Construction of counterexamples to the 2–jet determination Chern-Moser Theorem in higher codimension

Jan Gregorovič
(joint work with F. Meylan)
arXiv:2010.10220

University of Hradec Kralove and University of Vienna

Problem of jet determination

Real submanifold $M \subset \mathbb{C}^N$, biholomorphisms ϕ, ψ of $\mathbb{C}^N$ such that $\phi(M) \subset M, \psi(M) \subset M$

Problem of jet determination

Real submanifold $M \subset \mathbb{C}^N$, biholomorphisms ϕ, ψ of $\mathbb{C}^N$ such that $\phi(M) \subset M, \psi(M) \subset M$
For what r can we conclude that

$$j_z^r \phi = j_z^r \psi$$

implies $\phi = \psi$?

Problem of jet determination

Real submanifold $M \subset \mathbb{C}^N$, biholomorphisms ϕ, ψ of $\mathbb{C}^N$ such that $\phi(M) \subset M, \psi(M) \subset M$
For what r can we conclude that

$$j_z^r \phi = j_z^r \psi$$

implies $\phi = \psi$?

Equivalent to ask for what r can we conclude that $j_z^r(\phi \circ \psi^{-1}) = id$ implies $\phi \circ \psi^{-1} = id$.

Problem of jet determination

Real submanifold $M \subset \mathbb{C}^N$, biholomorphisms ϕ, ψ of $\mathbb{C}^N$ such that $\phi(M) \subset M, \psi(M) \subset M$
For what r can we conclude that

$$j_z^r \phi = j_z^r \psi$$

implies $\phi = \psi$?

Equivalent to ask for what r can we conclude that $j_z^r(\phi \circ \psi^{-1}) = id$ implies $\phi \circ \psi^{-1} = id$.
Nontrivial ϕ such that

$$j_z^r \phi = id$$

for $r > 0$ have nontrivial dynamics and pose restrictions, how M can look like.

Jet determination of CR automorphisms

(almost) CR structure ... M with distribution $\mathcal{D}$ and (almost) complex structure $\mathcal{I}$ on $\mathcal{D}$... CR automorphisms of M preserving $\mathcal{D}$ and $\mathcal{I}$

Jet determination of CR automorphisms

(almost) CR structure ... M with distribution $\mathcal{D}$ and (almost) complex structure $\mathcal{I}$ on $\mathcal{D}$... CR automorphisms of M preserving $\mathcal{D}$ and $\mathcal{I}$

Regular point of $M \subset \mathbb{C}^N$ ...

$$\mathcal{D} := TM \cap i(TM)$$

in the neighborhood of regular point ... CR structure of $M, \mathcal{D}$ with $\mathcal{I}$ induced by multiplication by i... biholomorphisms preserving M restrict to CR automorphisms on M

Jet determination of CR automorphisms

(almost) CR structure ... M with distribution $\mathcal{D}$ and (almost) complex structure $\mathcal{I}$ on $\mathcal{D}$... CR automorphisms of M preserving $\mathcal{D}$ and $\mathcal{I}$

Regular point of $M \subset \mathbb{C}^N$ ...

$$\mathcal{D} := TM \cap i(TM)$$

in the neighborhood of regular point ... CR structure of $M, \mathcal{D}$ with $\mathcal{I}$ induced by multiplication by i... biholomorphisms preserving M restrict to CR automorphisms on M
In general, not every CR structure can be embedded into $\mathbb{C}^N$. Not every CR automorphisms extends uniquely to biholomorphisms preserving M.

Jet determination of infinitesimal CR automorphisms

Topology comes to play ... CR automorphisms outside of connected component of identity can be determined by a higher jet.
In the connected component of identity, ..., jet determination of (complete) holomorphic vector fields Z whose flow preserve M:
For what r can we conclude that

$$j_z^r Z = 0$$

implies $Z = 0$?

Jet determination of infinitesimal CR automorphisms

Topology comes to play ... CR automorphisms outside of connected component of identity can be determined by a higher jet.
In the connected component of identity, ..., jet determination of (complete) holomorphic vector fields Z whose flow preserve M:
For what r can we conclude that

$$j^r_z Z = 0$$

implies $Z = 0$?

Infinitesimal CR automorphisms ... vector fields on M whose flows are CR automorphisms.
Jet determination of infinitesimal CR automorphisms.

Bounds on jet determination

What properties of M say about jet determination of biholomorphisms preserving M with such property?
I.e. we want to obtain bound on jet determination imposed by the properties of M.

Bounds on jet determination

What properties of M say about jet determination of biholomorphisms preserving M with such property?
I.e. we want to obtain bound on jet determination imposed by the properties of M.

Theorem (Cartan, Tanaka, Chern and Moser)

Let M be a real-analytic hypersurface through a point p in $\mathbb{C}^N$ with non-degenerate Levi form at p. Let F, G be two germs of biholomorphic maps preserving M. Then, if F and G have the same 2-jets at p, they coincide.

Bounds on jet determination

What properties of M say about jet determination of biholomorphisms preserving M with such property?
I.e. we want to obtain bound on jet determination imposed by the properties of M.

Theorem (Cartan, Tanaka, Chern and Moser)

Let M be a real-analytic hypersurface through a point p in $\mathbb{C}^N$ with non-degenerate Levi form at p. Let F, G be two germs of biholomorphic maps preserving M. Then, if F and G have the same 2-jets at p, they coincide.

The result becomes false without any hypothesis on the Levi form.
What about higher codimension?

Beloshapka's theory of quadric models

(Levi) non–degenerate quadric models $M_0 \subset \mathbb{C}^{n+k}$... 2-degree polynomial submanifolds given by

$$\operatorname{Im} w_1 = zH_1z^*, \cdots, \operatorname{Im} w_k = zH_kz^*, \tag{1}$$

where $z \in \mathbb{C}^n$, $w \in \mathbb{C}^k, 1 \le k \le n^2$,

1. the $n \times n$ Hermitian matrices H_j are linearly independent, and
2. the common kernel of all Hermitian matrices H_j is trivial, i.e., $zH_jz^* = 0$ for all j implies $z = 0$.

Beloshapka's theory of quadric models

(Levi) non–degenerate quadric models $M_0 \subset \mathbb{C}^{n+k}$... 2-degree polynomial submanifolds given by

$$\operatorname{Im} w_1 = zH_1z^*, \cdots, \operatorname{Im} w_k = zH_kz^*, \tag{1}$$

where $z \in \mathbb{C}^n$, $w \in \mathbb{C}^k, 1 \leq k \leq n^2$,

1. the $n \times n$ Hermitian matrices H_j are linearly independent, and
2. the common kernel of all Hermitian matrices H_j is trivial, i.e., $zH_jz^* = 0$ for all j implies $z = 0$.

Beloshapka investigated jet determination of real-analytic M that can be approximated by such M_0 and proposed that 2–jet determination in such situation.

Beloshapka's theory of quadric models

(Levi) non–degenerate quadric models $M_0 \subset \mathbb{C}^{n+k}$... 2-degree polynomial submanifolds given by

$$\operatorname{Im} w_1 = zH_1z^*, \cdots, \operatorname{Im} w_k = zH_kz^*, \tag{1}$$

where $z \in \mathbb{C}^n$, $w \in \mathbb{C}^k, 1 \leq k \leq n^2$,

1. the $n \times n$ Hermitian matrices H_j are linearly independent, and
2. the common kernel of all Hermitian matrices H_j is trivial, i.e., $zH_jz^* = 0$ for all j implies $z = 0$.

Beloshapka investigated jet determination of real-analytic M that can be approximated by such M_0 and proposed that 2–jet determination in such situation.

Baouendi, Ebenfelt and Rothschild ... there is a general bound $1 + k$ for jet determination in such situation.

Counterexample of F. Meylan

Let $M \subseteq \mathbb{C}^9$ be the real submanifold of (real) codimension 5 through 0 given in the coordinates $(z, w) = (z_1, \dots, z_4, w_1, \dots, w_5) \in \mathbb{C}^9$, by

$$\begin{cases} \operatorname{Im} w_1 = P_1(z, \bar{z}) = z_1\overline{z_2} + z_2\overline{z_1} \\ \operatorname{Im} w_2 = P_2(z, \bar{z}) = -iz_1\overline{z_2} + iz_2\overline{z_1} \\ \operatorname{Im} w_3 = P_3(z, \bar{z}) = z_3\overline{z_2} + z_4\overline{z_1} + z_2\overline{z_3} + z_1\overline{z_4} \\ \operatorname{Im} w_4 = P_4(z, \bar{z}) = z_1\overline{z_1} \\ \operatorname{Im} w_5 = P_5(z, \bar{z}) = z_2\overline{z_2} \end{cases} \tag{2}$$

Then there is the following holomorphic vector field whose flow preserves M

$$T = (-\frac{1}{2}{w_1}^2 + \frac{1}{2}{w_2}^2 + 2w_4w_5)i(-iz_1\frac{\partial}{\partial z_3} + iz_2\frac{\partial}{\partial z_4}) \\ + w_1w_2i(z_1\frac{\partial}{\partial z_3} + z_2\frac{\partial}{\partial z_4}) - 2w_2w_5i(z_1\frac{\partial}{\partial z_4}) - 2w_2w_4i(z_2\frac{\partial}{\partial z_3})$$

Aims of the talk

Analyze the counterexample of F. Meylan from viewpoint of Tanaka's prolongation theory.

Aims of the talk

Analyze the counterexample of F. Meylan from viewpoint of Tanaka's prolongation theory.

Formulate a general construction of counterexamples to 2–jet determination Chern-Moser Theorem in higher codimension.

Aims of the talk

Analyze the counterexample of F. Meylan from viewpoint of Tanaka's prolongation theory.

Formulate a general construction of counterexamples to 2–jet determination Chern-Moser Theorem in higher codimension.

Construct counterexamples with general order of jet determination.

Infinitesimal CR automorphisms of quadric models

Nondegenerate quadric models $\mathrm{Im}\, w_j = zH_jz^*$... weighted homogeneous for $[z_j] = 1, [w_j] = 2$ with corresponding Euler field

$$E := \sum_j z_j \frac{\partial}{\partial z_j} + 2 \sum_j w_j \frac{\partial}{\partial w_j}$$

Infinitesimal CR automorphisms of quadric models

Nondegenerate quadric models $\mathrm{Im}\, w_j = zH_jz^*$... weighted homogeneous for $[z_j] = 1, [w_j] = 2$ with corresponding Euler field

$$E := \sum_j z_j \frac{\partial}{\partial z_j} + 2\sum_j w_j \frac{\partial}{\partial w_j}$$

$=>$ Lie algebra of infinitesimal CR automorphisms decomposes as $\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \cdots \oplus \mathfrak{g}_{b-1} \oplus \mathfrak{g}_b$ such that $[\mathfrak{g}_c, \mathfrak{g}_d] \subset \mathfrak{g}_{c+d}$

$$\mathfrak{g}_{-2} = \{\sum_j q_j \frac{\partial}{\partial w_j}\}$$

$$\mathfrak{g}_{-1} = \{\sum_j p_j \frac{\partial}{\partial z_j} + 2i\sum_j zH_jp^* \frac{\partial}{\partial w_j}\},$$

where $p \in \mathbb{C}^n,\ q \in \mathbb{R}^k$ and the real span of the real parts of these vector fields on M_0 is TM_0.

Levi–Tanaka algebra

Definition

A non–degenerate Levi Tanaka algebra (of a nondegenerate quadric model) is a graded Lie algebra $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ together with complex structure$J$ on $\mathfrak{g}_{-1}$ satisfying

1. $[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = \mathfrak{g}_{-2}$,
2. $[X, \mathfrak{g}_{-1}] = 0$, $X \in \mathfrak{g}_{-1}$, implies $X = 0$
3. $[J(X), J(Y)] = [X, Y]$ for all $X, Y \in \mathfrak{g}_{-1}$.

Levi–Tanaka algebra

Definition

A non–degenerate Levi Tanaka algebra (of a nondegenerate quadric model) is a graded Lie algebra $\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ together with complex structure$J$ on $\mathfrak{g}_{-1}$ satisfying

1. $[\mathfrak{g}_{-1}, \mathfrak{g}_{-1}] = \mathfrak{g}_{-2}$,
2. $[X, \mathfrak{g}_{-1}] = 0$, $X \in \mathfrak{g}_{-1}$, implies $X = 0$
3. $[J(X), J(Y)] = [X, Y]$ for all $X, Y \in \mathfrak{g}_{-1}$.

$\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ in the Lie algebra of infinitesimal CR automorphism with the Lie bracket taken with the opposite sign defines a non–degenerate Levi Tanaka algebra (at $z = 0, w = 0$) of the nondegenerate quadric model with J induced by multiplication by i.

$$[(q, p), (\tilde{q}, \tilde{p})] = (2i(-pH_j\tilde{p}^* + \tilde{p}H_jp^*), 0)$$

in the above coordinates (q, p) of $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$.

Tanaka prolongation

Tanaka prolongation of Levi Tanaka algebra $(\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}, J)$ is the maximal graded Lie algebra $\mathfrak{g} = \mathfrak{m} \oplus \oplus_{i \geq 0} \mathfrak{g}_i$ such that

1. $\mathfrak{g}_0$ consists of grading preserving derivations of $\mathfrak{m}$ commuting with J,
2. for all $X \in \oplus_{i \geq 0} \mathfrak{g}_i$ the condition $[X, \mathfrak{g}_{-1}] = 0$ implies $X = 0$.

Tanaka prolongation

Tanaka prolongation of Levi Tanaka algebra $(\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}, J)$ is the maximal graded Lie algebra $\mathfrak{g} = \mathfrak{m} \oplus \oplus_{i \geq 0} \mathfrak{g}_i$ such that

1. $\mathfrak{g}_0$ consists of grading preserving derivations of $\mathfrak{m}$ commuting with J,
2. for all $X \in \oplus_{i \geq 0} \mathfrak{g}_i$ the condition $[X, \mathfrak{g}_{-1}] = 0$ implies $X = 0$.

The ith prolongation $\mathfrak{g}_i$ can be algebraically computed as

$$\mathfrak{g}_i := \{f \in \oplus_{j<0} \mathfrak{g}_j^* \otimes \mathfrak{g}_{j+i} : \ f([X, Y]) = [f(X), Y] + [X, f(Y)], \ X, Y \in \mathfrak{m}\}.$$

Tanaka prolongation

Tanaka prolongation of Levi Tanaka algebra $(\mathfrak{m} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}, J)$ is the maximal graded Lie algebra $\mathfrak{g} = \mathfrak{m} \oplus \oplus_{i \geq 0} \mathfrak{g}_i$ such that

1. $\mathfrak{g}_0$ consists of grading preserving derivations of $\mathfrak{m}$ commuting with J,
2. for all $X \in \oplus_{i \geq 0} \mathfrak{g}_i$ the condition $[X, \mathfrak{g}_{-1}] = 0$ implies $X = 0$.

The ith prolongation $\mathfrak{g}_i$ can be algebraically computed as

$$\mathfrak{g}_i := \{f \in \oplus_{j<0} \mathfrak{g}_j^* \otimes \mathfrak{g}_{j+i} : \; f([X, Y]) = [f(X), Y] + [X, f(Y)], \; X, Y \in \mathfrak{m}\}.$$

Theorem (Tanaka)

Suppose for all $X \in \mathfrak{g}_{-1}$ the condition $[X, \mathfrak{g}_{-1}] = 0$ implies $X = 0$. Then $\mathfrak{g}_l = 0$ for all l large enough and $\mathfrak{g}$ is finite dimensional Lie algebra.

Realization of Tanaka prolongation as infinitesimal CR automorphisms of M_0

Reconstruction of M_0 with the coordinates (w, z) of subalgebra $\mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$ of complexification $\mathfrak{g}_{\mathbb{C}}$ of Tanaka prolongation $\mathfrak{g}$:

$$\mathrm{Im}\, w := \frac{1}{4}[J(z), z].$$

Realization of Tanaka prolongation as infinitesimal CR automorphisms of M_0

Reconstruction of M_0 with the coordinates (w, z) of subalgebra $\mathfrak{n}_{-2} \oplus \mathfrak{n}_{-1}$ of complexification $\mathfrak{g}_{\mathbb{C}}$ of Tanaka prolongation $\mathfrak{g}$:

$$\mathrm{Im}\, w := \frac{1}{4}[J(z), z].$$

Construction of holomorphic vector fields corresponding to elements $X_b \in \mathfrak{g}_b$ whose flow preserves M_0:

$$\begin{aligned} &\sum_{c+2d=b+1,\ c,d\geq 0} \frac{(-1)^{c+d}}{(c+d)!}(\mathrm{ad}(z)^c(\mathrm{ad}(w)^d(X_b)))_{\mathfrak{n}_{-1},j}\frac{\partial}{\partial z_j} \\ +&\sum_{c+2d=b+2,\ c,d\geq 0} \frac{(-1)^{c+d}}{(c+d)!}(\mathrm{ad}(z)^c(\mathrm{ad}(w)^d(X_b)))_{\mathfrak{n}_{-2},j}\frac{\partial}{\partial w_j} \end{aligned} \tag{3}$$

where ad is Lie bracket on $\mathfrak{g}_{\mathbb{C}}$ and $_{\mathfrak{n}_{-i},j}$ is projection from $\mathfrak{g}_{\mathbb{C}}$ to jth–component of $\mathfrak{n}_{-i}$.

Levi decomposition of Tanaka prolongation

R ... the radical of the Tanaka prolongation $\mathfrak{g}$ of $(\mathfrak{m}, J)$.
Levi decomposition Theorem... the semisimple Lie algebra $\mathfrak{g}/R$ is isomorphic to $\mathfrak{s} \subset \mathfrak{g}$, i.e., $\mathfrak{g} = \mathfrak{s} \oplus_\rho R$, where $\rho : \mathfrak{s} \to \mathfrak{gl}(R)$ is the representation induced by the Lie bracket $[\mathfrak{s}, R] \subset R$.

Levi decomposition of Tanaka prolongation

R ... the radical of the Tanaka prolongation $\mathfrak{g}$ of $(\mathfrak{m}, J)$.
Levi decomposition Theorem... the semisimple Lie algebra $\mathfrak{g}/R$ is isomorphic to $\mathfrak{s} \subset \mathfrak{g}$, i.e., $\mathfrak{g} = \mathfrak{s} \oplus_\rho R$, where $\rho : \mathfrak{s} \to \mathfrak{gl}(R)$ is the representation induced by the Lie bracket $[\mathfrak{s}, R] \subset R$.
Medori and Nacinovich ... Levi decomposition of form

$$\mathfrak{s} = \mathfrak{s}_{-2} \oplus \mathfrak{s}_{-1} \oplus \mathfrak{s}_0 \oplus \mathfrak{s}_1 \oplus \mathfrak{s}_2$$
$$R = R_{-2} \oplus \cdots \oplus R_b,$$

with $J(\mathfrak{s}_{-1}) \subset \mathfrak{s}_{-1}$ and $J(R_{-1}) \subset R_{-1}$.

Levi decomposition of Tanaka prolongation

R ... the radical of the Tanaka prolongation $\mathfrak{g}$ of $(\mathfrak{m}, J)$.
Levi decomposition Theorem... the semisimple Lie algebra $\mathfrak{g}/R$ is isomorphic to $\mathfrak{s} \subset \mathfrak{g}$, i.e., $\mathfrak{g} = \mathfrak{s} \oplus_\rho R$, where $\rho : \mathfrak{s} \to \mathfrak{gl}(R)$ is the representation induced by the Lie bracket $[\mathfrak{s}, R] \subset R$.
Medori and Nacinovich ... Levi decomposition of form

$$\mathfrak{s} = \mathfrak{s}_{-2} \oplus \mathfrak{s}_{-1} \oplus \mathfrak{s}_0 \oplus \mathfrak{s}_1 \oplus \mathfrak{s}_2$$
$$R = R_{-2} \oplus \cdots \oplus R_b,$$

with $J(\mathfrak{s}_{-1}) \subset \mathfrak{s}_{-1}$ and $J(R_{-1}) \subset R_{-1}$.

=> decomposition of coordinates w of $\mathfrak{s}_{-2}$, w' of R_{-2}, z of $\mathfrak{s}_{-1}$ and z' of R_{-1} and

$$\operatorname{Im} w_j = zH_jz^*,$$
$$\operatorname{Im} w'_j = \mathrm{Re}(zP_j(z')^*) + z'Q_j(z')^*,$$

$zP(z')^* := -2\rho(z)(z')$ and H_j, Q_j depend only on brackets in $\mathfrak{s}, R$.

Analysis of Tanaka prolongation of counterexample of F. Meylan - I

The computation and decomposition of Tanaka prolongation $\mathfrak{g}$ can be done in Maple (package DifferentialGeometry).

Analysis of Tanaka prolongation of counterexample of F. Meylan - I

The computation and decomposition of Tanaka prolongation $\mathfrak{g}$ can be done in Maple (package DifferentialGeometry).

The Levi decomposition

$$\mathfrak{su}(2,3) \oplus_\rho (\mathbb{R} \oplus_\cdot V^{\lambda_2+\lambda_3}),$$

where

- $\mathfrak{su}(2,3)$ is 24 dimensional simple Lie algebra that commutes with $\mathbb{R}$ and acts on 75 dimensional vector space $V^{\lambda_2+\lambda_3}$ by a real representation with highest weight $\lambda_2+\lambda_3$,
- $\mathbb{R}$ in radical acts by multiplication $\cdot$ on $V^{\lambda_2+\lambda_3}$, and
- $V^{\lambda_2+\lambda_3}$ in radical is abelian.

Analysis of Tanaka prolongation of counterexample of F. Meylan - II

$\mathfrak{su}(2,3)$ is |2|–graded and corresponds to universal quadric model

$$\begin{cases} \operatorname{Im} w_1 &= z_1\overline{z_2} + z_2\overline{z_1} \\ \operatorname{Im} w_2 &= -iz_1\overline{z_2} + iz_2\overline{z_1} \\ \operatorname{Im} w_3 &= z_1\overline{z_1} \\ \operatorname{Im} w_4 &= z_2\overline{z_2} \end{cases} \tag{4}$$

$\mathbb{R} \subset \mathfrak{g}_0$

$dim_{\mathbb{R}}(V_{-2}) = 1$, $dim_{\mathbb{C}}(V_{-1}) = 2$ corresponds to additional equation

$$\operatorname{Im} w_1' = z_1'\overline{z_2} + z_2'\overline{z_1} + z_2\overline{z_1}' + z_1\overline{z_2}' \tag{5}$$

Analysis of Tanaka prolongation of counterexample of F. Meylan - III

$dim_{\mathbb{R}}(V_0)=8$, $dim_{\mathbb{R}}(V_1)=16$, $dim_{\mathbb{R}}(V_2)=17$,
$dim_{\mathbb{R}}(V_3)=16$,$dim_{\mathbb{R}}(V_4)=8$, $dim_{\mathbb{R}}(V_5)=4$, $dim_{\mathbb{R}}(V_6)=1$
In particular, the holomorphic vector field $T\in V_4$ and in V_6 is

$$
\begin{aligned}
&(-3w_1^4-6w_1^2w_2^2+24w_1^2w_3w_4-3w_2^4+24w_2^2w_3w_4-48w_3^2w_4^2)\frac{\partial}{\partial w_1'}\\
&+(2w_1^3+2w_1w_2^2-8w_1w_3w_4)(2w_1\frac{\partial}{\partial w_1'}+z_1\frac{\partial}{\partial z_1'}+z_2\frac{\partial}{\partial z_2'})\\
&+(2w_1^2w_2+2w_2^3-8w_2w_3w_4)(2w_2\frac{\partial}{\partial w_1'}-iz_1\frac{\partial}{\partial z_1'}+iz_2\frac{\partial}{\partial z_2'})\\
&+(-4w_1^2w_3-4w_2^2w_3+16w_3^2w_4)(2w_4\frac{\partial}{\partial w_1'}+z_2\frac{\partial}{\partial z_1'})\\
&+(-4w_1^2w_4-4w_2^2w_4+16w_3w_4^2)(2w_3\frac{\partial}{\partial w_1'}+z_1\frac{\partial}{\partial z_2'}),
\end{aligned}
$$

Conclusions from the counterexample

We can try to prescribe the Levi decomposition $\mathfrak{g} = \mathfrak{s} \oplus_\rho (\mathbb{K} \oplus V^\lambda)$:
|2|–grading of simple Lie algebras corresponding to simple Lie algebras are classified by Medori and Nacinovich.
Representations of simple Lie algebras are classified via the heighest weights λ.
$\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ for real, complex or quaternionic representations.

Conclusions from the counterexample

We can try to prescribe the Levi decomposition $\mathfrak{g} = \mathfrak{s} \oplus_\rho (\mathbb{K} \oplus V^\lambda)$:
|2|–grading of simple Lie algebras corresponding to simple Lie algebras are classified by Medori and Nacinovich.
Representations of simple Lie algebras are classified via the heighest weights λ.
$\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ for real, complex or quaternionic representations.

Proposition

Suppose $E_s \in \mathfrak{s}$ is the element providing the grading on $\mathfrak{s}$ with largest/smallest eigenvalues K_{max} and K_{min} on V^λ. Suppose $V^\lambda = V^\lambda_{-2} \oplus \cdots \oplus V^\lambda_c$. Then:

1. *V^λ_i is the $i + K_{min} + 2$ eigenspace of $\rho(E_s)$ in V^λ and $c = K_{max} - K_{min} - 2$.*
2. *Infinitesimal CR automorphisms in V^λ_c are at least K–jet determined, where K is $\frac{K_{max}-K_{min}}{2}$ rounded down.*

General construction of counterexamples

Additional assumptions are required to:
Find J on V_{-1} that would make $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ into Levi Tanaka algebra.
Check $\mathfrak{g} = \mathfrak{s} \oplus_\rho (\mathbb{K} \oplus V^\lambda)$ is in Tanaka prolongation of $(\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}, J)$.

General construction of counterexamples

Additional assumptions are required to:
Find J on V_{-1} that would make $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$ into Levi Tanaka algebra.
Check $\mathfrak{g} = \mathfrak{s} \oplus_\rho (\mathbb{K} \oplus V^\lambda)$ is in Tanaka prolongation of $(\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}, J)$.

Theorem

If

1. V_{-1} *is a complex representation of* $\mathfrak{s}_0$ *and the corresponding complex structure* J *on* $\mathfrak{g}_{-1}$ *is satisfying* $\rho(J(X)(J(Y)) = \rho(X)(Y)$ *for all* $X \in \mathfrak{s}_{-1}, Y \in V_{-1}$,
2. V_0 *acts complex linearly as a map from* $\mathfrak{s}_{-1}$ *to* V_{-1},

then $(\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}, J)$ *is a non–degenerate Levi Tanaka algebra and the nondegenerate quadric model* M *defined as above has Lie algebra* $\mathfrak{s} \oplus_\rho (\mathbb{K} \oplus V^\lambda)$ *of infinitesimal CR automorphisms. In particular, infinitesimal CR automorphisms in* $V_{K_{max}-K_{min}-2}$ *are at least* K*–jet determined, where* K *is* $\frac{K_{max}-K_{min}}{2}$ *rounded down.*

Counterexample in codimension 4 - I

Start with |2|–graded Lie algebra $\mathfrak{s} = \mathfrak{so}(3,5)$ of infinitesimal CR automorphisms of:

$$\begin{aligned}\operatorname{Im} w_1 &= -iz_1\bar{z}_2 + iz_2\bar{z}_1\\ \operatorname{Im} w_2 &= -iz_2\bar{z}_3 + iz_3\bar{z}_2\\ \operatorname{Im} w_3 &= -iz_1\bar{z}_3 + iz_3\bar{z}_1\end{aligned}$$

Pick $\lambda = \lambda_3 + \lambda_4$ which is real representation with $K_{max} = 3, K_{min} = -3$, $V = V_{-2} \oplus \cdots \oplus V_4$, $dim_{\mathbb{R}}(V_{-2}) = 1$.

Counterexample in codimension 4 - I

Start with |2|–graded Lie algebra $\mathfrak{s} = \mathfrak{so}(3,5)$ of infinitesimal CR automorphisms of:

$$\begin{aligned} \operatorname{Im} w_1 &= -iz_1\bar{z}_2 + iz_2\bar{z}_1 \\ \operatorname{Im} w_2 &= -iz_2\bar{z}_3 + iz_3\bar{z}_2 \\ \operatorname{Im} w_3 &= -iz_1\bar{z}_3 + iz_3\bar{z}_1 \end{aligned}$$

Pick $\lambda = \lambda_3 + \lambda_4$ which is real representation with $K_{max} = 3, K_{min} = -3$, $V = V_{-2} \oplus \cdots \oplus V_4$, $dim_{\mathbb{R}}(V_{-2}) = 1$.
V_{-1} is standard complex representation of $\mathfrak{g}_0 = \mathfrak{sl}(3,\mathbb{R}) \oplus \mathbb{C} =>$ condition (1) is satisfied and we get

$$\operatorname{Im} w_1' = z_1\bar{z}_1' + z_1'\bar{z}_1 + z_2\bar{z}_2' + z_2'\bar{z}_2 + z_3\bar{z}_3' + z_3'\bar{z}_3,$$

Analysis of wight spaces of V_0 implies that condition (2) is satisfied $=>$ we can apply our theorem

Counterexample in codimension 4 - II

The following submanifold in $\mathbb{C}^{10}$:

$$
\begin{aligned}
\operatorname{Im} w_1 &= -iz_1\bar{z}_2 + iz_2\bar{z}_1\\
\operatorname{Im} w_2 &= -iz_2\bar{z}_3 + iz_3\bar{z}_2\\
\operatorname{Im} w_3 &= -iz_1\bar{z}_3 + iz_3\bar{z}_1\\
\operatorname{Im} w_1' &= z_1\bar{z}_1' + z_1'\bar{z}_1 + z_2\bar{z}_2' + z_2'\bar{z}_2 + z_3\bar{z}_3' + z_3'\bar{z}_3
\end{aligned}
$$

has infinitesimal CR automorphism in V_4 that has weighted order 4 and is 3–jet determined:

$$
\begin{aligned}
&- w_1w_3(iz_3\frac{\partial}{\partial z_1'} + iz_1\frac{\partial}{\partial z_3'}) + w_2w_3(iz_2\frac{\partial}{\partial z_1'} + iz_1\frac{\partial}{\partial z_2'})\\
&- w_1w_2(iz_3\frac{\partial}{\partial z_2'} + iz_2\frac{\partial}{\partial z_3'}) + iw_3^2(z_1\frac{\partial}{\partial z_1'}) + iw_2^2(z_2\frac{\partial}{\partial z_2'}) + +iw_1^2(z_3\frac{\partial}{\partial z_3'}),
\end{aligned}
$$

where in the braces are rigid holomorphic vector fields in V_0.

Conclusion for 2–jet determination

Codimension $1, 2$ ... 2–jet determinations holds
Codimension 3 ... 2–jet determinations is open, our construction does not lead to any counterexample
Codimension > 3 ... adding equitation for quadrics in new variables to counterexamples in codimension 4,5 we get:

Conclusion for 2–jet determination

Codimension $1,2$ ... 2–jet determinations holds
Codimension 3 ... 2–jet determinations is open, our construction does not lead to any counterexample
Codimension > 3 ... adding equitation for quadrics in new variables to counterexamples in codimension 4,5 we get:

Theorem

For any codimension $k > 5$, there is a generic quadratic submanifold M in $\mathbb{C}^{2k-1}$ of codimension k such that 4–jets are required (and not less) to determine uniquely germs of biholomorphisms sending M to M.

Conclusion for 2–jet determination

Codimension $1, 2$ ... 2–jet determinations holds
Codimension 3 ... 2–jet determinations is open, our construction does not lead to any counterexample
Codimension > 3 ... adding equitation for quadrics in new variables to counterexamples in codimension 4,5 we get:

Theorem

For any codimension $k > 5$, there is a generic quadratic submanifold M in $\mathbb{C}^{2k-1}$ of codimension k such that 4–jets are required (and not less) to determine uniquely germs of biholomorphisms sending M to M.

CR dimension can be always made greater by adding quadrics with more new variables
There is counterexample to 2–jet determination in codimension 5 that does not have such a Levi decomposition.

Arbitrary high order of jet determination

Both of our counterexamples generalize to higher rank:

1. Codimension $\frac{(n-1)n}{2}$ submanifold in $\mathbb{C}^{\frac{n(n+1)}{2}}$ that has $|2|$–graded Lie algebra $\mathfrak{s} = \mathfrak{so}(n, n+2)$ with $\lambda = \lambda_n + \lambda_{n+1}$. This has $K_{max} = n, K_{min} = -n$, $dim_{\mathbb{R}}(V_{2n-2}) = 1$ and we get codimension $\frac{(n-1)n}{2} + 1$ submanifold in $\mathbb{C}^{\frac{(n+2)(n+1)}{2}}$ such that elements of V_{2n-2} are n–jet determined.

Arbitrary high order of jet determination

Both of our counterexamples generalize to higher rank:

1. Codimension $\frac{(n-1)n}{2}$ submanifold in $\mathbb{C}^{\frac{n(n+1)}{2}}$ that has $|2|$–graded Lie algebra $\mathfrak{s} = \mathfrak{so}(n, n+2)$ with $\lambda = \lambda_n + \lambda_{n+1}$. This has $K_{max} = n$, $K_{min} = -n$, $dim_{\mathbb{R}}(V_{2n-2}) = 1$ and we get codimension $\frac{(n-1)n}{2} + 1$ submanifold in $\mathbb{C}^{\frac{(n+2)(n+1)}{2}}$ such that elements of V_{2n-2} are n–jet determined.
2. For even $n = 2m$: Codimension m^2 submanifold in $\mathbb{C}^{m+m^2}$ that has $|2|$–graded Lie algebra $\mathfrak{s} = \mathfrak{su}(m, m+1)$ with $\lambda = \lambda_m + \lambda_{m+1}$. This has $K_{max} = n$, $K_{min} = -n$, $dim_{\mathbb{R}}(V_{2n-2}) = 1$ and we get codimension $m^2 + 1$ submanifold in $\mathbb{C}^{(m+1)^2}$ such that elements of V_{2n-2} are n–jet determined.

Conclusion for n–jet determination

Theorem

For any even $n = 2m$ and any $k > m^2$, there is a generic quadratic submanifold M in $\mathbb{C}^{2k-m^2+2m-1}$ of codimension k such that n–jets are required (and not less) to determine uniquely germs of biholomorphisms sending M to M.
For any odd n and any $k > \frac{(n-1)n}{2}$, there is a generic quadratic submanifold M in $\mathbb{C}^{2k-\frac{1}{2}n^2+\frac{5}{2}n-1}$ of codimension k such that n–jets are required (and not less) to determine uniquely germs of biholomorphisms sending M to M.

Conclusion for n–jet determination

Theorem

For any even $n = 2m$ and any $k > m^2$, there is a generic quadratic submanifold M in $\mathbb{C}^{2k-m^2+2m-1}$ of codimension k such that n–jets are required (and not less) to determine uniquely germs of biholomorphisms sending M to M.
For any odd n and any $k > \frac{(n-1)n}{2}$, there is a generic quadratic submanifold M in $\mathbb{C}^{2k-\frac{1}{2}n^2+\frac{5}{2}n-1}$ of codimension k such that n–jets are required (and not less) to determine uniquely germs of biholomorphisms sending M to M.

Observe that codimension grows quadraticaly w.r.t. to n.
It is not clear, how close to the bound $1 + k$–determination you can get.
We do not know how sharp these counterexamples are.