbol{x}\}$, $\boldsymbol{P}(\mathbf{0}) = 0$ and an injective linear form $\ell : \mathbb{N}^d \to \mathbb{R}^+$, $\ell(\boldsymbol{\alpha}) = \ell_1\alpha_1 + \cdots + \ell_d\alpha_d$, we can also write uniquely

$$\hat{f} = \sum_{n=0}^{\infty} \hat{f}_{\boldsymbol{P},\ell,n}(\boldsymbol{x})\boldsymbol{P}^n, \quad \hat{f}_{\boldsymbol{P},\ell,n}(\boldsymbol{x}) \in \Delta_\ell(\boldsymbol{P}, E).$$

The domains for asymptotic

A $\boldsymbol{P}-$*sector* is a set defined by the conditions

$$\Pi_{\boldsymbol{P}}(a,b;\boldsymbol{R}) = \left\{\boldsymbol{x}\in\mathbb{C}^d \mid \boldsymbol{P}(\boldsymbol{x})\neq 0, a<\arg(\boldsymbol{P}(\boldsymbol{x}))<b,\ 0<|x_j|<R_j\right\},$$

If $\boldsymbol{x}\in\Pi_{\boldsymbol{P}}(a,b;\boldsymbol{r})$ then $t=\boldsymbol{P}(\boldsymbol{x})\in\{z\in\mathbb{C}\mid 0<|z|<r,\ a<\arg(z)<b\}$, for some $r>0$.

Definition

Let $f \in \mathcal{O}(\Pi_{\boldsymbol{P}}, E)$, $\Pi_{\boldsymbol{P}} = \Pi_{\boldsymbol{P}}(a, b; \boldsymbol{R})$ and $\hat{f} \in E[[\boldsymbol{x}]]$. We will say that f has $\hat{f}$ as $\boldsymbol{P}-$*asymptotic expansion on* $\Pi_{\boldsymbol{P}}$ if for some $r > 0$ we have:

1. There is $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{O}_b(D_r^d, E)$ is a $\boldsymbol{P}-$asymptotic sequence for $\hat{f}$, i.e. $f_n \to f$ in the $\mathfrak{m}-$topology and $f_n \equiv \hat{f}(\bmod \boldsymbol{P}^n E[[\boldsymbol{x}]])$.
2. For every $N \in \mathbb{N}$ and $\Pi'_{\boldsymbol{P}} \subset \Pi_{\boldsymbol{P}}$ there exists $C_N(\Pi'_{\boldsymbol{P}}) > 0$ such that

$$\|f(\boldsymbol{x}) - f_N(\boldsymbol{x})\| \leq C_N(\Pi'_{\boldsymbol{P}})|\boldsymbol{P}(\boldsymbol{x})|^N, \quad \text{on } \Pi'_{\boldsymbol{P}} \cap D_r^d.$$

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$$\|f(\boldsymbol{x}) - f_N(\boldsymbol{x})\| \leq C_N(\Pi'_{\boldsymbol{P}})|\boldsymbol{P}(\boldsymbol{x})|^N, \quad \text{on } \Pi'_{\boldsymbol{P}} \cap D_r^d.$$

Fixing ℓ we can take $f_N = \sum_{n=0}^{N-1} f_{\boldsymbol{P},\ell,n}\boldsymbol{P}^j$ and for every $N \in \mathbb{N}$ and $\Pi'_{\boldsymbol{P}} \subset \Pi_{\boldsymbol{P}}$ there exists $L_N(\Pi'_{\boldsymbol{P}}) > 0$ such that

$$\left\| f(\boldsymbol{x}) - \sum_{n=0}^{N-1} f_{\boldsymbol{P},\ell,n}(\boldsymbol{x})\boldsymbol{P}(\boldsymbol{x})^n \right\| \leq L_N(\Pi'_{\boldsymbol{P}})|\boldsymbol{P}(\boldsymbol{x})|^N, \quad \text{on } \Pi'_{\boldsymbol{P}} \cap D_\rho^d.$$

Basic properties

- $\boldsymbol{P}-$asymptotic expansions are stable under addition and partial derivatives and products (if E is a Banach algebra).
- The $\boldsymbol{P}-$asymptotic expansion of a function on a $\boldsymbol{P}-$sector, if it exists, is unique. Indeed, if $f \sim^{\boldsymbol{P}} \hat{f} = \sum f_{\boldsymbol{\beta}} \boldsymbol{x}^{\boldsymbol{\beta}}$ on $\Pi_{\boldsymbol{P}}$ then

$$\lim_{\substack{\boldsymbol{x}\to\boldsymbol{0}\\ \boldsymbol{x}\in\Pi'_{\boldsymbol{P}}}} \frac{1}{\boldsymbol{\beta}!}\frac{\partial^{\boldsymbol{\beta}} f}{\partial \boldsymbol{x}^{\boldsymbol{\beta}}}(\boldsymbol{x}) = f_{\boldsymbol{\beta}}, \quad \Pi'_{\boldsymbol{P}} \subset \Pi_{\boldsymbol{P}}.$$

- Consider $\boldsymbol{P}, \boldsymbol{Q} \in \mathbb{C}\{\boldsymbol{x}\} \setminus \{0\}$ such that $\boldsymbol{Q} = U \cdot \boldsymbol{P}$ where U is a unit. If $f \sim^{\boldsymbol{P}} \hat{f}$ on $\Pi_{\boldsymbol{P}}(a, b; \boldsymbol{R})$ then $f \sim^{\boldsymbol{Q}} \hat{f}$ on $\Pi_{\boldsymbol{Q}}(a+\theta_1, b+\theta_2, \boldsymbol{R})$, if $\theta_1 < \arg(U(\boldsymbol{x})) < \theta_2$ and the polyradius $\boldsymbol{R}$ is taken small enough.

Gevrey type asymptotic in an analytic map

If $f \sim^{\boldsymbol{P}} \hat{f}$ on $\Pi_{\boldsymbol{P}}$ and furthermore:

1. The sequence $\{f_N\}_{N\in\mathbb{N}}$ satisfies $\|f_N(\boldsymbol{x})\| \leq KA^N N!^s$, for all $N \in \mathbb{N}$, $|\boldsymbol{x}| < r$.
2. There are constants $C, A > 0$ such that $C_N(\Pi'_{\boldsymbol{P}}) = CA^N N!^{1/k}$.

Then we say that the asymptotic expansion is of $\boldsymbol{P} - 1/k-$Gevrey type.

Gevrey type asymptotic in an analytic map

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Then we say that the asymptotic expansion is of $\boldsymbol{P} - 1/k-$Gevrey type.
As in the case of one variable we have:

1. $f \sim^{\boldsymbol{P}}_{1/k} \hat{0}$ on $\Pi_{\boldsymbol{P}}$ if and only if for every subsector $\Pi'_{\boldsymbol{P}} \subset \Pi_{\boldsymbol{P}}$ there are constants C, A such that

$$\|f(\boldsymbol{x})\| \leq C \exp(-1/A|\boldsymbol{P}(\boldsymbol{x})|^k), \quad \boldsymbol{x} \in \Pi'_{\boldsymbol{P}}.$$

2. *Watson's lemma*: If $f \sim^{\boldsymbol{P}}_{1/k} \hat{0}$ on $\Pi_{\boldsymbol{P}}(a, b; \boldsymbol{R})$ and $b - a > \pi/k$ then $f \equiv 0$.

$\boldsymbol{P} - k-$summability

Definition

Let $\hat{f} \in E[[\boldsymbol{x}]]$, $k > 0$ and θ be a direction.

1. The series $\hat{f}$ is called $\boldsymbol{P} - k-$*summable in direction* θ if we can find $f \in \mathcal{O}(\Pi_{\boldsymbol{P}}, E)$, $\Pi_{\boldsymbol{P}}(\theta - \frac{\pi}{2k} - \varepsilon, \theta + \frac{\pi}{2k} + \varepsilon, r)$ such that $f \sim^{\boldsymbol{P}}_{1/k} \hat{f}$ on $\Pi_{\boldsymbol{P}}$.
2. The series $\hat{f}$ is called $\boldsymbol{P} - k-$*summable*, if it is $\boldsymbol{P} - k-$summable in all directions up to a finite number of them mod. 2π.

The corresponding spaces are denoted by $E\{\boldsymbol{x}\}^{\boldsymbol{P}}_{1/k,\theta}$ and $E\{\boldsymbol{x}\}^{\boldsymbol{P}}_{1/k}$. If $\boldsymbol{P}(\boldsymbol{x}) = \boldsymbol{x}^{\boldsymbol{\alpha}}$ we simply write $E\{\boldsymbol{x}\}^{\boldsymbol{\alpha}}_{1/k,\theta}$ and $E\{\boldsymbol{x}\}^{\boldsymbol{\alpha}}_{1/k}$, respectively.

Tauberian theorems for summability in analytic functions

Theorem

Let $\boldsymbol{P}_j \in \mathbb{C}\{\boldsymbol{x}\} \setminus \{0\}$, $k_j > 0$ *for* $j = 0, 1, \ldots, n$. *For each* $j = 1, \ldots, n$ *consider a series* $\hat{f}_j \in E\{\boldsymbol{x}\}_{1/k_j}^{\boldsymbol{P}_j} \setminus E\{\boldsymbol{x}\}$. *Then*

$$\hat{f}_0 = \hat{f}_1 + \cdots + \hat{f}_n \in E\{\boldsymbol{x}\}_{1/k_0}^{\boldsymbol{P}_0},$$

if and only if there are $p_j \in \mathbb{N}^+$ *and units* U_j *such that* $\boldsymbol{P}_0^{p_0} = U_j \boldsymbol{P}_j^{p_j}$ *and* $p_0/k_0 = p_j/k_j$ *for all* $j = 1, \ldots, n$.

In particular, $E\{\boldsymbol{x}\}_{1/k_0}^{\boldsymbol{P}_0} = E\{\boldsymbol{x}\}_{1/k_1}^{\boldsymbol{P}_1}$ *if and only if there are* $p_0, p_1 \in \mathbb{N}^+$ *and a unit* U *such that* $p_0/k_0 = p_1/k_1$ *and* $\boldsymbol{P}_1^{p_1} = U \cdot \boldsymbol{P}_0^{p_0}$.

An example from Pfaffian systems

Consider a system of PDEs the form:

$$\begin{cases} x_1^{p+1}\dfrac{\partial \boldsymbol{y}}{\partial x_1} = \boldsymbol{F}_1(x_1, x_2, \boldsymbol{y}), \\ x_2^{q+1}\dfrac{\partial \boldsymbol{y}}{\partial x_2} = \boldsymbol{F}_2(x_1, x_2, \boldsymbol{y}), \end{cases}$$

Theorem (Gérard-Sibuya)

If the system is completely integrable and $\frac{\partial \boldsymbol{F}_1}{\partial \boldsymbol{y}}(0,0,\mathbf{0})$, $\frac{\partial \boldsymbol{F}_2}{\partial \boldsymbol{y}}(0,0,\mathbf{0})$ *and are invertible then it admits a unique analytic solution at the origin* $\boldsymbol{y}$ *such that* $\boldsymbol{y}(0,0) = \mathbf{0}$.

Borel-Laplace analysis for monomial summability

If $f \sim^{\boldsymbol{\alpha}}_{1/k} \hat{f}$ on $\Pi_{\boldsymbol{\alpha}}$ it follows that $\hat{f} \in E[[\boldsymbol{x}]]^{\boldsymbol{\alpha}}_{1/k}$, i.e.

$$\|a_{\boldsymbol{\beta}}\| \leq CA^{|\boldsymbol{\beta}|} \min_{1\leq j\leq d} \beta_j!^{1/k\alpha_j}, \quad \boldsymbol{\beta} \in \mathbb{N}^d.$$

The *formal $k-$Borel transform associated to the monomial $\boldsymbol{x}^{\boldsymbol{\alpha}}$ with weight* $\boldsymbol{s} \in \overline{\sigma_d}$ is defined by

$$\begin{aligned} \hat{\mathcal{B}}_{\boldsymbol{\lambda}} : & E[[\boldsymbol{x}]] \longrightarrow \boldsymbol{\xi}^{-k\boldsymbol{\alpha}} E[[\boldsymbol{\xi}]] \\ \boldsymbol{x}^{\boldsymbol{\beta}} & \longmapsto \frac{\boldsymbol{\xi}^{\boldsymbol{\beta}-k\boldsymbol{\alpha}}}{\Gamma\left(\langle \boldsymbol{\beta}, \boldsymbol{\lambda}\rangle\right)}. \end{aligned}$$

Here and below, $\overline{\sigma_d} := \{\boldsymbol{s} \in (\mathbb{R}_{\geq 0})^d \mid s_1 + \cdots + s_d = 1\}$ and $\boldsymbol{\lambda} = \left(\frac{s_1}{\alpha_1 k}, \ldots, \frac{s_d}{\alpha_n k}\right)$.

Let $\hat{f} \in E[[\boldsymbol{x}]]^{\boldsymbol{\alpha}}_{1/k}$, $\boldsymbol{s} \in \overline{\sigma_d}$ and θ a direction. We will say that $\hat{f}$ is *$k - \boldsymbol{s}-$Borel summable in the monomial $\boldsymbol{x}^{\boldsymbol{\alpha}}$ in direction θ* if:

1. $\hat{\mathcal{B}}_{\boldsymbol{\lambda}}(\hat{f})$ can be analytically continued, say as $\varphi_{\boldsymbol{s}}$, to an unbounded monomial sector containing θ.
2. The extension satisfies

$$\|\varphi(\boldsymbol{\xi})\| \leq C \exp\left(B \max\{|\xi_1|^{\frac{\alpha_1 k}{s_1}}, \ldots, |\xi_n|^{\frac{\alpha_n k}{s_n}}\}\right).$$

In this case the *$k - \boldsymbol{s}-$Borel sum of* $\hat{f}$ in direction θ is defined as

$$\begin{aligned} f(\boldsymbol{x}) :=& \mathcal{L}_{\boldsymbol{\lambda}}(\varphi_{\boldsymbol{s}})(\boldsymbol{x}) \\ =& \boldsymbol{x}^{k\boldsymbol{\alpha}} \int_0^{e^{i\theta}\infty} f\left(x_1 \xi^{\frac{s_1}{\alpha_1 k}}, \ldots, x_n \xi^{\frac{s_n}{\alpha_n k}}\right) e^{-\xi} d\xi. \end{aligned}$$

Equivalence of the methods

Theorem

Let $\hat{f}$ be a $1/k-$Gevrey series in the monomial $\boldsymbol{x}^{\boldsymbol{\alpha}}$. Then it is equivalent:

1. $\hat{f} \in E\{\boldsymbol{x}\}^{\boldsymbol{\alpha}}_{1/k,\theta}$, *i.e.* $\hat{f}$ *is* $\boldsymbol{x}^{\boldsymbol{\alpha}} - k-$*summable in direction* θ.
2. *There is* $\boldsymbol{s} \in \overline{\sigma_d}$ *such that* $\hat{f}$ *is* $k - \boldsymbol{s}-$*Borel summable in the monomial* $\boldsymbol{x}^{\boldsymbol{\alpha}}$ *in direction* θ.
3. *For all* $\boldsymbol{s} \in \overline{\sigma_d}$, $\hat{f}$ *is* $k - \boldsymbol{s}-$*Borel summable in the monomial* $\boldsymbol{x}^{\boldsymbol{\alpha}}$ *in direction* θ.

In all cases the corresponding sums coincide.

For each $\boldsymbol{\alpha} \in \mathbb{N}^d \setminus \{\mathbf{0}\}$, $k > 0$ and $\boldsymbol{s} \in \overline{\sigma_d}$ we have the following monomorphism between the structures

$$\left(E[[\boldsymbol{x}]]^{\boldsymbol{\alpha}}_{1/k}, +, \times, X_{\boldsymbol{\lambda}}\right) \xrightarrow{\widehat{\mathcal{B}}_{\boldsymbol{\lambda}}} \left(\boldsymbol{\xi}^{-k\boldsymbol{\alpha}} E\{\boldsymbol{\xi}\}, +, *_{\boldsymbol{\lambda}}, \boldsymbol{\xi}^{k\boldsymbol{\alpha}}(\cdot)\right),$$

where

$$X_{\boldsymbol{\lambda}} = \frac{\boldsymbol{x}^{k\boldsymbol{\alpha}}}{k}\left(\frac{s_1}{\alpha_1} x_1 \frac{\partial}{\partial x_1} + \cdots + \frac{s_d}{\alpha_d} x_d \frac{\partial}{\partial x_d}\right), \quad \boldsymbol{\lambda} = \left(\frac{s_1}{\alpha_1 k}, \ldots, \frac{s_d}{\alpha_d k}\right),$$

and the convolution is given by

$$(f *_{\boldsymbol{\lambda}} g)(\boldsymbol{x}) = \boldsymbol{x}^{k\boldsymbol{\alpha}} \int_0^1 f(x_1 \tau^{\frac{s_1}{\alpha_1 k}}, \ldots, x_d \tau^{\frac{s_d}{\alpha_d k}}) g(x_1(1-\tau)^{\frac{s_1}{\alpha_1 k}}, \ldots, x_d(1-\tau)^{\frac{s_d}{\alpha_d k}}) d\tau.$$

Applications to singularly perturbed PDEs

Consider the singularly perturbed PDE

$$\boldsymbol{x}^{\boldsymbol{\alpha}}\boldsymbol{\varepsilon}^{\boldsymbol{\alpha}'}\left(\lambda_1 x_1\frac{\partial \boldsymbol{y}}{\partial x_1}+\cdots+\lambda_n x_n\frac{\partial \boldsymbol{y}}{\partial x_n}\right)=\boldsymbol{F}(\boldsymbol{x},\boldsymbol{\varepsilon},\boldsymbol{y}),$$

where $\boldsymbol{x}\in\mathbb{C}^n, \boldsymbol{\varepsilon}\in\mathbb{C}^m$, $\boldsymbol{\alpha}\in(\mathbb{N}^+)^n$, $\boldsymbol{\alpha'}\in(\mathbb{N}^+)^m$, $\boldsymbol{\lambda}=(\lambda_1,\ldots,\lambda_n)\in(\mathbb{R}^+)^n$ and $\boldsymbol{F}$ analytic at the origin.

Theorem

If $A=\frac{\partial \boldsymbol{F}}{\partial y}(0,0,\mathbf{0})$ is an invertible matrix the above problem has a unique formal power series solution $\hat{\boldsymbol{y}}\in\mathbb{C}[[\boldsymbol{x},\boldsymbol{\varepsilon}]]^N$ and it is $1-\boldsymbol{x}^{\boldsymbol{\alpha}}\boldsymbol{\varepsilon}^{\boldsymbol{\alpha}'}-$summable. The singular directions are determined by the equation

$$\det\left(\langle\boldsymbol{\lambda},\boldsymbol{\alpha}\rangle\,\boldsymbol{\xi}^{\boldsymbol{\alpha}}\boldsymbol{\eta}^{\boldsymbol{\alpha}'}I_N-A_0\right)=0,$$

in the $(\boldsymbol{\xi},\boldsymbol{\eta})-$Borel space.

For $\mu > 0$ we work in the space

$$\mathcal{A}_\mu^N(S) := \{\boldsymbol{f} \in \mathcal{O}(S, \mathbb{C}^N) \mid \boldsymbol{f}(\boldsymbol{0}, \boldsymbol{\eta}) = \boldsymbol{0},\ \|\boldsymbol{f}\|_{N,\mu} := \max_{1\leq j\leq N} \|f_j\|_\mu < +\infty\},$$

$$\|f\|_\mu := M_0 \sup_{(\boldsymbol{\xi},\boldsymbol{\eta})\in S} |f(\boldsymbol{\xi}, \boldsymbol{\eta})|(1 + R(\boldsymbol{\xi})^2)e^{-\mu R(\boldsymbol{\xi})}, \qquad f \in \mathcal{O}(S),$$

$$R(\boldsymbol{\xi}) = R_{\boldsymbol{\lambda}'}(\boldsymbol{\xi}) = \max_{1\leq j\leq n} \{|\xi_j|^{\alpha_j/s_j}\},$$

$$S \subset \{(\boldsymbol{\xi}, \boldsymbol{\eta}) \in \mathbb{C}^n \times \mathbb{C}^m \mid \boldsymbol{\xi}^{\boldsymbol{\alpha}} \boldsymbol{\eta}^{\boldsymbol{\alpha}'} \neq \lambda_j \text{ for all } j = 1, \ldots, N\}.$$

The problem of multisummability

How to mix the possible summability methods we have at hand?

Monomial acceleration operators

We formally compute the composition of a Borel and Laplace transform for different indexes. If $\boldsymbol{\alpha},\boldsymbol{\beta}\in(\mathbb{N}^+)^d$, $k,k'>0$, $\boldsymbol{s},\boldsymbol{s}'\in\sigma_d$ and $\boldsymbol{\lambda}=\left(\frac{s_1}{\alpha_1 k},\dots,\frac{s_d}{\alpha_d k}\right)$, $\boldsymbol{\mu}=\left(\frac{s_1'}{\beta_1 k'},\dots,\frac{s_d'}{\beta_d k'}\right)$, then

$$\begin{aligned}\mathcal{B}_{\boldsymbol{\mu}}\circ\mathcal{L}_{\boldsymbol{\lambda}}(\varphi)(\boldsymbol{\xi})&=\boldsymbol{\xi}^{k\boldsymbol{\alpha}-k'\boldsymbol{\beta}}\int_0^{e^{i\theta}\infty}\varphi(\xi_1\tau^{\frac{s_1}{\alpha_1 k}},\dots,\xi_d\tau^{\frac{s_d}{\alpha_d k}})C_\Lambda(\tau)d\tau,\\&:=\mathfrak{A}_{\boldsymbol{\mu},\boldsymbol{\lambda}}(\varphi)(\boldsymbol{\xi}).\end{aligned}$$

1. The parameters must satisfy the relations

$$\Lambda := \frac{s_1/\alpha_1 k}{s_1'/\beta_1 k'} = \cdots = \frac{s_d/\alpha_d k}{s_d'/\beta_d k'} > 1.$$

2. Given $\boldsymbol{s} \in \sigma_d$ then $\boldsymbol{s}'$ is given by

$$s_j' = \frac{s_j \beta_j/\alpha_j}{s_1\beta_1/\alpha_1 + \cdots + s_d\beta_d/\alpha_d}, \qquad j = 1, \ldots, d.$$

   This holds if

$$\max_{1\leq j\leq d} \frac{\alpha_j}{\beta_j} < \frac{k'}{k}.$$

3. $\mathfrak{A}_{\boldsymbol{\mu},\boldsymbol{\lambda},\theta}(\varphi)$ is well-defined for functions φ with

$$\|f(\boldsymbol{\xi})\| \leq C\exp\left(M\max_{1\leq j\leq d}|\xi_j|^{\kappa_j}\right), \quad \frac{1}{\kappa_j} = \frac{s_j}{\alpha_j k} - \frac{s_j'}{\beta_j k'}, \quad j = 1, \ldots, d.$$

Goals

1. Prove that a multisummable series can we decomposed as sum of summable series, each one of a different level. Prove that the set of multisummable series is an algebra stable by partial derivatives.
2. Apply the methods of multisummability to treat formal solutions of systems of type

$$\text{diag}\{\epsilon^{q_1}x^{p_1}I^{(1)}, \epsilon^{q_2}x^{p_2}I^{(2)}\}x\frac{d\boldsymbol{y}}{dx} = A_0\boldsymbol{y} + xG(x;\epsilon;\boldsymbol{y}),$$

where $I^{(j)}$ denotes the identity matrix of dimension $n_j \in \mathbb{N}$, $N = n_1 + n_2$, $\boldsymbol{y} \in \mathbb{C}^N$, $A_0 = \text{diag}\{\lambda_1, \ldots, \lambda_n\}$, and g is analytic at $(0, 0, \mathbf{0}) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^N$.

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Thanks for your attention.