# CR geometry for beginners 

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## 1 A short crash course in SCV

We denote by $\mathbb{C}^{N} N$-dimensional complex space, that is, the vector space $\left\{\left(Z_{1}, \ldots, Z_{N}\right): Z_{j} \in \mathbb{C}\right\}$. This is a complex vector space with the usual addition and (complex) scalar multiplication. We denote the coordinates of a vector $Z$ by $Z_{j}$, and identify the coordinate functions with the vectors in the usual way. The norm of a vector is given by $\|Z\|=\sum_{j}\left|Z_{j}\right|^{2}$; with this norm, $\mathbb{C}^{N}$ becomes an inner product space, the inner product given by $\langle Z, W\rangle=\sum_{j} Z_{j} \bar{W}_{j}$.

The underlying real vector space is $\mathbb{R}^{2 N}$; real coordinates in this space are given by $Z_{j}=x_{j}+i y_{j}$, i.e.

$$
x_{j}=\operatorname{Re} Z_{j}=\frac{Z_{j}+\overline{Z_{j}}}{2}, \quad y_{j}=\operatorname{Im} Z_{j}=\frac{Z_{j}-\overline{Z_{j}}}{2 i}
$$

### 1.1 Some linear algebra: complexification of vector spaces

Given a real vector space $F$, its complexification is denoted by $\mathbb{C} F=\mathbb{C} \otimes F$. In terms of a universal property, every real linear map from $F$ into a complex vector space has a unique complex linear extension to $\mathbb{C} F$; in terms of a basis $f_{j}$ of $F, \mathbb{C} F$ is given by all complex linear combinations

$$
\sum_{j} \alpha_{j} f_{j}, \quad \alpha_{j} \in \mathbb{C}
$$

$F$ is isomorphic to the real subspace given by $\left\{\operatorname{Im} \alpha_{j}=0\right\}=1 \otimes F$. Note that $F \oplus i F=\mathbb{C} F$; in general, a real subspace of a complex vector space with this property is said to be maximally totally real.

If we consider the complexification of the underlying real vector space $V_{\mathbb{R}}$ of a complex vector space $V$ of complex dimension $N$, it is natural to ask whether we can find $V$ as a complex subspace of $W=\mathbb{C} V_{\mathbb{R}}$. Now if we denote the real linear map on $V_{\mathbb{R}}$ given by multiplication by $i$ in $V$ by $J, J$ extends to a complex linear map on $W$; since $J^{2}=-$ id, its eigenvalues are $i$ and $-i$. We denote the eigenspace of the eigenvalue $i$ by $W^{(1,0)}$, and the eigenspace of $-i$ by $W^{(0,1)}$. These are two complex subspaces of $W$, each of complex dimension $N$, and $W=W^{(1,0)} \oplus W^{(0,1)}$. Note that complex conjugation (the real linear map given by $v+i w \mapsto v-i w$ for $\left.\in V_{\mathbb{R}} \oplus i V_{\mathbb{R}}\right)$ interchanges these two spaces. Explicit isomorphisms $V \rightarrow W^{(1,0)}$ and $V \rightarrow W^{(0,1)}$ are respectively given by

$$
v \mapsto \frac{(v-i J v)}{2} \text { and } v \mapsto \frac{(v+i J v)}{2}
$$

which are also formulas for projections of $W$ onto these spaces. We also have that

$$
\operatorname{Re} w=\frac{w+\bar{w}}{2} \text { and } \operatorname{Im} w=\frac{w-\bar{w}}{2 i}
$$

are both in $V_{\mathbb{R}}$, and of course $w=\operatorname{Re} w+i \operatorname{Im} w$.
If we consider the complexification of the dual space $W^{*}$, its $(1,0)$ and $(0,1)$ parts can be identified with the space of complex linear and complex antilinear forms with values in $\mathbb{C}$. That is, if we have a mapping $\lambda \in \mathbb{C} W^{*}$ (i.e. a real linear map $W \rightarrow \mathbb{C}$ ), its decomposition is $\lambda=\lambda^{(1,0)}+\lambda^{(0,1)}$ where

$$
\lambda^{(1,0)}(w)=\frac{\lambda(w)-i \lambda(J w)}{2}, \quad \lambda^{(0,1)}(v)=\frac{\lambda(v)-i \lambda(J v)}{2}
$$

and for $\alpha \in \mathbb{C}, \lambda^{(1,0)}(\alpha w)=\alpha \lambda^{(1,0)}(w)$ and $\lambda^{(0,1)}(\alpha w)=\bar{\alpha} \lambda^{(0,1)}(w)$.
The decomposition of $\mathbb{C} W=W^{(1,0)} \oplus W^{(0,1)}$ gives rise to decompositions of associated spaces, like exterior products; e.g.,

$$
\Lambda^{n} \mathbb{C} W=\bigoplus_{p+q=n} \Lambda^{p} W^{(1,0)} \wedge \Lambda^{q} W^{(0,1)}=\bigoplus_{p+q=n} W^{(p, q)}
$$

### 1.2 Tangent spaces: the flat case

The tangent space $T_{p} \mathbb{C}^{N}$ is the usual real tangent space. As a vector space, it is spanned by the partial derivatives

$$
\left.\frac{\partial}{\partial x_{j}}\right|_{p},\left.\quad \frac{\partial}{\partial y_{j}}\right|_{p}, \quad j=1, \ldots, N
$$

For $p \in \mathbb{C}^{N}$, the real tangent space $T_{p} \mathbb{C}^{N}$ is a real vector space of dimension $2 N$. We declare a complex structure operator $J$ on it by

$$
\left.J \frac{\partial}{\partial x_{j}}\right|_{p}=\left.\frac{\partial}{\partial y_{j}}\right|_{p},\left.\quad J \frac{\partial}{\partial y_{j}}\right|_{p}=-\left.\frac{\partial}{\partial x_{j}}\right|_{p}, \quad j=1, \ldots, N .
$$

With this, $T_{p} \mathbb{C}^{N}$ becomes a complex vector space of dimension $N$, which we will denote by $T_{p}^{c} \mathbb{C}^{N} . T \mathbb{C}^{N}$ and $T^{c} \mathbb{C}^{N}$ denote the flat vector bundles over $\mathbb{C}^{N}$ (whose fiber over $p \in \mathbb{C}^{N}$ is $T_{p} \mathbb{C}^{N}$ and $T_{p}^{c} \mathbb{C}^{N}$, respectively).

The real cotangent space $T_{p}^{*} \mathbb{C}^{N}$ is just the dual space to $T_{p} \mathbb{C}^{N}$. The complex structure described above gives rise to a complex structure on this space.

Let now $\Omega \subset \mathbb{C}^{N}$ be open. A smooth vector field $X$ on $\Omega$ is a section of $T \mathbb{C}^{N}$ over $\Omega$, for which we write $X \in \Gamma\left(\Omega, T \mathbb{C}^{N}\right)$; in terms of the basis vectors introduced before,

$$
X=\sum_{j=1}^{N} a_{j} \frac{\partial}{\partial x_{j}}+\sum_{j=1}^{N} b_{j} \frac{\partial}{\partial y_{j}}
$$

where $a_{j}$ and $b_{j}$ are smooth (real-valued) functions on $\Omega$; other regularity classes are similarly defined.
$X$ acts on smooth functions on $\Omega$ by differentiation. The (exterior) differential $d \varphi$ of a smooth (realvalued) function $\varphi: \Omega \rightarrow \mathbb{R}$ is a section of the cotangent bundle $T^{*} \mathbb{C}^{N}$ over $\Omega$, defined by $d \varphi(X)=X \varphi$. In general, section of this bundle are called 1-forms; it is easy to check that the dual basis to our coordinate basis $\frac{\partial}{\partial x_{j}}, \frac{\partial}{\partial y_{j}}$ of $T \mathbb{C}^{N}$ is given by the differentials of the coordinate functions $d x_{j}, d y_{j}$. Smooth 1-forms on $\Omega$ are therefore expressions of the form

$$
\omega=\sum_{j} a_{j} d x_{j}+\sum_{j} b_{j} d y_{j}
$$

where again $a_{j}$ and $b_{j}$ are smooth (real-valued) functions on $\Omega$. The exterior differential is then defined for forms of arbitrary degree $n$, that is, sections of $\Lambda^{n} T^{*} \mathbb{C}^{N}$ by requiring that $d$ fulfills the Leibniz rule, i.e. for an $n$-form $\omega$ and a smooth function $a$, we have $d a \omega=d a \wedge \omega+a d \omega$.

We now turn to the complexification of the tangent bundles. Just as tangent vectors act on real-valued functions, complexified tangent vectors act on complex-valued functions. $\mathbb{C} T \mathbb{C}^{N}$ can be decomposed into
its $(1,0)$ and $(0,1)$ parts, which are denoted by $T^{(1,0)} \mathbb{C}^{N}$ and $T^{(0,1)} \mathbb{C}^{N}$. The coordinate basis of $T^{(1,0)} \mathbb{C}^{N}$ which is associated to the basis $\frac{\partial}{\partial x_{j}}$ in the complex vector bundle $T^{c} \mathbb{C}^{N}$ is usually denoted by

$$
\frac{\partial}{\partial z_{j}}=\frac{1}{2}\left(\frac{\partial}{\partial x_{j}}-i \frac{\partial}{\partial y_{j}}\right), \quad j=1, \ldots, N
$$

and likewise we have a coordinate basis of $T^{(0,1)} \mathbb{C}^{N}$ given by

$$
\frac{\partial}{\partial \bar{z}_{j}}=\frac{1}{2}\left(\frac{\partial}{\partial x_{j}}+i \frac{\partial}{\partial y_{j}}\right), \quad j=1, \ldots, N
$$

Sections of $T^{(1,0)} \mathbb{C}^{N}$ over $\Omega$ are commonly referred to as $(1,0)$-vector fields; they are thus expressions of the form

$$
X=\sum_{j=1}^{N} a_{j} \frac{\partial}{\partial z_{j}}
$$

where $a_{j}$ are smooth functions on $\Omega$.
We also have the complexification $\mathbb{C} T^{*} \mathbb{C}^{N}$ of the cotangent bundle. Its decomposition into ( 1,0 )- and $(0,1)$-parts gives rise to $(1,0)$ - and $(0,1)$-forms. A coordinate basis is given by the differentials of the coordinate functions $z_{j}$ and $\bar{z}_{j}$, i.e. the $d z_{j}=d x_{j}+i d y_{j}$ span $\mathbb{C} T^{*(1,0)} \mathbb{C}^{N}$ and the $d \bar{z}_{j}=d x_{j}-i d y_{j}$ span $\mathbb{C} T^{*(0,1)} \mathbb{C}^{N}$. This is actually the dual basis to the basis $\frac{\partial}{\partial z_{j}}$ and $\frac{\partial}{\partial \bar{z}_{j}}$. Also, the exterior differential splits into $d=\partial+\bar{\partial}$; specifically, we have for a smooth function $\varphi: \Omega \rightarrow \mathbb{C}$

$$
d \varphi=\partial \varphi+\bar{\partial} \varphi=\sum_{j=1}^{N} \frac{\partial \varphi}{\partial z_{j}} d z_{j}+\sum_{j=1}^{N} \frac{\partial \varphi}{\partial \bar{z}_{j}} d \bar{z}_{j}
$$

### 1.3 Formal power series

Definition 1. A formal power series $A:\left(\mathbb{C}^{N}, 0\right) \rightarrow \mathbb{C}$ is an expression of the form

$$
A(Z)=\sum_{\alpha \in \mathbb{N}^{N}} A_{\alpha} Z^{\alpha}, \quad A_{\alpha} \in \mathbb{C}
$$

Here, we use standard multi-index notation; $\alpha=\left(\alpha_{1}, \ldots, \alpha_{N}\right)$ and $Z^{\alpha}=Z_{1}^{\alpha_{1}} \ldots Z_{N}^{\alpha_{N}}$. The ring of formal power series is denoted by $\mathbb{C}[[Z]]$.
Problem 1. Prove that $\mathbb{C}[[Z]]$ is a ring. Find a condition which ensures that the composition of two formal power series is well-defined.

We can also consider formal power series at every other point $p \in \mathbb{C}^{N}$; this ring is denoted by $\mathbb{C}[[Z-p]]$. For convenience, we shall usually only deal with $p=0$. Note that it does not make sense to speak about the "value" of a formal power series (besides of its value at 0 ).
Example 1. If $f$ is a germ of a smooth function at $a \in \mathbb{C}^{N}$, its Taylor series $T_{a} f$ is the series $T_{a} f(Z, \bar{Z}) \in$ $\mathbb{C}[[Z-a, \overline{Z-a}]]$ given by

$$
\begin{equation*}
T_{a} f(Z, \bar{Z})=\sum_{\alpha, \beta \in \mathbb{N}^{N}} \frac{1}{\alpha!\beta!} \frac{\partial^{|\alpha+\beta|} f}{\partial Z^{\alpha} \bar{Z}^{\beta}}(a)(Z-a)^{\alpha}(\overline{Z-a})^{\beta} \tag{1.1}
\end{equation*}
$$

Similarly, we can define formal maps $A:\left(\mathbb{C}^{N}, 0\right) \rightarrow \mathbb{C}^{m}$ as expressions of the form

$$
A(Z)=\sum_{\alpha \in \mathbb{N}^{N}} A_{\alpha} Z^{\alpha}, \quad A_{\alpha} \in \mathbb{C}^{m}
$$

hence, formal maps can be thought of as elements of $\mathbb{C}[[Z]]^{m}$.

Differentiation is well-defined on the ring of formal power series; we have

$$
\frac{\partial^{|\beta|}}{\partial Z^{\beta}}\left(\sum_{\alpha \in \mathbb{N}^{N}} A_{\alpha} Z^{\alpha}\right)=\sum_{\gamma} \frac{(\gamma+\beta)!}{\gamma!} A_{\gamma+\beta} Z^{\gamma}
$$

We often denote differentiation by subscripts; thus, we have in particular

$$
A_{Z^{\alpha}}(0)=\alpha!A_{\alpha}
$$

It is a good exercise to prove the
Theorem 1 (Formal implicit function theorem). Let $A(z, w)=\left(A^{1}, \ldots A^{n}\right):\left(\mathbb{C}^{n} \times \mathbb{C}^{d}, 0\right) \rightarrow\left(\mathbb{C}^{n}, 0\right)$ be a formal map, satisfying $A(0)=0$ and $\operatorname{det} A_{z}(0) \neq 0$. Here, $A_{z}$ denotes the matrix

$$
A_{z}=\left(\begin{array}{ccc}
A_{z_{1}}^{1} & \ldots & A_{z_{n}}^{1} \\
\vdots & & \vdots \\
A_{z_{1}}^{n} & \ldots & A_{z_{n}}^{n}
\end{array}\right)
$$

Then there exists a unique formal map $\varphi(w):\left(\mathbb{C}^{d}, 0\right) \rightarrow\left(\mathbb{C}^{n}, 0\right)$ which satisfies $A(\varphi(w), w)=0$.
We also have a useful operation on formal power series given by truncating all terms of order higher or equal to some $k \in \mathbb{N}$. The resulting object (a polynomial of degree at most $k$ ) is often referred to as a "jet": Definition 2. The space of $k$-jets of formal power series, denoted by $J_{0}^{k}\left(\mathbb{C}^{N}\right)$, is the space of all polynomials of degree at most $k$ in $N$ variables. The mapping $j_{0}^{k}: \mathbb{C}[[Z]] \rightarrow J_{0}^{k}\left(\mathbb{C}^{N}\right)$ is defined by

$$
j_{0}^{k}\left(\sum_{\alpha \in \mathbb{N}^{N}} A_{\alpha} Z^{\alpha}\right)=\sum_{|\alpha| \leq k} A_{\alpha} Z_{\alpha}
$$

$j_{0}^{k} A$ is referred to as the $k$-jet of $A$.
Similarly, we can define the space $J_{0,0}\left(\mathbb{C}^{N}, \mathbb{C}^{n}\right)$ of $k$-jets of formal maps $\left(\mathbb{C}^{N}, 0\right) \rightarrow\left(\mathbb{C}^{n}, 0\right)$, and the associated truncation operation.
Problem 2. We denote by $\mathcal{M}$ the maximal ideal in $\mathbb{C}[[Z]]$, that is, $\mathcal{M}=\left(Z_{1}, \ldots, Z_{N}\right)$. Show that

$$
J_{0}^{k}\left(\mathbb{C}^{N}\right)=\frac{\mathbb{C}[[Z]]}{\mathcal{M}^{k+1}}
$$

The rank of a formal map $A:\left(\mathbb{C}^{N}, 0\right) \rightarrow\left(\mathbb{C}^{m}, 0\right)$, denoted by $\mathrm{rk} A$, is defined as the rank of the matrix $A_{Z}$ over the quotient field of $\mathbb{C}[[Z]]$. Hence, rk $A=r$ if there exists a minor of $A_{Z}$ of size $r$ which has nonzero determinant, but all determinants of minors of $A_{Z}$ of bigger size vanish.

Full-rank maps carry over to the formal setting some of the properties of full-rank maps in the usual setting. We will record here one such instance (although in rather rough form).

Proposition 1. Let $A:\left(\mathbb{C}^{N}, 0\right) \rightarrow\left(\mathbb{C}^{N}, 0\right)$ be a formal map of full rank, and assume that $f:\left(\mathbb{C}^{n}, 0\right) \rightarrow$ $\left(\mathbb{C}^{N}, 0\right)$ is a formal map satisfying

$$
\operatorname{det} A^{\prime}(f(z)) \equiv \equiv 0
$$

Then there exists an integer $k$ such that if $g:\left(\mathbb{C}^{n}, 0\right) \rightarrow\left(\mathbb{C}^{N}, 0\right)$ is a formal map satisfying

$$
A(f(z))=A(g(z)), \quad j_{0}^{k} f=j_{0}^{k} g
$$

then $f(z)=g(z)$.

Proof. We note that we can write

$$
A(Y)-A(X)=\int_{0}^{1} A^{\prime}(t Y+(1-t) X) d t(Y-X)
$$

by the fundamental theorem of calculus (which we can apply in this formal setting - why?). Thus, we have

$$
A(Y)-A(X)=B(X, Y)(X-Y)
$$

where $B$ is a formal map in the variables $X$ and $Y$ taking values in the $N \times N$ matrices, and satisfies $B(Y, Y)=A^{\prime}(Y)$. We can thus write

$$
0=A(f(z))-A(g(z))=B(f(z), g(z))(f(z)-g(z))
$$

Recall that for any $N \times N$-matrix $M$, its classical adjoint $M^{c}$ is defined by $M^{c}=\left(M_{j, k}^{c}\right)$, where $(-1)^{j+k} M_{j, k}^{c}$ is the determinant of the matrix obtained from $M$ by deleting its $k$-th row and its $j$-th column. For example,

$$
\left(\begin{array}{ll}
a & b \\
c & d
\end{array}\right)=\left(\begin{array}{cc}
d & -c \\
-b & a
\end{array}\right)
$$

The main property of the classical adjoint is that $M M^{c}=M^{c} M=(\operatorname{det} M) I$. Since $\operatorname{det} A^{\prime}(f(z)) \neq 0$, there exists a $k$ such that for any $g$ with $j_{0}^{k} g=j_{0}^{k} f$, we have that $\operatorname{det} B(f(z), g(z)) \equiv 0$. Thus, multiplying by the classical adjoint of $B(f(z), g(z))$, we have that

$$
(\operatorname{det} B(f(z), g(z)))(f(Z)-g(Z))=0
$$

and since the determinant is nonzero, $f(Z)=g(Z)$.

### 1.4 Convergent power series

Definition 3. A formal power series $A(Z) \in \mathbb{C}[[Z]]$ is convergent at $\zeta \in \mathbb{C}^{N}$ if the series

$$
\sum_{\alpha} A_{\alpha} \zeta^{\alpha}
$$

converges to a finite value, which we will denote by $A(\zeta)$.
Given $\left(r_{1}, \ldots, r_{N}\right) \in \mathbb{R}_{+}^{N}$ and $a \in \mathbb{C}^{N}$ we denote by $P(a, r)$ the polydisc centered at a with multiradius $r$, that is,

$$
P(a, r)=\left\{Z \in \mathbb{C}^{N}:\left|Z_{j}-a_{j}\right|<r_{j} .\right\}
$$

Lemma 1. If $A(Z) \in \mathbb{C}[[Z]]$ converges at $\zeta \in \mathbb{C}^{N}$, then it converges uniformly and absolutely on compact subsets of $P\left(0,\left(\left|\zeta_{1}\right|, \ldots,\left|\zeta_{N}\right|\right)\right)$.

Proof. Let us write $r_{j}=\left|\zeta_{j}\right|$. Note that since the series $\sum a_{\alpha} \zeta^{\alpha}$ converges, there exists a constant $C$ such that

$$
\left|a_{\alpha}\right| r^{\alpha} \leq C, \quad \alpha \in \mathbb{N}^{N}
$$

For all $\eta \in \mathbb{C}^{N}$ with $\left|\eta_{j}\right| \leq R_{j}<r_{j}$ we have that

$$
\sum_{\alpha}\left|a_{\alpha}\right|\left|\eta^{\alpha}\right| \leq C \sum_{\alpha} \frac{R^{\alpha}}{r^{\alpha}}=\prod_{j} \frac{1}{1-\frac{R_{j}}{r_{j}}}<\infty
$$

so the uniform convergence of this series follows from the convergence of the geometric series.
Lemma 2. Assume that the power series $A(Z)$ converges absolutely uniformly on $\overline{P(0, r)}$. Then

$$
\begin{equation*}
r^{\alpha} A_{\alpha}=r^{\alpha} \frac{A_{Z^{\alpha}}(0)}{\alpha!}=\frac{1}{2^{N} \pi^{N}} \int_{[0,2 \pi]^{N}} A\left(r_{1} e^{i t_{1}}, \ldots, r_{N} e^{i t_{N}}\right) e^{-i\left(\alpha_{1} t_{t}+\cdots+\alpha_{N} t_{N}\right)} d t_{1} \cdots d t_{N} \tag{1.2}
\end{equation*}
$$

Problem 3. Prove Lemma 2.
Lemma 3 (The Cauchy Estimates). Assume that the power series $A(Z)$ converges absolutely uniformly on $\overline{P(0, r)}$. Then

$$
\begin{equation*}
r^{\alpha}\left|A_{\alpha}\right| \leq \max \left\{|A(\zeta)|:\left|\zeta_{j}\right|=r_{j}, j=1, \ldots, N\right\} \tag{1.3}
\end{equation*}
$$

This follows by a brute estimate of (1.2).
The subring of $\mathbb{C}[[Z]]$ containing all convergent power series is denoted by $\mathbb{C}\{Z\}$. It carries a natural inductive limit topology (which we will discuss in more detail later on) as the limit of the spaces $\mathcal{H}_{r}$ of power series convergent on the polydisc $P(0, r)$ as $r \rightarrow 0$. Differentiation is a continuous map in this topology; and if $A(Z)$ converges on $P(0, r), A_{Z^{\alpha}}(Z)$ also converges on $P(0, r)$.
Problem 4. Show that the domain of convergence $D(A)$ (i.e. the interior of the set of points at which $A$ converges) of a power series $A$ is a complete Reinhardt domain, that is, $Z \in D(A)$ implies $\lambda Z \in D(A)$ for all $\lambda \in \mathbb{C}$ with $|\lambda|<1$ which is also logarithmically convex.

### 1.5 Holomorphic functions

Definition 4. Let $\Omega \subset \mathbb{C}^{N}$ be an open set. A smooth function $f: \Omega \rightarrow \mathbb{C}$ is holomorphic if $\bar{\partial} f=0$ on $\Omega$. From the linear algebra above, we thus have that $f$ is holomorphic on $\Omega$ if and only if $d f$ is complex linear at every point in $\Omega$. The space of holomorphic functions on $\Omega$ is denoted by $\mathcal{H}(\Omega)$.
Remark 1. $\mathcal{H}(\Omega)$ is a Frechet space with the topology of uniform convergence on compact subsets of $\Omega$. This topology is generated by the norms

$$
\|f\|_{K}=\max _{Z \in K}|f(Z)|
$$

where $K$ varies over all compact subsets of $\Omega$. By choosing a compact exhaustion of $\Omega$, we see that a countable number of these norms generates the topology.

Our first goal is to derive the power series expansion of a holomorphic function. Let us first recall the basic Cauchy formula from one-dimensional complex analysis.

Proposition 2 (The inhomogeneous Cauchy formula). Let $D \subset \mathbb{C}$ be a smoothly bounded domain, $f \in$ $C^{1}(\bar{D})$. Then for any $z \in D$,

$$
\begin{equation*}
f(z)=\frac{1}{2 \pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta-z} d \zeta+\frac{1}{2 \pi i} \int_{D} \frac{\frac{\partial f}{\partial \bar{\zeta}}(\zeta)}{\zeta-z} d \zeta \wedge d \bar{\zeta} \tag{1.4}
\end{equation*}
$$

Corollary 1. Let $D \subset \mathbb{C}$ be a domain with smooth boundary. If $f \in C^{1}(\bar{D})$ is holomorphic in $D$, then for any $z \in D$ we have

$$
\begin{equation*}
f(z)=\frac{1}{2 \pi i} \int_{\partial D} \frac{f(\zeta)}{\zeta-z} d \zeta \tag{1.5}
\end{equation*}
$$

Proposition 3. Let $\Omega \subset \mathbb{C}^{N}$ be an open set, $f$ a holomorphic function on $\Omega$. Then for any $a \in \Omega$, the Taylor series $T_{a} f$ of $f$ converges to $f$ on any polydisc $P(a, r) \subset \Omega$.
Proof. We assume w.l.o.g. $a=0$. Let $r$ be any multiradius such that $\overline{P(0, r)} \subset \Omega$. Fix $Z=\left(Z_{1}, \ldots, Z_{N}\right) \in$ $P(0, r)$. Now we apply the Cauchy formula (1.5) to the holomorphic function of one variabe $\zeta \mapsto f\left(\zeta, Z_{2}, \ldots, Z_{N}\right)$ and obtain

$$
f(Z)=\frac{1}{2 \pi i} \int_{|\zeta|=r_{1}} \frac{f\left(\zeta, Z_{2}, \ldots, Z_{N}\right)}{\zeta-Z_{1}} d \zeta
$$

Repeated application of this argument gives us the formula

$$
f(Z)=\left(\frac{1}{2 \pi i}\right)^{N} \int_{\left|\zeta_{1}\right|=r_{1}} \cdots \int_{\left|\zeta_{N}\right|=r_{N}} \frac{f\left(\zeta_{1}, \ldots, \zeta_{N}\right)}{\left(\zeta_{1}-Z_{1}\right) \cdots\left(\zeta_{N}-Z_{N}\right)} d \zeta_{1} \cdots d \zeta_{N}
$$

We now develop this integral in a series using the geometric series formula (why is the interchange justified?) to obtain

$$
f(Z)=\sum_{\alpha} a_{\alpha} Z^{\alpha}
$$

where

$$
a_{\alpha}=\left(\frac{1}{2 \pi i}\right)^{N} \int_{\left|\zeta_{1}\right|=r_{1}} \cdots \int_{\left|\zeta_{N}\right|=r_{N}} \frac{f\left(\zeta_{1}, \ldots, \zeta_{N}\right)}{\zeta_{1}^{\alpha_{1}+1} \cdots \zeta_{N}^{\alpha_{N}+1}} d \zeta_{1} \cdots d \zeta_{N}
$$

The term in the denominator of the integrand above will be denoted by $\zeta^{\alpha+1}$ (just as in the one-dimensional case); since every function represented as a power series is its own Taylor expansion, the Proposition is fully proved. We also obtain the following useful formula

$$
\begin{equation*}
f_{Z^{\alpha}}(a)=\left(\frac{\alpha!}{2 \pi i}\right)^{N} \int_{\left|\zeta_{1}-a_{1}\right|=r_{1}} \cdots \int_{\left|\zeta_{N}-a_{N}\right|=r_{N}} \frac{f\left(\zeta_{1}, \ldots, \zeta_{N}\right)}{(\zeta-a)^{\alpha+1}} d \zeta_{1} \cdots d \zeta_{N} \tag{1.6}
\end{equation*}
$$

Now that we know that holomorphic functions are exactly functions locally represented as a power series, we obtain a couple of useful conclusions.

Proposition 4 (The Cauchy estimates, again). Assume that $K \subset \subset L^{\circ} \subset \subset \Omega$. Then there exists a constant c (depending on $K$ and $L$ ) such that

$$
\begin{equation*}
\left\|f_{Z^{\alpha}}\right\|_{K} \leq \frac{\alpha!}{c^{|\alpha|}}\|f\|_{L} \tag{1.7}
\end{equation*}
$$

For the proof, we apply Lemma 3 and note that since $K$ is relatively compact in $L^{\circ}$, there exists a radius $r$ such that $P(a, r) \subset L^{\circ}$ for all $a \in K$. One conclusion from the Cauchy estimates is
Theorem 2. $M \subset \mathcal{H}(\Omega)$ is compact if and only if it is closed and bounded.
In other words, the Montel theorem states that if a sequence $f_{n} \in \mathcal{H}(\Omega)$ is uniformly bounded on compact subsets of $\Omega$, then there exists a subsequence $f_{n_{k}}$ which converges uniformly on compact subsets of $\Omega$.

Proof. We prove the Montel theorem in the second formulation; that this implies the theorem as stated follows from general functional analysis. We will use the Ascela-Arzoli theorem: Every sequence in a family $F$ of continuous functions on a compact set $K$ contains a uniformly convergent subsequence if and only if $F$ is bounded and equicontinuous.

We first choose a compact exhaustion $K_{j}$ of $\Omega$; that is, each $K_{j}$ is compact, $K_{j} \subset K_{j+1}^{\circ}$, and $\cup K_{j}=\Omega$. Given a sequence $f_{n} \in \mathcal{H}(\Omega)$, we apply the Cauchy estimates to the pair $K_{j}$ and $K_{j+1}$; we obtain that there exists a constant $C_{j}$ such that

$$
\left\|f_{Z_{k}}\right\|_{K_{j}} \leq C_{j}\|f\|_{K_{j+1}}, \quad k=1, \ldots, N
$$

Since $f_{n}$ is bounded on compacts, we see that the right hand side of this inequality is bounded by some constant $R_{j}$. We claim that this implies that $f_{n}$ is equicontinuous on compacts.

Indeed, by the fundamental theorem of calculus,

$$
f(\zeta)-f(\eta)=\int_{0}^{1} f_{Z}(t \zeta+(1-t) \eta) \cdot(\zeta-\eta) d t
$$

and so

$$
|f(\zeta)-f(\eta)| \leq R_{j}\|\zeta-\eta\|
$$

if $\zeta, \eta$, and the line connecting them are contained within $K_{j}$. We now cover $K_{j-1}$ by balls whose radius $r$ is chosen so small that balls of the double radius still stay inside $K_{j}$. Thus, if $|\zeta-\eta|<r$, the line connecting $\zeta$ and $\eta$ fully stays inside $K_{j}$ and we can apply the above estimate. The Ascela-Arzoli theorem implies that we can extract a uniformly convergent subsequence, and a diagonal argument gives the result.

Proposition 5 (The identity principle). Let $\Omega \subset \mathbb{C}^{N}$ be a domain (an open and connected set). If $f \in \mathcal{H}(\Omega)$ vanishes on an open subset of $\Omega$, it vanishes on all of $\Omega$.
Problem 5. Prove the identity principle. (Hint: Show that the set of all points in $\Omega$ which have an open neigbourhood on which $f$ vanishes is both open and closed)
Definition 5. The ring of germs of holomorphic functions at 0 , denoted by $\mathcal{O}$ (or $\mathcal{O}_{N}$, if we want to denote the dimension), is the limit of the spaces $\mathcal{H}(\Delta)$ as $\Delta$ varies over all open neighbourhoods of 0 . That is, it is the set of all equivalence classes of holomorphic functions defined near 0 , where two functions are equivalent if they agree on some open neighbourhood of 0 .
Problem 6. Show that $\mathcal{O}=\mathbb{C}\{Z\}$ (as sets; they are actually isomorphic as topological vector spaces).
Definition 6. The space of bounded holomorphic functions, denoted by $\mathcal{H}^{\infty}(\Omega)$, is the space of all holomorphic functions on $\Omega$ which are also bounded, endowed with the norm

$$
\|f\|_{\Omega}=\sup _{Z \in \Omega}|f(Z)|
$$

Problem 7. Show that $\mathcal{H}^{\infty}(\Omega)$ is a Banach space.

### 1.6 Domains of holomorphy: just an aside

In one complex variable, if we are given a domain $\Omega \subset \mathbb{C}$, then there exists a holomorphic function which cannot be extended to any domain strictly larger than $\Omega$. This is not true in several complex variables; there exist domains $\Omega \subset \mathbb{C}^{N}$ for which every $f \in \mathcal{H}(\Omega)$ automatically extend to some strictly larger open set.

An example is the "Hartogs figure" $\Omega$, the set given by

$$
\Omega=\left\{(z, w) \in \mathbb{C}^{2}: 1 / 2<|z|<1,|w| \leq 1 / 2\right\} \cup\left\{(z, w) \in \mathbb{C}^{2}: 1 / 2 \leq|w|<1\right\}
$$

We claim that every function holomorphic on $\Omega$ extends to the polydisc $P(0,(1,1))$. Given $f \in \mathcal{H}(\Omega)$, we define a function $g$ by

$$
g(z, w)=\frac{1}{2 \pi i} \int_{\zeta=3 / 4} \frac{f(\zeta, w)}{\zeta-z}, \quad|z|<3 / 4
$$

$g$ is holomorphic on $P(0,(3 / 4,1))$, and agrees with $f$ on the intersection with $\Omega$ by the Cauchy formula. We can thus extend $f$ by setting it equal to $g$ on $P(0,(3 / 4,1))$.

The characterization of domains of holomorphy (i.e. domains on which there exists a function not extendible to any strictly larger domain) has been one of the leading problems in the analysis of several complex variables. The solution to the so-called Levi problem gives us that a domain is a domain of holomorphy if and only if it is pseudoconvex. We will explore this notion in more detail in the case of domains with smooth boundaries; it turns out that in this case, pseudoconvexity is actually a local holomorphic boundary invariant.

### 1.7 The Cartan uniqueness theorem

In the 1930s, H. Cartan proved his important uniqueness theorem, which generalizes the Schwarz Lemma in one complex variable. Loosely stated it says that biholomorphisms of a bounded domain in $\mathbb{C}^{N}$ are determined by their value and the value of their derivative at any given point. For a domain $\Omega \subset \mathbb{C}^{N}$, we denote by $\operatorname{Aut}(\Omega)$ the set of all holomorphic mappings $\Omega \rightarrow \Omega$ which are one-to-one and onto.
Theorem 3. Let $\Omega \subset \mathbb{C}^{N}$ be a bounded domain. If $H \in \operatorname{Aut}(\Omega)$ satisfies $H(p)=p$ and $H^{\prime}(p)=I$, then $H(Z)=Z$ for $Z \in \Omega$.
Proof. W.l.o.g. we can assume that $p=0$. Assume that $H(Z)=Z+p(Z)+\ldots$, where $p(Z)$ is the lowest order homogeneous polynomial in the Taylor series expansion of $H$ which does not vanish. Note that composing $H$ with itself $k$ times gives us the Taylor expansion of $H^{(k)}$ as

$$
H^{(k)}(Z)=Z+k p(Z)+\ldots ;
$$

but since $\Omega$ is bounded, the Cauchy estimates imply that the derivatives of any automorphism at 0 of any fixed order are uniformly bounded. Hence, $p(Z)=0$, and so $H(Z)=Z$.

## 2 The Weierstrass theorem and its consequences

### 2.1 The Weierstrass division theorem

The Weierstrass division theorem paves the way to understanding the most fundamental local properties of analytic functions. In what follows, we will write $Z=\left(Z^{\prime}, Z_{N}\right) \in \mathbb{C}^{N-1} \times \mathbb{C}$. We say that a (germ of a) holomorphic function $D(Z)$ defined near 0 is $k$-regular in the $Z_{N}$ direction if the function $Z_{N} \mapsto D\left(0, Z_{N}\right)$ has a zero of order $k$ at the origin. In terms of a power series expansion

$$
D(Z)=\sum_{j} D_{j}\left(Z^{\prime}\right) Z_{N}^{j}
$$

this means that

$$
D_{j}(0)=0, \quad j<k, \quad D_{k}(0) \neq 0
$$

Problem 8. If $D$ is any holomorphic function, then there exists a $k$ and a holomorphic coordinate system $Z$ such that $D$ is $Z_{N}$-regular in that coordinate system.

A function $p$ is polynomial in $Z_{N}$ of degree at most $k$ if

$$
p(Z)=\sum_{j=0}^{k} p_{j}\left(Z^{\prime}\right) Z_{N}^{j}
$$

We are now ready to state the Weierstrass theorem. It generalizes the well-known division algorithm for polynomials in the sense that we can divide by any $k$-regular function:

Theorem 4 (The Weierstrass Division Theorem). Let $D$ be a germ of a holomorphic function near 0, which is $k$-regular in the $Z_{N}$ direction. Then there exists a polydisc $P=P(0, r)$ such that any bounded holomorphic function $h$ on $P$ can be written as

$$
\begin{equation*}
h(Z)=D(Z) q(Z)+r(Z), \tag{2.1}
\end{equation*}
$$

where $r(Z)$ is polynomial in $Z_{N}$ of degree at most $k-1$. Furthermore, this decomposition is unique, and there exists a constant $C$ such that

$$
\begin{equation*}
\|q\|_{P} \leq C\|h\|_{P} \tag{2.2}
\end{equation*}
$$

Proof. Note that w.l.o.g. we can assume that $D_{k}\left(Z^{\prime}\right)=1$ (by dividing $D$ by $D_{k}$ if necessary). We write $P=P^{\prime} \times \Delta_{r_{N}}$. Given $f$, we write

$$
f(Z)=f_{0}\left(Z^{\prime}\right)+Z_{N}(T f(Z)), \quad T f(Z)=\frac{f(Z)-f\left(Z^{\prime}, 0\right)}{Z_{N}}
$$

We first claim that $T$ is continuous on $\mathcal{H}^{\infty}(P)$ and

$$
\|T f\|_{P} \leq \frac{2}{r_{N}}\|f\|_{P}
$$

Indeed, this follows from the fact that $\left\|f\left(Z^{\prime}, 0\right)\right\|_{P^{\prime}} \leq\|f\|_{P}$ and

$$
\left\|Z_{N} \varphi(Z)\right\|_{P}=r_{N}\|\varphi(Z)\|_{P}
$$

which follows from the (one-dimensional) maximum principle.

We can thus write

$$
f(Z)=\sum_{j=0}^{k-1} f_{k}\left(Z^{\prime}\right) Z_{N}^{j}+Z_{N}^{k} T^{k} f(Z)
$$

where

$$
\left\|T^{k} f\right\|_{P} \leq \frac{C}{r_{N}^{k}}\|f\|_{P}
$$

Consider the operator $S$ defined by

$$
S f=\left(f(Z)-Z_{N}^{k} T^{k} f(Z)\right)+D(Z) T^{k} f(Z)
$$

We claim that with an appropriate choice of $r=\left(r^{\prime}, r_{N}\right)$, this operator is bijective.
To see this, compute $S f-f=E(Z) T^{k} f(Z)$, where $E(Z)=\sum_{j=0}^{k-1} E_{j}\left(Z^{\prime}\right) Z_{N}^{j}+Z_{N}^{k+1} \tilde{E}(Z)$, where $E_{j}(0)=0$ for $j<k$ by our assumptions on $D$. We can thus estimate

$$
\|S f-f\|_{P} \leq \frac{2^{k}\left(\varepsilon\left(r^{\prime}\right)+K r_{N}^{k+1}\right)}{r_{N}^{k}}\|f\|_{P}
$$

where $\varepsilon\left(r^{\prime}\right) \rightarrow 0$ as $r^{\prime} \rightarrow 0$, and $K$ does not depend on $r$. We choose $r$ so that this ratio is smaller than $1 / 2$, and see that on this $P$ we now have

$$
\|S f-f\|_{P} \leq \frac{1}{2}\|f\|_{P}
$$

This implies that $S$ is bijective, its inverse being given by the series

$$
S^{-1}=\sum_{j=0}^{\infty}(I-S)^{j}
$$

which is bounded in norm by 2 .
Thus, given any $h$, we have a unique $f$ with $S f=h$, or equivalently

$$
h(Z)=D(Z) q(Z)+r(Z)
$$

where $q(Z)=T^{k} f(Z)$ and $r(Z)=f(Z)-Z_{N}^{k} T^{k} f$ satisfy the assumptions of the theorem, their uniqueness being guaranteed by the uniqueness of $f$. Furthermore,

$$
\|q\|_{P} \leq \frac{2^{k+1}}{r_{N}^{k}}\|h\|_{P}
$$

If we divide $Z_{N}^{k}$ by a function $D$ which is $k$-regular in the $Z_{N}$ direction, we obtain a representation of $D$ as $D(Z)=u(Z) W(Z)$ where $W$ is a Weierstrass polynomial of degree $k$ in $Z_{N}$, that is

$$
W(Z)=Z_{N}^{k}+W_{1}\left(Z^{\prime}\right) Z_{N}^{k-1}+\cdots+W_{k}\left(Z^{\prime}\right)
$$

where $W_{j}(0)=0$.
Corollary 2 (The Weierstrass Preparation Theorem). Let $D \in \mathcal{O}$ be a function which is $k$-regular in the $Z_{N}$-direction. Then there exists a unique unit $u(Z) \in \mathcal{O}$ and a unique Weierstrass polynomial $W(Z)$ of degree $k$ in $Z_{N}$ such that $D(Z)=u(Z) W(Z)$.

### 2.2 Properties of the ring of germs

In one complex variable, the ring of germs $\mathcal{O}_{1}$ is a principal ideal domain; indeed, every ideal is of the form $I=\left(z^{k}\right)$ for some integer $k \in \mathbb{N}$. Thus, it is a unique factorization domain. This means that for any $f \in \mathcal{O}_{1}$, there exist finitely many distinct irreducible elements $p_{j} \in \mathcal{O}_{1}$ and integers $n_{j}$ such that

$$
f=\prod p_{j}^{n_{j}}
$$

and this representation is unique up to multiplication by units.
The unique factorization property actually carries over to the rings $\mathcal{O}_{N}$ for all $N$; before we prove that, we give the following useful property of Weierstrass polynomials.

Lemma 4. If the product of two polynomials $W, V \in \mathcal{O}_{N-1}\left[Z_{N}\right]$ is a Weierstrass polynomial, so are $W$ and $V$.

Proof. Assume that

$$
W(Z)=\sum_{j=1}^{k} W_{j}\left(Z^{\prime}\right) Z_{N}^{j}, \quad V(Z)=\sum_{j=1}^{m} V_{j}\left(Z^{\prime}\right) Z_{N}^{j}
$$

Choose $j_{W}$ minimal with $W_{j_{w}}(0) \neq 0$, and $j_{V}$ minimal with $V_{j_{V}}(0) \neq 0$. Consider the term of order $j_{W}+j_{V}$ in $W V$; it has as a coefficient

$$
\sum_{a+b=j_{W}+j_{V}} W_{a}\left(Z^{\prime}\right) V_{b}\left(Z^{\prime}\right)
$$

which evaluated at 0 is equal to $W_{j_{w}}(0) V_{j_{V}}(0) \neq 0$, so $W V$ is not Weierstrass unless $j_{W}=k$ and $j_{V}=m$.
Proposition 6. The ring $\mathcal{O}_{N}$ is a unique factorization domain.
Proof. We assume by induction that $\mathcal{O}_{N-1}$ is a unique factorization domain (it's true for $N=2$; see above). We will use the fact that if a ring $U$ is a unique factorization domain, then so is the polynomial ring $U[x]$ (which is a standard fact from algebra).

Now let $f \in \mathcal{O}_{N}$ be a germ of a holomorphic function, $f \neq 0$. Thus, w.l.o.g., we can assume that $f$ is $k$-regular in $Z_{N}$ for some $k \geq 1$. By the Preparation Theorem, we can write $f=u W$, with $u$ a unit in $\mathcal{O}_{N}$ and $W \in \mathcal{O}_{N-1}\left[Z_{N}\right]$ a Weierstrass polynomial of degree $k$ in $Z_{N}$. By unique factorization in $\mathcal{O}_{N-1}\left[Z_{N}\right]$ we can thus write $W=\prod_{j} p_{j}^{n_{j}}$, where the $p_{j}$ are distinct irreducible Weierstrass polynomials in $\mathcal{O}_{N-1}\left[Z_{N}\right]$ (see Lemma 4).

We first claim that the $p_{j}$ are also irreducible in $\mathcal{O}_{N}$. Indeed, assume that $p$ is a Weierstrass polynomial and $p=g_{1} g_{2}$ in $\mathcal{O}_{N}$. Then necessarily both $g_{j}$ are regular in $Z_{N}$ of some order; and we can apply the preparation theorem to each in turn, obtaining $g_{j}=u_{j} W_{j}$. Thus, $p=u_{1} u_{2} W_{1} W_{2}$. Since $p$ is Weierstrass, $u_{1} u_{2}=1$, and $p=W_{1} W_{2}$ already splits in $\mathcal{O}_{N-1}\left[Z_{N}\right]$.

Thus, every $f \in \mathcal{O}_{N}$ has a representation as a product of irreducible factors; that this representation is unique up to units follows from the uniqueness of factorization in $\mathcal{O}_{N-1}\left[Z_{N}\right]$.

We will have to deal with sets of the form $V=Z \in \mathbb{C}^{N}: f_{\alpha}(Z)=0, \alpha \in A$ where $A$ is some index set and the $f_{\alpha}$ are holomorphic functions (such sets are called analytic varieties). One of the most important simplifications we will use is the fact that any such set can be defined by only finitely many of the $f_{\alpha}$. In fact, every ideal $I \subset \mathcal{O}_{N}$ is finitely generated, i.e. $\mathcal{O}_{N}$ is Noetherian:
Definition 7. A ring $R$ (we will always assume that our rings are commutative and with unity) is called Noetherian if any ideal $I \subset R$ is finitely generated, or equivalently, if $R$ satisfies the ascending chain condition: if $I_{1} \subset I_{2} \subset \cdots \subset I_{j} \subset \cdots \subset R$ is an ascending chain of ideals, then $I_{j}=I_{j+1}=I_{j+2}=\ldots$ for some index $j$.

Note that $\mathcal{O}_{1}$ is Noetherian, since it is a principal ideal domain - indeed, every ideal is of the form $\left(z^{k}\right)$ for some $k \in \mathbb{N}$, as already noted above. The Noetherian property also carries over to the rings $\mathcal{O}_{N}$; the proof is based on what is known as the "Hilbert basis theorem":

Theorem 5. If $R$ is Noetherian, so is $R[x]$.
Proof. Let $I \subset R[x]$ be an ideal. We choose a sequence of elements $p_{j} \in I$ by letting $p_{1}$ be any polynomial of minimum degree in $I$, and inductively, $p_{j}$ any polynomial of minimal degree in $I \backslash\left(p_{1}, \ldots, p_{j-1}\right)$.

We now consider the ideal $\tilde{I} \subset R$ of initial coefficients of the $p_{j}$ : if $p_{j}=a_{j} x^{d_{j}}+O\left(d_{j}-1\right)$, then $\tilde{I}=\left(a_{1}, a_{2}, \ldots\right)$. Since $R$ is Noetherian, there exists an $n$ such that $\tilde{I}=\left(a_{1}, \ldots, a_{n}\right)$. We claim that $I=\left(p_{1}, \ldots, p_{n}\right)$. If not, there exists a polynomial $p_{n+1}$ of minimal degree in $I \backslash\left(p_{1}, \ldots, p_{n}\right)$.

But since $I=\left(a_{1}, \ldots, a_{n}\right)$, we can write $a_{n+1}=\sum_{j \leq n} r_{j} a_{j}$ for some $r_{j} \in R$. Then the polynomial $p_{n+1}-\sum_{j \leq n} r_{j} x^{d_{j+1}-d_{j}} p_{j} \in I \backslash\left(p_{1}, \ldots p_{n}\right)$ has strictly lower degree than $p_{n+1}$, contradicting the minimality of the degree of $p_{n+1}$.

Proposition 7. The ring $\mathcal{O}_{N}$ is a Noetherian ring.
Proof. We prove this by induction on $N$. For $N=1$, it follows from the observation above. Now assume that $N>1$ and let $I \subset \mathcal{O}_{N}$ be an ideal. We can choose variables in such a way that there exists a function $D \in \mathcal{O}_{N-1}\left[Z_{N}\right]$ which is $k$-regular in the $Z_{N}$-direction. For any $f \in I$, we denote by $r_{f}$ the remainder of $f$ upon division by $D$, i.e. $f=D q+r_{f}$. Consider the ideal $I_{D} \subset \mathcal{O}_{N-1}\left[Z_{N}\right]$ generated by $\left\{r_{f}: f \in I\right\}$. $I_{D}$ is finitely generated by the induction assumption and Theorem 5, and hence, so is $I$ (by $D$ and a set of generators of $I_{D}$ ).

### 2.3 Local structure of complex hypersurfaces

A complex variety $V \subset \mathbb{C}^{N}$ is a set given locally by the vanishing of a family of holomorphic functions; i.e. for any $p \in V$, there exists an open neighbourhood $U$ of $p$ in $\mathbb{C}^{N}$ and a family of functions $S \subset \mathcal{H}(U)$ such that $V \cap U=\{f=0, f \in S\}$. A germ of a complex variety at $p$ is an equivalence class of complex varieties through $p$, the equivalence relation being given by agreeing on a neighbourhood of $p$. A germ of a variety is thus defined by a family $S \subset \mathcal{O}$ (and we will as usual assume that $p=0$ ). Note that we can always replace $S$ by the ideal $I(S)$ generated by $S$. Thus, the Noetherian property of $\mathcal{O}$ implies that any germ of a variety is defined by the vanishing of a finite family $S \subset \mathcal{O}$. If we can choose $S$ such that it contains of a single element, we say that $V$ is a germ of a complex hypersurface. It is irreducible if it cannot be written as the proper union of two complex hypersurfaces.

Lemma 5. If $V$ is the germ of a complex hypersurface, it can be written uniquely as a union $V_{1} \cup \cdots \cup V_{j}$ of pairwise nonequal complex hypersurfaces.

Lemma 6. If $V$ is the germ of an irreducible complex hypersurface, there exists an integer $k$, complex coordinates $Z=\left(Z^{\prime}, Z_{N}\right)$, a polydisc $P(0, r) \subset \mathbb{C}^{N}$, a complex hypersurface $E \subset P\left(0, r^{\prime}\right)$ such that $V \cap P(0, r)$ has the following properties, where $\pi(Z)=Z^{\prime}$ is the projection onto the first $N-1$ coordinates:

1. $\pi(V)=P\left(0, r^{\prime}\right)$;
2. $\left.\pi\right|_{V \cap \pi^{-1}(E)}: V^{\prime} \rightarrow \mathbb{C}^{N-1}$ is a $k$-fold covering map.

In particular, the set of points where a complex hypersurface is not smooth is thin; also, we have that its complement, the set of regular points, is connected. This can be proved by induction from Lemma 6.

## 3 CR-manifolds

### 3.1 Basic notions for real submanifolds of $\mathbb{R}^{k}$

### 3.1.1 Definitions and short review

Let us recall that a smooth real submanifold of $\mathbb{R}^{k}$ of codimension $d$ is given (locally) by the simultaneous vanishing of $d$ real-valued, smooth "defining functions". That is, for any $p \in M$, there exists a neighbourhood
$U$ of $p$ in $\mathbb{R}^{k}$ and smooth functions $\rho_{1}, \ldots, \rho_{d}$ on $U$ such that $M \cap U=x \in U: \rho_{1}(x)=\cdots=\rho_{d}(x)=0$ and $d \rho_{1}(x) \wedge \cdots \wedge d \rho_{d}(x) \neq 0$ for $x \in U$. The tangent space at $p$ is the kernel of the defining functions:

$$
\begin{equation*}
T_{p} M=\left\{X \in T_{p} \mathbb{R}^{k}: d \rho_{j}(p)(X)=0, j=1, \ldots, d\right\} \tag{3.1}
\end{equation*}
$$

In coordinates, this means that $T_{p} M$ consists of all tangent vectors

$$
X=\left.\sum_{j} a_{j} \frac{\partial}{\partial x_{j}}\right|_{p}
$$

which annihilate the $\rho_{\ell}$, that is, $\sum_{j} a_{j} \frac{\partial \rho_{\ell}}{\partial x_{j}}(p)=0$, for $\ell=1, \ldots, d$, and we identify an $X$ as in (3.1) with the vector $v_{X}=\left(a_{1}, \ldots, a_{k}\right) \in \mathbb{R}^{k}$.

If we use a rotation to choose coordinates $x=\left(x_{1}, x_{2}\right) \in \mathbb{R}^{k-d} \times \mathbb{R}^{d}$ such that the tangent space $T_{p} M$ corresponds to $x_{2}=0$, the defining equations necessarily satisfy det $\frac{\partial \rho}{\partial x_{2}}(p) \neq 0$, so we can apply the implicit function theorem to write the defining equations in the form $x_{2}=\varphi\left(x_{1}\right)$; we also have that $d \varphi(p)=0$.

A further transformation shows that a smooth real submanifold as above is locally diffeomorphic to a coordinate hyperplane $x_{2}=0$. Another way to present a real submanifold of $\mathbb{R}^{k}$ is by a parametrization near $p$, that is, a one-to-one smooth map $\xi: \mathbb{R}^{k-d} \supset U \rightarrow \mathbb{R}^{k}$ which satisfies $M=\xi(U) \cap V$ for a neighbourhood $V$ of $p$ and where $\xi^{\prime}$ is of full rank $k$ on $U$. A graphing function as in the preceding paragraph gives rise to such a parametrization by $x_{1} \mapsto\left(x_{1}, \varphi\left(x_{1}\right)\right)$. In that case, $T_{p} M=$ image $\xi^{\prime}\left(\xi^{-1}(p)\right)$.
Example 2. Consider the real submanifold $M \subset \mathbb{R}^{4}$ given by

$$
M=\left\{(x, y, s, t) \in \mathbb{R}^{4}: t=x^{2}+y^{2}\right\}
$$

Its tangent space $T_{x, y, s, x^{2}+y^{2}} M$ is spanned by the tangent vectors

$$
X=\frac{\partial}{\partial x}+2 x \frac{\partial}{\partial t}, \quad Y=\frac{\partial}{\partial y}+2 y \frac{\partial}{\partial t}, \quad S=\frac{\partial}{\partial s}
$$

A parametrization is given by $(x, y, s) \mapsto\left(x, y, s, x^{2}+y^{2}\right)$.

### 3.1.2 The Lie bracket

A vector field $X$ is a smooth mapping which associates to any point $p \in M$ a tangent vector $X(p)=X_{p} \in$ $T_{p} M$ (i.e. it is a smooth section of $T M$ ). The set of smooth vector fields on $M$ is denoted by $\mathfrak{X}(M)$. On this set, the Lie bracket defines a bilinear map by

$$
[X, Y]=X Y-Y X
$$

where the right hand side is understood in the operator notation, i.e. it corresponds to the differential operator $X Y-Y X$. It turns out that this is again a first-order operator, and so it defines a vector field.

### 3.1.3 The Frobenius Theorem

A particularly important class of submanifolds arise as the solutions to systems of ordinary differential equations. If we have a family of smooth vector fields $\left\{X_{1}, \ldots, X_{r}\right\} \subset \mathfrak{X}(\Omega)$, where $\Omega \subset \mathbb{R}^{k}$ is open, which is linearly independent at each point, when is there through any $p \in \Omega$ a smooth $r$-dimensional manifold $M$ such that for any $q \in M, T_{q} M=\operatorname{span}\left\{\mathrm{X}_{1}(\mathrm{q}), \ldots, \mathrm{X}_{\mathrm{r}}(\mathrm{q})\right\}$ ? The answer is given by the Frobenius Theorem, which states that the "obvious" necessary condition that the system $\left\{X_{1}, \ldots, X_{r}\right\}$ satisfies that the Lie bracket of every two vector fields can be expressed in terms of the system is also sufficient.
Theorem 6. Assume that $\left\{X_{1}, \ldots, X_{r}\right\} \subset \mathfrak{X}(\Omega)$, where $\Omega \subset \mathbb{R}^{k}$ is open, is linearly independent at each $p \in \Omega$, and that for every $j, k=1, \ldots, r$ we have

$$
\begin{equation*}
\left[X_{j}, X_{k}\right]=\sum_{\ell=1}^{r} a_{j, k, \ell} X_{\ell} \tag{3.2}
\end{equation*}
$$

for some smooth functions $a_{j, k, \ell}$ on $\Omega$. Then for any $p \in \Omega$, there exists a neighbourhood $U \subset \Omega$ of $p$ and smooth coordinates $\left(x_{1}, \ldots, x_{k}\right)$ on $U$ in which

$$
\operatorname{span}\left\{X_{j}: j \leq r\right\}=\operatorname{span}\left\{\frac{\partial}{\partial x_{j}}: j \leq r\right\}
$$

### 3.2 Biholomorphically invariant geometry: First order

### 3.2.1 Complex tangent spaces and CR manifolds

We will now assume that $M$ is a smooth, real submanifold of $\mathbb{C}^{N}$. Our goal will be to describe the fundamental (that is, first order) invariant associated to the geometry of such a real object when considering biholomorphic transformations as symmetries. The first one is the complex tangent space. Since $M \subset \mathbb{C}^{N}$, we have for any $p \in M$ the real subspace $T_{p} M \subset T_{p} \mathbb{C}^{N}$. Recall the complex structure operator $J$ introduced in the first section. There exists a maximal subspace of $T_{p} M$ which is left invariant by $J$, given by $T_{p} M \cap J T_{p} M$. This space is called the complex tangent space of $M$ at $p$ and denoted by $T_{p}^{c} M$. It is a complex vector space of some (complex) dimension $n=n(p)$.
Example 3. Consider the manifold given by the equation $w=|z|^{2}$ in $\mathbb{C}^{2}$. It is a real submanifold of real codimension 2. In this case, $\operatorname{dim}_{\mathbb{C}} T_{0}^{c} M=1$, and $\operatorname{dim}_{\mathbb{C}} T_{p}^{c} M=0$ for all $0 \neq p \in M$.
Example 4. Consider the manifold $M$ from Example 2. The complex tangent space is spanned over $\mathbb{R}$ by

$$
Z_{1}=X+2 y S, \quad Z_{2}=Y-2 x S
$$

(Check that $\left.J Z_{1}=Z_{2}, J Z_{2}=-Z_{1}\right)$.
Definition 8. A real submanifold $M \subset \mathbb{C}^{N}$ is CR (or Cauchy-Riemann) if the dimension of its complex tangent spaces is constant. The common complex dimension of these spaces is referred to as the $C R$ dimension of $M$.
Example 5. Every real hypersurface in $\mathbb{C}^{N}$ is CR of CR-dimension $N-1$. Indeed, in every complex vector space $V$ of complex dimension $N$, the dimension of the maximal complex subspace of a real $2 N-1$ dimensional real subspace $E$ is $N-1$. This follows since $E+J E=V$ and so, as real vector spaces, we have $E /(E \cap J E) \equiv(E+J E) / J E$. So the real codimension of $E \cap J E$ in $E$ is equal to the real dimension of the space on the right hand side, which is one.

Since we have identified the real tangent space with a subspace of $\mathbb{R}^{k}$, we need to give a description of the complex tangent space as a subspace of the ambient space $\mathbb{C}^{k}$. We can achieve this in terms of the defining functions:
Lemma 7. Assume that $\rho=\left(\rho_{1}, \ldots, \rho_{d}\right): \mathbb{C}^{N} \supset U \rightarrow \mathbb{R}^{d}$ is a defining function for a real submanifold $M$ near $p \in U$. Then for $q \in M$ near $p$, we have

$$
T_{q}^{c} M=\left\{Z \in \mathbb{C}^{N}: \partial \rho_{j}(q)(Z)=0\right\}
$$

In other words, $T_{q}^{c} M$ is the kernel of the matrix

$$
\rho_{Z}=\left(\begin{array}{ccc}
\frac{\partial \rho_{1}}{\partial Z_{1}} & \cdots & \frac{\partial \rho_{1}}{\partial Z_{N}} \\
\vdots & & \vdots \\
\frac{\partial \rho_{d}}{\partial Z_{1}} & \cdots & \frac{\partial \rho_{d}}{\partial Z_{N}}
\end{array}\right),
$$

evaluated at $q$.
Proof. Let $X \in T_{q}^{c} M$ be any vector. Hence, also $J X=i X \in T_{q}^{c} M$. At $q$, we thus have $d \rho(X)=\partial \rho(X)+$ $\bar{\partial} \rho(X)=0$, and $-i d \rho(i X)=\partial \rho(X)-\bar{\partial} \rho(X)=0$. Add the two equations to get $\partial \rho(X)=0$. On the other hand, if $\partial \rho(X)=0$, we have $\bar{\partial} \rho(X)=0$; indeed, we have

$$
\overline{\partial \rho(X)}=\bar{\partial} \bar{\rho}(X)=\bar{\partial} \rho(X)
$$

for $X \in T_{q} \mathbb{C}^{N}=T_{q} \mathbb{R}^{2 N}$.

Corollary 3. A connected real submanifold $M \subset \mathbb{C}^{N}$ is $C R$ if for any $p \in M$ there exists a local defining function $\rho$ near $p$ such that the rank of $\rho_{Z}$ is constant on $M$ near $p$. The dimension of the complex tangent spaces $n(p)$ depends upper-semicontinuously on $p$.

### 3.2.2 Generic and totally real submanifolds

A special case appears if the matrix $\rho_{Z}$ is of maximal rank $d$. These manifolds are of special importance:
Definition 9. A real submanifold $M \subset \mathbb{C}^{N}$ of real codimension $d$ is generic if it can be written as the intersection of $d$ smooth real hypersurfaces whose complex tangent spaces are in general position. $M$ is generic at $p \in M$ if there exists a neighbourhood $U$ of $p$ in $M$ such that $U$ is generic.

We have already met a class of real subspaces of complex vector spaces which are of special importance, the totally real ones. This concept carries over to submanifolds:
Definition 10. A real submanifold $M \subset \mathbb{C}^{N}$ is totally real if the dimension of its complex tangent spaces is 0 . An equivalent condition is that $T_{p} M \cap J T_{p} M=\{0\}$ for $p \in M . M$ is maximally totally real if its real dimension is $N$, or equivalently, if $T_{p} M \oplus J T_{p} M=T_{p} \mathbb{C}^{N}$.
Problem 9. Every totally real submanifold is CR. A totally real submanifold is maximal if and only if it is generic.
Lemma 8. Let $M \subset \mathbb{C}^{N}$ be a real submanifold of real codimension $d$ and $C R$-dimension $n$. Then the following are equivalent:
(i) $M$ is generic;
(ii) $N=n+d$;
(iii) $T_{p} M+J T_{p} M=\mathbb{C}^{N}$ for every $p \in M$.

Proof. i $\Leftrightarrow$ ii: If the rank of $\rho_{Z}$ is $\tilde{d} \leq d$ and the dimension of the kernel of $\rho_{Z}$ is $n$ we have (since $\left.N=\operatorname{rk} \rho_{Z}+\operatorname{dim} \operatorname{ker} \rho_{Z}\right), N=n+\tilde{d}$. So $M$ is generic if and only if $N=n+d$.
ii $\Leftrightarrow$ iii: Note that for any $E \subset T_{p} M$ with $T_{p} M=T_{p}^{c} M \oplus E$, we necessarily have $J E \cap T_{p} M=\{0\}$. Indeed, if $f \in J E \cap T_{p} M$, then $e=-J f \in T_{p}^{c} M$, so $e=0$, and $f=0$. Hence, $T_{p} M+J T_{p} M=T_{p}^{c} M \oplus E \oplus J E$. So $N=n+d$ iff $T_{p} M+J T_{p} M=\mathbb{C}^{N}$.

Lemma 9. If $M$ is generic, then for every $p \in M$, if $f$ is a germ of a holomorphic function on $\mathbb{C}^{N}$ and $f$ vanishes on $M$, necessarily $f=0$.

Proof. Assume that $f$ is a nontrivial holomorphic function near $p$ which vanishes on $M$. Note that we can choose $f$ such that $\partial f$ does not vanish on $M$ (we can choose $f$ to be a Weierstrass polynomial in some variable of least degree with the property that it vanishes on a neighbourhood of $p$ on $M$, then this is automatic). So there exists a smooth point $q$ of the complex hypersurface $F=\{f=0\} \subset \mathbb{C}^{N}$ on $M$. By assumption $M \subset F$. So $T_{q} M \subset T_{q} F$. But $T_{q} F$ is a complex affine hypersurface, and so $T_{q} M+J T_{q} M \subset T_{q} F+J T_{q} F=T_{q} F \neq \mathbb{C}^{N}$. Hence, condition iii in Lemma 8 does not hold.

### 3.2.3 Special coordinate choices for generic and totally real submanifolds

Proposition 8. If $M \subset \mathbb{C}^{N}$ is a generic submanifold of real codimension $d$, $p \in M$, then there exist holomorphic coordinates $Z=(z, w) \in \mathbb{C}^{n} \times \mathbb{C}^{d}$ in which $p=0$ and $M$ is given near 0 by a defining equation of the form

$$
\operatorname{Im} w=\varphi(z, \bar{z}, \operatorname{Re} w)
$$

where $\varphi(z, \bar{z}, s): \mathbb{C}^{n} \times \mathbb{R}^{d} \rightarrow \mathbb{R}^{d}$ is a smooth function satisfying $\nabla \varphi(0)=0$.
Proof. We choose complex coordinates in which $T_{p}^{c} M \equiv \mathbb{C}^{n} \times\{0\}$. By the proof of Lemma 8, ii $\Rightarrow$ iii, we see that any real subspace $E$ with the property that $T_{p}^{c} M \oplus E=T_{p} M$ is necessarily totally real, and $T_{p}^{c} M \oplus E \oplus J E=\mathbb{C}^{N}$. With a choice of (real) basis $e_{1}, \ldots, e_{d}$ of $E$, we get thus a complex basis of $\mathbb{C}^{d}=E \oplus J E$. Choosing coordinates $w=\left(w_{1}, \ldots, w_{d}\right)$ from the basis homomorphism $w \mapsto w_{1} e_{1}+\cdots+w_{d} e_{d}$ we get coordinates which satisfy the conclusion of Proposition 8.

Proposition 9. If $M \subset \mathbb{C}^{N}$ is a totally real submanifold of real dimension $r$, then for any $p \in M$ there exist coordinates $Z=(z, w) \in \mathbb{C}^{r} \times \mathbb{C}^{N-r}$ and smooth functions $\varphi: \mathbb{R}^{d} \supset U \rightarrow \mathbb{R}^{d}, \psi: \mathbb{R}^{d} \supset U \rightarrow \mathbb{C}^{d}$ such that in these coordinates, $M$ is given (near 0$)$ by $\operatorname{Im} z=\varphi(\operatorname{Re} z)$, $w=\psi(\operatorname{Re} z)$.

Proof. To get these coordinates, just choose a complex complement to the complex subspace $T_{p} M \oplus J T_{p} M$, which we turn into $\mathbb{C}^{r} \times\{0\}$ by a complex affine change of coordinates.

### 3.2.4 Holomorphic submanifolds

Definition 11. A smooth submanifold $M \subset \mathbb{C}^{N}$ is holomorphic if for any $p \in M$ there exists a neighbourhood $U$ of $p$ in $\mathbb{C}^{N}$ and holomorphic coordinates $Z=(z, w) \in \mathbb{C}^{n} \times \mathbb{C}^{d}$ in $U$, centered at $p$, such that in $U M$ is given by $w=0$.
Lemma 10. For a smooth $C R$ submanifold $M \subset \mathbb{C}^{N}$, the following are equivalent:
(i) $M$ is holomorphic;
(ii) $T_{p} M=T_{p}^{c} M$ for all $p \in M$;
(iii) $J T_{p} M=T_{p} M$ for all $p \in M$;
(iv) $2 N=2 n+d$.

Proof. Assuming i, ii-iv hold by simple computations; that they are all equivalent follows by some linear algebra. Now assume that $T_{p} M=T_{p}^{c} M$ holds for all $p \in M$. For any fixed $p \in M$, we choose holomorphic coordinates in which $T_{p} M=\mathbb{C}^{n} \times\{0\}$. Setting $Z=(z, w) \in \mathbb{C}^{n} \times \mathbb{C}^{d}$, $w=s+i t \in \mathbb{R}^{d} \oplus i \mathbb{R}^{d}$, we can thus write defining equations $s-\varphi(z, \bar{z}), t-\psi(z, \bar{z})$, where $\phi, \psi: \mathbb{C}^{n} \supset U \rightarrow \mathbb{R}^{d}$ are smooth functions, vanishing together with their first derivatives at the origin. We claim that $\varphi+i \psi: \mathbb{C}^{n} \supset U \rightarrow \mathbb{C}^{d}$ is holomorphic.

To see this, note that the rank of the $\rho_{\bar{Z}}$ is $d$ on $M$, where $\rho=(\varphi, \psi)$. Computing, we see that

$$
\rho_{\bar{Z}}=\left(\begin{array}{cc}
-\bar{\partial} \varphi & \frac{1}{2} I \\
-\bar{\partial} \psi & -\frac{i}{2} I
\end{array}\right)
$$

Now add $-i$ times the lower block to the upper block to see that the matrix above is row equivalent to

$$
\left(\begin{array}{cc}
-\bar{\partial}(\varphi+i \psi) & 0 \\
-\bar{\partial} \psi & -\frac{i}{2} I
\end{array}\right)
$$

the rank of which is obviously equal to $d+\operatorname{rk}(\bar{\partial}(\varphi+i \psi))$. So $\operatorname{rk}(\bar{\partial}(\varphi+i \psi))=0$, and $\Phi=\varphi+i \psi$ is holomorphic.
We can now choose coordinates as required in Definition 11 by using the the transformation $(z, w) \mapsto$ $(z, w-\Phi(z))$, which is a biholomorphism near $(0,0)$, since its derivative there is the identity.

### 3.3 A preview: Second order invariants

For a smooth CR submanifold $M \subset \mathbb{C}^{N}$, we have the subspaces $T_{p}^{c} M \subset T_{p} M$. Choose a family of smooth vector fields $\left\{X_{1}, \ldots, X_{2 n}\right\}$ which at each point $p \in U \subset M$ spans $T_{p}^{c} M$. We can choose $U$ so small that it is diffeomorphic to $\mathbb{R}^{2 N-d} \times\{0\}$. We can then ask whether the subset $\left\{X_{1}, \ldots, X_{2 n}\right\} \subset \mathfrak{X}(U)$ is integrable, as in Theorem 6. This is measured by whether the Lie bracket

$$
\left[X_{j}, X_{k}\right](p) \in T_{p}^{c} M
$$

or not. But if we have a smooth submanifold $N \subset M$ of real dimension $2 n$ which satisfies the conclusions of Theorem 6, we can apply Lemma 10 to it to see that for any $p \in U$ there exists a holomorphic submanifold $N \subset M$ of (complex) dimension $n$. This is pretty special! The bracket operation above can actually be expressed by a hermition form

$$
T_{p}^{c} M \times T_{p}^{c} M \rightarrow T_{p} M / T_{p}^{c} M
$$

In order to do this most easily, we need to introduce some more concepts. Before we do this, we will first define real-analytic functions and discuss some of their properties.

### 3.4 Real analytic functions and maps

Definition 12. A function $f: \mathbb{R}^{k} \supset U \rightarrow \mathbb{C}$ is real analytic if for each $p \in U$ there exists a neighbourhood $V$ of $U \subset \mathbb{C}^{k}$ and a holomorphic function $F: V \rightarrow \mathbb{C}$ with $\left.F\right|_{U}=f$.

If we consider real analytic functions on $\mathbb{C}^{N}$, instead of considering $\mathbb{C}^{N}$ (with coordinates $Z_{j}$ ) as a subset of $\mathbb{C}^{2} N$ by means of mapping $Z$ to $\left(\operatorname{Re} Z_{1}, \operatorname{Im} Z_{1}, \ldots, \operatorname{Re} Z_{N}, \operatorname{Im} Z_{N}\right)$, it is often advantageous to consider it as a totally real subspace of $\mathbb{C}^{2 N}$ by mapping $Z$ to $(Z, \bar{Z})$. We can the write any real analytic function $f$ in a power series of $Z$ and $\bar{Z}$ :

$$
f(Z, \bar{Z})=\sum_{\alpha, \beta} f_{\alpha, \beta}(Z-p)^{\alpha}(\bar{Z}-\bar{p})^{\beta}
$$

for any $p \in U$, and the set $V$ can be chosen to be of the form $V=W \times \bar{W}$, where we can also assume that the series $f(Z, \zeta)$ converges on $W \times \bar{W}$. Since the subspace $\zeta=\bar{Z}$ of $\mathbb{C}^{2 N}$ with coordinates $(Z, \zeta)$ is totally real, we see that for any real analytic function $f$, if $f(Z, \bar{Z})=0$, then $f(Z, \zeta)=0$.

A real analytic submanifold $M \subset \mathbb{C}^{N}$ is a smooth submanifold which can locally be defined by real analytic defining functions.

### 3.5 CR vectors, the CR bundle, and intrinsic definition of CR manifolds

### 3.5.1 Definition of the CR bundle and abstract CR manifolds

If we now consider the complexified tangent bundle $\mathbb{C} T M$ of a CR manifold, it contains the $(0,1)$ - and (1,0)-parts of $\mathbb{C} T^{c} M$. These are, respectively, the $C R$-bundle and the anti-CR-bundle of $M$; we will denote these subbundles by $\mathcal{V}$ and $\overline{\mathcal{V}}$, respectively. In coordinates, sections of $\mathcal{V}$ are the tangent $(0,1)$-vector fields, that is, tangent vector fields of the form

$$
\sum_{j=1}^{N} a_{j} \frac{\partial}{\partial \bar{Z}_{j}}
$$

which are referred to as $C R$-vector fields, while sections of $\overline{\mathcal{V}}$ are all tangent vector fields of the form

$$
\sum_{j=1}^{N} b_{j} \frac{\partial}{\partial Z_{j}}
$$

which are referred to as anti-CR-vector fields.
We have that $n=\operatorname{dim} \mathcal{V}_{p}$ is the CR dimension of $M$. The subbundle $\mathcal{V}$ is formally integrable, that is, the Lie bracket of every two vector fields with values in $\mathcal{V}$ is again in $\mathcal{V}$; we denote this by

$$
\begin{equation*}
[\mathcal{V}, \mathcal{V}] \subset \mathcal{V} \tag{3.3}
\end{equation*}
$$

$\mathcal{V}$ also has the property that

$$
\begin{equation*}
\mathcal{V} \cap \overline{\mathcal{V}}=\{0\} \tag{3.4}
\end{equation*}
$$

These are the defining properties of an abstract CR structure:
Definition 13. An abstract CR structure on a smooth manifold $M$ is given by a subbundle $\mathcal{V} \subset \mathbb{C} T M$ satisfying (3.3) and (3.4).

The CR-dimension of $M$ is $n=\operatorname{dim}_{\mathbb{C}} \mathcal{V}$. Noting that by (3.4), we have $2 n \leq \operatorname{dim}_{\mathbb{R}} M$, we denote by $d=\operatorname{dim}_{\mathbb{R}} M-2 n$ the CR-codimension of $M$. We write $N=n+d$ (which is the dimension of the ambient space for generic manifolds).

Our considerations above show that every smooth CR submanifold of $\mathbb{C}^{N}$ carries an abstract CR structure. On the other hand, an abstract CR structure is defined without referring to any embedding into $\mathbb{C}^{N}$. Whether or not such a structure is embeddable is a fascinating and quite hard question. We first give the following definition.

Definition 14. An abstract CR structure is integrable if for any point $p \in M$ there exist functions $Z_{1}, \ldots, Z_{N}$ which are annihilated by all CR vetor fields and for which $d Z_{1} \wedge \cdots \wedge d Z_{N}(p) \neq 0$.

Such functions are referred to as basic solutions of the CR structure. The question of local embeddability is equivalent to the question of the existence of basic solutions.

### 3.5.2 CR vector fields for embedded manifolds

As we already said, sections of the CR bundle are the CR vector fields. For an embedded manifold $M$, the CR vector fields are vector fields of the form

$$
L=\sum_{j=1}^{N} a_{j}(Z, \bar{Z}) \frac{\partial}{\partial \bar{Z}_{j}}
$$

where the $a_{j}(Z, \bar{Z})$ are smooth functions defined in some neighbourhood of $M$, which satisfy that $L f=0$ on $M$ whenever $f=0$ on $M$. Equivalently, they annihilate the defining functions of $M$. By the general linear algebra done in the beginning, CR vector fields are in one-to-one correspondence with sections of the bundle $T^{c} M$, as are the anti-CR vector fields. Also, the map $L \mapsto \operatorname{Re} L$ is a (real-linear) homomorphism of $\mathcal{V}$ onto $T^{c} M$.
Example 6. Consider the unit sphere, given by

$$
\left|Z_{1}\right|^{2}+\cdots+\left|Z_{N}\right|^{2}=1
$$

We have CR vector fields of the form

$$
L_{j, k}=Z_{k} \frac{\partial}{\partial \bar{Z}_{j}}-Z_{j} \frac{\partial}{\partial \bar{Z}_{k}}
$$

on any open set $U_{k}$ of the form $\left\{Z_{k} \neq 0\right\}$, the vector fields $L_{j, k}$ for $j \neq k$ constitute a basis of the CR vector fields, i.e. any CR vector field $L$ on $U_{k}$ is of the form

$$
L=\sum_{j \neq k} a_{j}(Z, \bar{Z}) L_{j, k}
$$

for some smooth functions $a_{j}(Z, \bar{Z})$.
More generally, given any defining function $\rho(Z, \bar{Z})$, near a point $p_{0}$ where-say- $\rho_{Z_{k}}\left(p_{0}, \bar{p}_{0}\right) \neq 0$, the CR vector fields are spanned by the vector fields

$$
L_{j, k}=\rho_{\bar{Z}_{j}}(Z, \bar{Z}) \frac{\partial}{\partial \bar{Z}_{k}}-\rho_{\bar{Z}_{k}}(Z, \bar{Z}) \frac{\partial}{\partial \bar{Z}_{j}}
$$

### 3.5.3 Holomorphic and characteristic forms

A form $\omega$ with values in $\mathbb{C} T^{*} M$ is said to be holomorphic if it annihilates all CR vector fields, that is, if $\omega_{p} \in \mathcal{V}_{p}^{\perp}=T_{p}^{\prime} M$. A characteristic form $\theta$ is a form which annihilates both CR and anti CR vector fields, that is, $\theta_{p} \in\left(\mathcal{V}_{p} \oplus \overline{\mathcal{V}_{p}}\right)^{\perp}=T_{p}^{0} M$, and which is real (that is, it is real valued on real elements of $\mathbb{C} T M$ ).
Lemma 11. Assume that $M \subset \mathbb{C}^{N}$ is a generic real submanifold, and denote $\iota: M \rightarrow \mathbb{C}^{N}$ the embedding of $M$ into $\mathbb{C}^{N}$. Then the space of holomorphic forms is spanned by $\iota^{*} d Z_{1}, \ldots \iota^{*} d Z_{N}$. If $\rho=\left(\rho_{1}, \ldots, \rho_{d}\right)$ is a real-valued defining function of $M$, then a basis of the space of characteristic forms is given by

$$
i \iota^{*}(\partial-\bar{\partial}) \rho_{1}, \ldots, i \iota^{*}(\partial-\bar{\partial}) \rho_{d}
$$

Proof. First, note that since $\operatorname{dim} \mathbb{C} T_{p} M=2 n+d$, and $\operatorname{dim} \mathcal{V}_{p}=n$, the dimension of the space of holomorphic forms is $N=n+d$. It is clear from the explicit description of the CR vectors as tangent linear combinations of the $\frac{\partial}{\partial Z_{j}}$ that they are all annihilated by the $\iota^{*} d Z_{j}$. We claim that those are linearly independent. If they are not, there exists a complex linear combination $\sum a_{j} d Z_{j}$ which annihilates all vectors in $\mathbb{C} T_{p} M$; in particular, it annihilates all real tangent vectors. But this implies that all real tangent vectors are in the complex plane $\sum a_{j} Z_{j}=0$, which contradicts the genericity of $M$.

Again, the fact that the vectors in $\mathcal{V}$ are tangent linear combinations of the $\frac{\partial}{\partial Z_{j}}$ implies that all of the forms $\theta_{j}=i \iota^{*}(\partial-\bar{\partial}) \rho_{j}$ annihilate $\mathcal{V}$ and $\overline{\mathcal{V}}$. The genericity of $M$ implies that they are linearly independent. We are left with checking that they are also real. But when acting on real elements, $\overline{\theta_{j}}=\theta_{j}$, which is exactly what is needed.

### 3.5.4 The Levi-form (and the Levi-map)

Here goes a definition of that thing, plus basic properties.

### 3.5.5 CR functions

A smooth function $f$ on $M$ is a CR function if it is annihilated by all CR vector fields. The prototypical example of a CR function on an embedded manifold is the restriction of a holomorphic function to that manifold. However, not all CR functions are of this form. On the other hand, a holomorphic function is given by a convergent power series at any point; we can associate to a CR function its formal holomorphic series (which neither necessarily converges nor fulfills the identity principle). But for some constructions, it gives us a good tool to work with.

Proposition 10. Let $M$ be an abstract $C R$ manifold, which is integrable, and assume that $Z_{1}, \ldots, Z_{N}$ are basic solutions near $p \in M$. Then for each $C R$ function $f$ defined near $p$, there exists a unique formal power series $T_{p} f=\sum A_{\alpha} Z^{\alpha} \in \mathbb{C}[[Z]]$ which satisfies that for each $k$,

$$
f(q)-\sum_{|\alpha| \leq k} A_{\alpha} Z(q)^{\alpha}
$$

vanishes to order $k$ at $p$.
Proof. We proceed by induction and start by choosing $A_{0}=f(p)$. Now assume that we have already chosen the $A_{\alpha}$ for $|\alpha|<k$, and that they have the property that $\left.D\left(f-\sum_{|\alpha|<k} A_{\alpha} Z^{\alpha}\right)\right|_{p}=0$ for every differential operator of order less than $k$.

We let $L_{1}, \ldots, L_{n}$ be a basis near $p$ of the space of CR vector fields, and choose vector fields $X_{1}, \ldots, X_{N}$ such that $L_{1}, \ldots, L_{n}, X_{1}, \ldots, X_{N}$ forms a local basis of sections of $\mathbb{C} T M$. Since $Z_{1}, \ldots, Z_{N}$ are basic solutions, their differentials are linearly independent, and so for each $k$, the matrix $\left(\left.X^{\alpha} Z^{\beta}\right|_{p}\right)_{|\alpha|=|\beta|=k}$ is invertible.

We define the $A_{\beta}$ for $|\beta|=k$ as the solution to the equations

$$
\left.X^{\alpha}\left(f-\sum_{|\gamma|<k} A_{\gamma} Z^{\gamma}\right)\right|_{p}=\left.\sum_{|\beta|=k} A_{\beta} X^{\alpha} Z^{\beta}\right|_{p}
$$

which exists and is unique. We still have to verify that the solution thus obtained satisfies that its difference with $f$ annihilates all differential operators of order less or equal to $k$. For differential operators of length less than $k$, this follows from the induction hypothesis and the product rule. If we have a differential operator of length $k$, it agrees with an operator whose terms of degree $k$ are actually of the form $X^{\alpha} L^{\beta}$, and if $|\beta|>0$, those annihilate the difference because the difference is CR.

## 4 Segre varieties and maps

Let $M \subset \mathbb{C}^{N}$ be a generic, real-analytic submanifold. We assume that near $p_{0} \in M, M$ is given by a defining function $\varrho(Z, \zeta)$, and that $\varrho(Z, \zeta)$ is defined on a set of the form $U \times \bar{U}$, and also, that the rank of $\varrho_{Z}(Z, \zeta)$ (and $\left.\varrho_{\zeta}(Z, \zeta)\right)$ remains maximal $(=d)$ throughout $U \times \bar{U}$.

For $q \in U$, we define the Segre-variety of $q$ by

$$
S_{q}(U)=\{Z \in U: \varrho(Z, \bar{q})=0\} .
$$

Note that $S_{q}$ is a complex submanifold (of dimension $n$ ) of $U$. While the definition is dependent on the neighbourhood $U$, we will henceforth suppress $U$ from the notation, and understand that all statements are understood locally near some reference point $p$. We first state two basic properties of Segre-varieties, whose proofs are immediate.

Lemma 12. For $p$ and $q$ near $p_{0}$, the following holds.
(i) $q \in S_{p} \Leftrightarrow p \in S_{q}$;
(ii) $p \in S_{p} \Leftrightarrow p \in M$;

For the next property, which loosely stated implies that Segre-varieties transform nicely under holomorphic mappings, we introduce the following terminology: Let $M \subset \mathbb{C}^{N}$ and $M^{\prime} \subset \mathbb{C}^{N^{\prime}}$ be real-analytic hypersurfaces, $p \in M, p^{\prime} \in M^{\prime}$. We say that a holomorphic map $H: \mathbb{C}^{N} \supset U \rightarrow \mathbb{C}^{N^{\prime}}$ defined on a neighbourhood $U$ of $p$ takes $(M, p)$ into $\left(M^{\prime}, p^{\prime}\right)$ and write $H:(M, p) \rightarrow\left(M^{\prime}, p^{\prime}\right)$ if $H(p)=p^{\prime}$ and there exists a neighbourhood $V$ of $p$ with $H(M \cap V) \subset M^{\prime}$. We shall denote the Segre varieties associated to $M^{\prime}$ by $S_{q^{\prime}}^{\prime}$.
Lemma 13. If $H:(M, p) \rightarrow\left(M^{\prime}, p^{\prime}\right)$ is a holomorphic map taking $(M, p)$ into $\left(M^{\prime}, p^{\prime}\right)$, then for $q$ close by $p, H\left(S_{q}\right) \subset S_{H(q)}^{\prime}$.

Proof. Choosing defining function $\varrho$ and $\varrho^{\prime}$, we see that the assumption that $H:(M, p) \rightarrow\left(M^{\prime}, p^{\prime}\right)$ is equivalent to $\varrho^{\prime}(H(Z), \bar{H}(\zeta))=A(Z, \zeta) \varrho(Z, \zeta)$ for $(Z, \zeta)$ close by $(p, \bar{p})$. Thus, if $Z \in S_{\bar{\zeta}}, \varrho^{\prime}(H(Z), \bar{H}(\zeta))=0$, which means that $H(Z) \in S_{H(\bar{\zeta})}^{\prime}$.

Problem 10. Let $\mathbb{S}^{2 N-1}$ be the unit sphere in $\mathbb{C}^{N}$. For an entire map $H$ mapping $\mathbb{S}^{2 N-1}$ into $\mathbb{S}^{2 N^{\prime}-1}$ and a hyperplane $E \subset \mathbb{C}^{N}$, determine a hyperplane $E^{\prime} \subset \mathbb{C}^{N^{\prime}}$ with $H(E) \subset E^{\prime}$.

A lot of information about the geometry of $M$ is encoded in its Segre-varieties. First, we will discuss the notion of type.

### 4.1 Finite type

Let $M$ be an abstract CR manifold, $p \in M$. We say that $M$ is of finite type at $p$ if the Lie-algebra generated by the CR and the anti-CR vector fields, evaluated at $p$, gives $\mathbb{C} T_{p} M$.

## Example 7.

??? From here Michael included the topics we treated in Bernhard's lecture in summer term 2011, starting with the section about real-analytic functions on p. 67 of Michael's notes ???
??? The definition of a real-analytic function is already defined in a previous section ???

## 5 Real-Analytic Functions and their Complexification

Definition 15. Let $U \subset \mathbb{R}^{k}$ be open. We say that the function $f: U \rightarrow \mathbb{C}$ is real-analytic, if for each $p \in U$ there exists a neighborhood $V \subset \mathbb{C}^{k}$ of $p$ and a holomorphic function $F: V \rightarrow \mathbb{C}$ such that $\left.F\right|_{V \cap \mathbb{R}^{k}}=f$.
Remark 2. (i) An equivalent definition that a function $f: \mathbb{R}^{k} \supset U \rightarrow \mathbb{C}$ is real-analytic is the following: for all $p \in U$ there exist constants $M, r>0$ and complex numbers $\left(a_{\alpha}\right)_{\alpha \in \mathbb{N}^{k}}$ with $\left|a_{\alpha}\right| \leq \frac{M}{r^{|\alpha|}}$ such that in a neighborhood of $p$ we have $f(x)=\sum_{\alpha} a_{\alpha}(x-p)^{\alpha}$.
(ii) The radius of convergence of a real-analytic function essentially depends on $V \subset \mathbb{C}^{k}$ as comparing the power series expansion of the function $x \mapsto \frac{1}{1+x^{2}}$ at 0 and 1 shows.
(iii) In Definition 15 we could realize $\mathbb{R}^{k}$ in $\mathbb{C}^{k}$ by choosing coordinates $z=\left(z_{1}, \ldots, z_{k}\right)$ in $\mathbb{C}^{k}$ and identifying $\mathbb{R}^{k}$ in $\mathbb{C}^{k}$ as the set $\left\{z \in \mathbb{C}^{k}: \operatorname{Im} z=0\right\}$, but other maximally totally real subspaces of $\mathbb{C}^{k}$ can be used. Choosing any basis $\left\{v_{1}, \ldots, v_{k}\right\}$ of $\mathbb{C}^{k}$ we realize $\mathbb{R}^{k}$ in $\mathbb{C}^{k}$ via $\mathbb{R}^{k} \ni\left(x_{1}, \ldots, x_{k}\right) \mapsto \sum_{j=1}^{k} x_{j} v_{j}$. More generally if $L: \mathbb{R}^{k} \rightarrow \mathbb{C}^{k}$ is a real-linear mapping with real-linearly independent components, then the image of $L$ under $\mathbb{R}^{k}$ is a maximally totally real subspace of $\mathbb{C}^{k}$. The same works if we take $k=2 N$ in Definition 15, i.e., if we start with a real-analytic function $f: \mathbb{C}^{N} \supset U \rightarrow \mathbb{C}$. Similarly we can realize $\mathbb{C}^{N}$ as a maximally totally real subspace of $\mathbb{C}^{2 N}$ by taking $L: \mathbb{C}^{N} \rightarrow \mathbb{C}^{2 N}$, a real-linear mapping with real-linearly independent components. This leads to the definition of real-analyticity of functions in complex spaces.
Definition 16. Let $\widetilde{L}: \mathbb{C}^{N} \rightarrow \mathbb{C}^{2 N}$ be a real-linear mapping with real-linearly independent components. Define $E:=\operatorname{image}(\widetilde{L})$ and let $\Omega \subset \mathbb{C}^{N}$ be open. Then $f: \Omega \rightarrow \mathbb{C}$ is called real-analytic, if there exists $\widetilde{\Omega} \subset \mathbb{C}^{2 N}$ with $\widetilde{\Omega} \cap E=\Omega$ and a holomorphic function $F: \widetilde{\Omega} \rightarrow \mathbb{C}$ such that $F \circ \widetilde{L}=f$.
Remark 3. This definition gives us the freedom to take a convenient subspace for $E$, i.e, the realization of $\mathbb{C}^{N}$ in $\mathbb{C}^{2 N}$ can be seen as follows:
(i) As in the beginning we identify $\mathbb{C}^{N} \cong \mathbb{R}^{2 N}$ by setting $x_{j}=\operatorname{Re}\left(z_{j}\right)$ and $y_{j}=\operatorname{Im}\left(z_{j}\right)$. Then we consider the real-linear mapping $\left(x_{1}, y_{1}, \ldots, x_{N}, y_{N}\right) \mapsto\left(x_{1}+\mathrm{i} y_{1}, x_{1}-\mathrm{i} y_{1}, \ldots, x_{N}+\mathrm{i} y_{N}, x_{N}-\mathrm{i} y_{N}\right)$ and recognize $z_{j}=x_{j}+\mathrm{i} y_{j}$ and $\bar{z}_{j}=x_{j}-\mathrm{i} y_{j}$. So we realized $\mathbb{C}^{N}$ in $\mathbb{C}^{2 N}$ via $\widetilde{L}\left(z_{1}, \ldots, z_{N}\right)=\left(z_{1}, \bar{z}_{1}, \ldots, z_{N}, \bar{z}_{N}\right)$ and hence $E=\left\{(Z, \zeta)=\left(z_{1}, \ldots, z_{N}, \zeta_{1}, \ldots, \zeta_{N}\right) \in \mathbb{C}^{2 N}: \zeta=\bar{Z}\right\}$.
(ii) There is also an argument from Algebra to get to the above $\widetilde{L}$. If we consider $\mathbb{C}$ as the field expansion of $\mathbb{R} \subset \mathbb{C}$, then the group of automorphisms of $\mathbb{C}$ fixing $\mathbb{R}$ is a Galois group and consists of the identity and the complex conjugation, such that $\widetilde{L}\left(z_{1}, \ldots, z_{N}\right)=\left(z_{1}, \ldots, z_{N}, \bar{z}_{1}, \ldots, \bar{z}_{N}\right)$.
(iii) These choices for $\widetilde{L}$ and $E$ have the following consequence for real-analytic functions $f: \mathbb{C}^{N} \rightarrow \mathbb{C}$ : As above we write $z_{j}=x_{j}+\mathrm{i} y_{j}$ and $Z=\left(z_{1}, \ldots, z_{N}\right)$, then $f(Z)=\sum_{\alpha, \beta} a_{\alpha \beta} x^{\alpha} y^{\beta}=\sum_{\gamma, \delta} b_{\gamma \delta} z^{\gamma} \bar{z}^{\delta}=$ $f(Z, \bar{Z})$, where we wrote $x_{j}=\frac{z_{j}+\bar{z}_{j}}{2}$ and $y_{j}=\frac{z_{j}-\bar{z}_{j}}{2 \mathrm{i}}$ for the second equality. Hence the holomorphic function $F$ from Definition 16 is given by $F(Z, \zeta):=f(Z, \zeta)$. Realizing a function defined on $\mathbb{C}^{N}$ as a function on $\mathbb{C}^{2 N}$ is called complexification. Since $E$ is a maximally totally real subspace we also have the complexification principle, which says that if a real-analytic function $f$ satisfies $f(Z, \bar{Z}) \equiv 0$, then also $f(Z, \zeta) \equiv 0$. We can apply this principle to real-analytic submanifolds.
Definition 17. Let $M$ be a real-analytic submanifold of $\mathbb{C}_{Z}^{N}$. Then there is a complex submanifold $\mathcal{M}$ of $\mathbb{C}_{(Z, \zeta)}^{2 N}$ with $\mathcal{M} \cap\left\{(Z, \zeta) \in \mathbb{C}^{2 N}: \zeta=\bar{Z}\right\}=M$. We call $\mathcal{M}$ the complexification of $M$.
Remark 4. (i) One can show that $M$ is a maximally totally real submanifold of $\mathcal{M}$ and the germ of $\mathcal{M}$ is unique near $\left\{(Z, \zeta) \in \mathbb{C}^{2 N}: \zeta=\bar{Z}\right\}$.
(ii) Again we have the complexification principle for real-analytic functions on $M$, i.e., if $f(Z, \bar{Z})=0$ on $M$, then $f(Z, \zeta)=0$ on $\mathcal{M}$.
(iii) Note that we have the following structure on $\mathcal{M}$. If $\rho$ is a local defining function for $M$, then we have by the reality of $\rho$ on $M$ that $\rho(Z, \bar{Z})=\bar{\rho}(\bar{Z}, Z)$ and thus on $\mathcal{M}$ we have $\rho(Z, \zeta)=\bar{\rho}(\zeta, Z)$.
Example 8. We conclude this section by applying the complexification principle to holomorphic mappings of real-analytic submanifolds. Let $(M, 0) \subset \mathbb{C}^{N}$ and $\left(M^{\prime}, 0\right) \subset \mathbb{C}^{N^{\prime}}$ be germs of real-analytic submanifolds with defining functions $\rho$ and $\rho^{\prime}$ and consider the associated complexifications $\mathcal{M}$ and $\mathcal{M}^{\prime}$. Let $H: M \subset U \rightarrow \mathbb{C}^{N^{\prime}}$ be a holomorphic mapping with $H(0)=0$ and $H(U \cap M) \subset M^{\prime}$. The last property means that $\rho^{\prime}(H(Z), \overline{H(Z)})=0$ if $\rho(Z, \bar{Z})=0$. This is equivalent by saying that there exists a real-analytic, real matrix-valued function $A$ such that $\rho^{\prime}(H(Z), \overline{H(Z)})=A(Z, \bar{Z}) \rho(Z, \bar{Z})$ as can be seen as follows: We write $Z=\left(x_{1}, \ldots, x_{2 N}\right)$ and after rotating and reordering the coordinates, we can assume that $\left(\left(\rho_{j}^{\prime}\right)_{x_{k}}(0)\right)_{1 \leq j \leq d, 2 N-d+1 \leq k \leq 2 N}$ is of full rank, hence by the Implicit Function Theorem we can choose coordinates $y_{j}:=x_{j}$ for $1 \leq j \leq 2 N-d$ and $y_{2 N-d+j}:=\rho_{j}^{\prime}(Z)$ for $1 \leq j \leq d$. In these new coordinates the
function $f:=\rho^{\prime} \circ H$ satisfies $f\left(y_{1}, \ldots, y_{2 N-d}, 0, \ldots, 0\right)=0$ and thus

$$
\begin{aligned}
f\left(y_{1}, \ldots, y_{2 N}\right)= & f\left(y_{1}, \ldots, y_{2 N}\right)-f\left(y_{1}, \ldots, y_{2 N-1}, 0\right)+f\left(y_{1}, \ldots, y_{2 N-1}, 0\right)-f\left(y_{1}, \ldots, y_{2 N-2}, 0,0\right) \\
& +\ldots+f\left(y_{1}, \ldots, y_{2 N-d+1}, 0, \ldots, 0\right)-f\left(y_{1}, \ldots, y_{2 N-d}, 0, \ldots, 0\right) \\
= & \sum_{l=1}^{d} \int_{0}^{1} y_{2 N-d+l} f_{y_{2 N-d+l}}\left(y_{1}, \ldots, y_{2 N-d}, y_{2 N-d+1}, \ldots, y_{2 N-d+l-1}, t y_{2 N-d+l}, 0, \ldots, 0\right) d t \\
= & \sum_{l=1}^{d} y_{2 N-d+l} a_{l}\left(y_{1}, \ldots, y_{2 N}\right)
\end{aligned}
$$

where $\left(a_{l}\right)_{1 \leq l \leq d}$ is the $d^{\prime} \times d$-matrix $A$. Then the complexified version of $\rho^{\prime}(H(Z), \overline{H(Z)})=A(Z, \bar{Z}) \rho(Z, \bar{Z})$ is $\rho^{\prime}(H(Z), \bar{H}(\zeta))=A(Z, \zeta) \rho(Z, \zeta)$ and we obtain the corresponding mapping $\mathcal{H}(Z, \zeta)=(H(Z), \bar{H}(\zeta))$, which sends $\mathcal{M}$ to $\mathcal{M}^{\prime}$. We will refer to equations of such form as mapping equations.

## 6 The Segre Variety

Definition 18. Let $M$ be a generic and real-analytic submanifold of $\mathbb{C}^{N}$ and $\rho$ be a real-analytic defining function for $M$ defined in some $V \times \bar{V} \subset \mathbb{C}^{2 N}$. The Segre variety $\mathcal{S}_{p}$ of $p \in V$ is the complex variety given by

$$
\mathcal{S}_{p}:=\mathcal{S}_{p}(U):=\{Z \in U: \rho(Z, \bar{p})=0\},
$$

for $U \subset V$ a small neighborhood around $p$.
Remark 5. (i) If $p \in V$, there is a small neighborhood $U \subset V$ of $p$, such that $\mathcal{S}_{p}(U)$ is nonempty and $\mathcal{S}_{p}$ is uniquely determined: Let $p \in U$ and $d$ denote the codimension of $M$. Since we assume that $M$ is generic, we can reorder the components $\rho_{1}, \ldots, \rho_{d}$ of $\rho$ and the coordinates $\left(z_{1}, \ldots, z_{N}\right)$ of $\mathbb{C}^{N}$, such that

$$
\left|\begin{array}{ccc}
\rho_{1 z_{n+1}} & \cdots & \rho_{1 z_{N}} \\
\vdots & & \vdots \\
\rho_{d z_{n+1}} & \cdots & \rho_{d z_{N}}
\end{array}\right|(p, \bar{p}) \neq 0
$$

Now we complexify the equation $\rho(Z, \bar{p})=0$, i.e., we consider $\bar{p}$ as fixed and write $\rho_{\bar{p}}(Z):=\rho(Z, \bar{p})$. Further we denote $Z=\left(Z^{\prime}, Z^{\prime \prime}\right) \in \mathbb{C}^{n} \times \mathbb{C}^{d}$. Since $\rho_{\bar{p}}(p)=0$ and $\frac{\partial}{\partial Z^{\prime \prime}} \rho_{\bar{p}}\left(Z^{\prime}, Z^{\prime \prime}\right)$ is invertible for $\left(Z^{\prime}, Z^{\prime \prime}\right) \in W \subset V$, a neighborhood of $p \in V$, we apply the Implicit Function Theorem. This gives the existence of neighborhoods $\left(U^{\prime}, U^{\prime \prime}\right) \subset \mathbb{C}^{n} \times \mathbb{C}^{d}$ of $p^{\prime}$ and $p^{\prime \prime}$, respectively, and a holomorphic mapping $Z^{\prime \prime}: U^{\prime} \rightarrow U^{\prime \prime}$, such that $\left(q^{\prime}, Z^{\prime \prime}\left(q^{\prime}\right)\right) \in U^{\prime} \times U^{\prime \prime} \subset W$, satisfies $\rho_{\bar{p}}\left(q^{\prime}, Z^{\prime \prime}\left(q^{\prime}\right)\right)=0$ for all $q^{\prime} \in U^{\prime}$. To sum up, there exists $Z=\left(q^{\prime}, Z^{\prime \prime}\left(q^{\prime}\right)\right)$ for $q^{\prime}$ being in a neighborhood of $p^{\prime}$, such that $\rho(Z, \bar{p})=0$ and the neighborhood $U$ can be set to $U^{\prime} \times U^{\prime \prime}$. Note that in general the Segre set is not open, but we have shown that $\mathcal{S}_{p}$ is a complex submanifold of codimension $d$. Further, since the coefficients of $\rho_{\bar{p}}$ depend real-analytically on $\bar{p}$, we have a real-analytic dependence of $Z$ on $\bar{p}$.
The uniqueness of $\mathcal{S}_{p}$ follows from the fact that local defining functions for $\mathcal{S}_{p}$ only differ by a unit in the space of holomorphic functions.
(ii) If $\mathcal{M}$ is the complexification of $M$, then $\mathcal{S}_{\bar{\zeta}}=\left\{Z \in \mathbb{C}^{N}:(Z, \zeta) \in \mathcal{M}\right\}$.

Proposition 11. The Segre variety $\mathcal{S}_{p}$ has the following properties:
(i) $p \in \mathcal{S}_{q} \Leftrightarrow q \in \mathcal{S}_{p}$
(ii) $p \in \mathcal{S}_{p} \Leftrightarrow p \in M$
(iii) If $H$ is a holomorphic mapping sending $(M, p)$ to $\left(M^{\prime}, p^{\prime}\right)$ with $H(p)=p^{\prime}$, then we have $H\left(\mathcal{S}_{p}\right) \subset \mathcal{S}_{H(p)}^{\prime}$.

Proof. (i) and (ii) are immediately from the definition of $\mathcal{S}_{p}$. For (iii) we take $\rho$ and $\rho^{\prime}$ defining functions for $M$ and $M^{\prime}$ near $p$ and $p^{\prime}$. As in Example $8, H$ has to satisfy $\varrho^{\prime}(H(Z), \bar{H}(\zeta))=A(Z, \zeta) \varrho(Z, \zeta)$ for $(Z, \zeta)$ close by $(p, \bar{p})$. Thus, if $Z \in \mathcal{S}_{\bar{\zeta}}, \varrho^{\prime}(H(Z), \bar{H}(\zeta))=0$, which means that $H(Z) \in \mathcal{S}_{H(\bar{\zeta})}^{\prime}$.

Example 9. Geometrically Proposition 11 (iii) can be viewed as a reflection principle, which can easily be observed in $\mathbb{C}$. We define $\Omega_{+}:=\{z \in \mathbb{C}: \operatorname{Im} z>0\}$ and $\Omega_{-}:=\{z \in \mathbb{C}: \operatorname{Im} z<0\}$. Let $f \in \mathcal{H}\left(\Omega_{-}\right) \cap \mathcal{C}\left(\bar{\Omega}_{-}\right)$ with $f(\mathbb{R}) \subset \mathbb{R}$. Since $\mathbb{R} \subset \mathbb{C}$ is given by $\rho(z, \bar{z}):=z-\bar{z}$ as a real-analytic submanifold in $\mathbb{C}$, we have $\mathcal{S}_{w}=\{\bar{w}\}$ and Proposition 11 (iii) says that $\{f(\bar{w})\} \subset\{\overline{f(w)}\}$. This suggests to define the holomorphic extension of $f$ to $\Omega_{+}$by $f(\bar{z}):=\overline{f(z)}$, which is known as the Schwarz reflection principle.

## 7 Normal Coordinates

Motivation 1. Before we continue with iterating the Segre varieties, we introduce special coordinates for real-analytic submanifolds such that the Segre varieties are given by easy-to-handle equations.

In Proposition 8 we introduced coordinates for a generic and real-analytic submanifold $M \subset \mathbb{C}^{N}$ which are given by $w-\tau=2 \mathrm{i} \varphi\left(z, \chi, \frac{w+\tau}{2}\right)$ in the complexified form. Applying the Implicit Function Theorem and solving for $w$, we obtain that $M$ is given by $w=Q(z, \chi, \tau)$. Then $\mathcal{S}_{\left(0, w_{0}\right)}$ is given by $w=Q\left(z, 0, w_{0}\right)$. Intuitively we want to change coordinates by setting $\widetilde{w}=w+Q\left(z, 0, w_{0}\right)$ to obtain that $\mathcal{S}_{\left(0, w_{0}\right)}$ is "flat", i.e., to have $\mathcal{S}_{\left(0, w_{0}\right)}=\left\{(z, w) \in \mathbb{C}^{N}: w=\bar{w}_{0}\right\}$. This implies that $Q\left(z, 0, \bar{w}_{0}\right)=\bar{w}_{0}$ or, since $z \in \mathcal{S}_{\bar{w}}$ if and only if $w \in \mathcal{S}_{\bar{z}}, Q\left(0, \chi, \bar{w}_{0}\right)=\bar{w}_{0}$. Also note that in this case $\mathcal{S}_{0}=T_{p}^{c}(M)$ for $p$ near 0 and geometrically this means that we graphed $M$ over its complex tangent space.
Definition 19. $(z, w) \in \mathbb{C}^{n} \times \mathbb{C}^{d}$ are called normal coordinates for a generic and real-analytic submanifold $M$ at 0 , if for $w_{0}$ near 0 we have $\mathcal{S}_{\left(0, w_{0}\right)}=\left\{w=\bar{w}_{0}\right\}$ or equivalent, for the complexification of $M$, there exists a holomorphic $Q: \mathbb{C}^{n} \times \mathbb{C}^{n} \times \mathbb{C}^{d} \rightarrow \mathbb{C}^{d}$ with $Q(z, 0, \tau)=\tau=Q(0, \chi, \tau)$.
Remark 6. (i) If $\rho$ is a defining function for $\mathcal{M}$, then yet another equivalent condition for ( $z, w$ ) being normal coordinates is that $\rho(z, w, 0, w)=0$.
(ii) The property $Q(z, 0, \tau)=\tau=Q(0, \chi, \tau)$ is called normality condition.
(iii) In normal coordinates we have the following reality condition: Together with $w=Q(z, \chi, \tau)$ we have $\tau=\bar{Q}(\chi, z, w)$, thus $w=Q(z, \chi, \bar{Q}(\chi, z, w))$ as well as $\tau=\bar{Q}(\chi, z, Q(z, \chi, \tau))$.

Lemma 14. For a generic and real-analytic submanifold $M$ there always exist normal coordinates.
Proof. Since $M$ is generic, by Proposition 8, we can find coordinates $\left(z^{\prime}, w^{\prime}\right) \in \mathbb{C}^{n} \times \mathbb{C}^{d}$ such that $M=$ $\left\{\operatorname{Im} w^{\prime}=\varphi\left(z^{\prime}, \bar{z}^{\prime}, \operatorname{Re} w^{\prime}\right)\right\}$, where $\varphi$ is a real-analytic function with $\nabla \varphi(0)=0$. Complexifying this equation we obtain the defining function $\rho^{\prime}\left(z^{\prime}, w^{\prime}, \chi^{\prime}, \tau^{\prime}\right):=w^{\prime}-\tau^{\prime}-2 \mathrm{i} \varphi\left(z^{\prime}, \chi^{\prime}, \frac{w^{\prime}+\tau^{\prime}}{2}\right)$ for $\mathcal{M}$. Since there are no linear terms in $\varphi$ we have $\rho_{w^{\prime}}(0)=I_{d \times d}-2 \mathrm{i} \varphi_{w^{\prime}}(0)=I_{d \times d}$ and we can solve for $w^{\prime}$ by the Implicit Function Theorem to obtain that $\mathcal{M}$ is given by $\rho^{\prime \prime}\left(z^{\prime}, w^{\prime}, \chi^{\prime}, \tau^{\prime}\right):=w^{\prime}-Q\left(z^{\prime}, \chi^{\prime}, \tau^{\prime}\right)$. To prove the Lemma we need to determine a holomorphic change of coordinates $\left(z^{\prime}, w^{\prime}\right)=(f(z, w), g(z, w))$, such that in the new coordinates $(z, w)$ we have $Q(z, 0, \tau)=\tau=Q(0, \chi, \tau)$. An equivalent formulation is that $\rho(z, w, \chi, \tau):=$ $\rho^{\prime \prime}(f(z, w), g(z, w), \bar{f}(\chi, \tau), \bar{g}(\chi, \tau))=g(z, w)-Q(f(z, w), \bar{f}(\chi, \tau), \bar{g}(\chi, \tau))$ has to satisfy $\rho(z, w, 0, w)=0$. Since there are no linear terms in $\varphi$, the only linear term in $Q(z, \chi, \tau)$ is $\tau$. We write $Q(z, \chi, \tau)=\tau+\widetilde{Q}(z, \chi, \tau)$, where in $\widetilde{Q}$ only terms of order 2 in $(z, \chi, \tau)$ occur. Let's see what conditions we get for $(f, g)$, if we require that $(z, w)$ are normal coordinates:

$$
\begin{align*}
0=\rho(z, w, 0, w) & =g(z, w)-Q(f(z, w), \bar{f}(0, w), \bar{g}(0, w))  \tag{7.1}\\
& =g(z, w)-\bar{g}(0, w)-\widetilde{Q}(f(z, w), \bar{f}(0, w), \bar{g}(0, w))
\end{align*}
$$

From above we see that good choices for $f$ and $g$ are $g(z, w)=g(w)$ and $f(z, w)=z$ to simplify matters, then we need to assume for $g$ :

$$
\begin{equation*}
g(w)-\bar{g}(w)=\widetilde{Q}(z, 0, \bar{g}(w)) \tag{7.2}
\end{equation*}
$$

Since we are dealing with transformations we need $g_{w}(0) \neq 0$, hence we set $g(w)=w+\widetilde{G}(w)$, where $\widetilde{G}$ is of order 2 . Then we note that $(7.2)$ is a condition for $\operatorname{Im}(\widetilde{G})$, hence we take $\widetilde{G}=\mathrm{i} G$, where $G$ only consists of real coefficients, i.e., $G(w)=\bar{G}(w)$. Now we need to check, if such a holomorphic $G$ actually exists: We plug in our choices for $g$ into (7.2), set $z=0$ to obtain, since we require $G(w)=\bar{G}(w)$, that

$$
w+\mathrm{i} G(w)-(w-\mathrm{i} \bar{G}(w))=\widetilde{Q}(0,0, w+\mathrm{i} \bar{G}(w)) \Leftrightarrow 2 \mathrm{i} G(w)=\widetilde{Q}(0,0, w+\mathrm{i} \bar{G}(w))
$$

and since the left-hand side is of order 2 in $G(w)$ we can solve this equation by the Implicit Function Theorem to obtain a holomorphic $G$. Finally we need to check if the so attained $G$ satisfies $G(w)=\bar{G}(w)$, which was our assumption in the first place. Since our transformations $f$ and $g$ satisfy the condition in (7.1), we plug $\bar{g}(\chi, \tau)=g(z, w)=w+\mathrm{i} G(w)$ and $\bar{f}(\chi, \tau)=f(z, w)=z$ into $\rho$ and set $z=0$ to obtain

$$
w+\mathrm{i} G(w)=Q(0,0, w-\mathrm{i} G(w))
$$

and after conjugating this equation and replacing $\bar{w}$ by $w$ we obtain

$$
w-\mathrm{i} \bar{G}(w)=\bar{Q}(0,0, w+\mathrm{i} \bar{G}(w))
$$

Finally we combine the two previous equations and use the reality condition to obtain $G(w)=\bar{G}(w)$.
Remark 7. Note that normal coordinates are not unique, since they depend on the choices of $f$ and $\operatorname{Re}(g)$.

## 8 Iterated Segre Varieties and Segre Mappings

??? Fill in Bernhard's draft to illustrate the iterations ???
We can actually iterate the Segre varieties as follows:
Definition 20. We define

$$
\begin{aligned}
& \mathcal{S}_{p}^{1}:=\mathcal{S}_{p}, \\
& \mathcal{S}_{p}^{k}:=\bigcup_{q \in \mathcal{S}_{p}^{k-1}} \mathcal{S}_{q},
\end{aligned}
$$

for $k \geq 2$, called the $k$-th iterated Segre variety $\mathcal{S}_{p}^{k}$ at $p$.
Remark 8. In what follows we want to give a description of $\mathcal{S}_{p}^{k}$ in terms of normal coordinates. If $M$ is given in normal coordinates $(z, w)$ such that $w=Q(z, \chi, \tau)$, we also have $\tau=\bar{Q}(\chi, z, w)$ and $Q: \mathbb{C}^{2 n+d} \supset$ $U \rightarrow \mathbb{C}^{d}$ is holomorphic satisfying the normality condition. We also assume $U$ to be small enough such that $Q: \mathbb{C}^{2 n+d} \supset V \times V \times W \rightarrow W$.
Then we have $\mathcal{S}_{p} \cap\{V \times W\}=\left\{\left(z_{1}, w_{1}\right) \in V \times W: w_{1}=Q\left(z_{1}, \bar{p}\right), p \in V \times W\right\}=\left\{\left(z_{1}, w_{1}\right) \in V \times W:\right.$ $\left.\bar{w}_{1}=\bar{Q}\left(\bar{z}_{1}, p\right), p \in V \times W\right\}$. Note that we have written $\mathcal{S}_{p}$ as the image of the holomorphic mapping $S^{1}(z ; p): V \ni z \mapsto(z, Q(z, \bar{p}))$.
To start with the iteration process, let $q:=\left(z_{1}, w_{1}\right) \in \mathcal{S}_{p}$, i.e., $w_{1}=Q\left(z_{1}, \bar{p}\right)$ and $\mathcal{S}_{q}=\{(z, w) \in V \times W$ : $w=Q(z, \bar{q})\}$ and

$$
\begin{aligned}
\mathcal{S}_{p}^{2}=\bigcup_{q \in \mathcal{S}_{p}} \mathcal{S}_{q} & =\bigcup_{z_{1} \in V}\left\{(z, w) \in V \times W: w=Q\left(z, \bar{z}_{1}, \bar{Q}\left(\bar{z}_{1}, p\right)\right)\right\} \\
& =\text { image }\left\{\left(z, z_{1}\right) \mapsto\left(z, Q\left(z, \bar{z}_{1}, \bar{Q}\left(\bar{z}_{1}, p\right)\right)\right):\left(z, z_{1}\right) \in V \times V\right\}
\end{aligned}
$$

If we complexify the real-analytic mapping in the previous equation, we get that $\mathcal{S}_{p}^{2}$ is the image of the holomorphic mapping

$$
\left(z, \chi_{1}\right) \mapsto S^{2}\left(z, \chi_{1} ; \zeta\right):=\left(z, Q\left(z, \chi_{1}, \bar{Q}\left(\chi_{1}, p\right)\right)\right)
$$

where $\left(z, \chi_{1}\right) \in V \times V$. Similar, if we go one step further, we obtain that $\mathcal{S}_{p}^{3}$ is the image of the holomorphic mapping

$$
\left(z, \chi_{1}, z_{1}\right) \mapsto S^{3}\left(z, \chi_{1}, z_{1} ; \zeta\right):=\left(z, Q\left(z, \chi_{1}, \bar{Q}\left(\chi_{1}, z_{1}, Q\left(z_{1}, \bar{p}\right)\right)\right)\right)
$$

where $\left(z, \chi_{1}, z_{1}\right) \in V^{3}$. For higher iterations we introduce the following notation.
Definition 21. We define for $x^{m} \in \mathbb{C}^{n}$, where $m \geq 1, x^{[k]}:=\left(x^{k}, \ldots, x^{1}\right) \in \mathbb{C}^{k n}, t \in \mathbb{C}^{N}$ and $j \geq 2$ ??? Define $S$ without the first component or give the second component a separate name, only needed once ???

$$
\begin{aligned}
S^{0}(t) & :=(0, t) \\
S^{1}\left(x^{1} ; t\right) & :=\left(x^{1}, Q\left(x^{1}, t\right)\right) \\
S^{j}\left(x^{[j]} ; t\right) & :=\left(x^{j}, Q\left(x^{j}, \bar{S}^{j-1}\left(x^{[j-1]} ; t\right)\right)\right),
\end{aligned}
$$

the $j$-th Segre mapping $S^{j}\left(x^{[j]} ; t\right)$.
Remark 9 (Segre sets as images of Segre maps). With the above notation we have that the image of

$$
\left(z, \chi_{1}, z_{1}, \chi_{2}, \ldots, z_{k-1}, \chi_{k}\right) \mapsto S^{2 k}\left(z, \chi_{1}, z_{1}, \chi_{2}, \ldots, z_{k-1}, \chi_{k} ; p\right)
$$

is $\mathcal{S}_{p}^{2 k}$ for $k \geq 1$ and the image of

$$
\left(z, \chi_{1}, z_{1}, \chi_{2}, \ldots, \chi_{k}, z_{k}\right) \mapsto S^{2 k+1}\left(z, \chi_{1}, z_{1}, \chi_{2}, \ldots, \chi_{k}, z_{k} ; \bar{p}\right)
$$

is $\mathcal{S}_{p}^{2 k+1}$ for $k \geq 0$.
Remark 10 (Segre sets via submanifolds). Another way to describe the iterated Segre sets is, if we associate the following submanifolds $\mathcal{M}^{(k)}$ of codimension $k d$ to the complexification of $M$. We will skip the neighborhoods involved.
We start with $\mathcal{M}^{(1)}:=\mathcal{M}$ the complexification of $M$, where we write $(Z, \zeta)$ for points on $\mathcal{M}$. Next we replace $\zeta$ by points $\left(\zeta^{1}, Z^{1}\right) \in \mathcal{M}$, attach them to $\mathcal{M}$ according to Definition 20, and define

$$
\mathcal{M}^{(2)}:=\left\{\left(Z, \zeta^{1}, Z^{1}\right) \in \mathbb{C}^{3 N}:\left(Z, \zeta^{1}\right) \in \mathcal{M},\left(Z^{1}, \zeta^{1}\right) \in \mathcal{M}\right\}
$$

The next step is to replace $Z^{1}$ by $\left(Z^{1}, \zeta^{2}\right) \in \mathcal{M}$ and attach them to $\mathcal{M}^{(2)}$ to obtain

$$
\mathcal{M}^{(3)}:=\left\{\left(Z, \zeta^{1}, Z^{1}, \zeta^{2}\right) \in \mathbb{C}^{4 N}:\left(Z^{1}, \zeta^{2}\right) \in \mathcal{M},\left(Z, \zeta^{1}, Z^{1}\right) \in \mathcal{M}^{(2)}\right\}
$$

and more generally we have

$$
\begin{aligned}
\mathcal{M}^{(2 j-1)} & :=\left\{\left(Z, \zeta^{1}, Z^{1}, \ldots, Z^{j-1}, \zeta^{j}\right) \in \mathbb{C}^{2 j N}:\left(Z, \zeta^{1}, Z^{1}, \ldots, Z^{j-1}\right) \in \mathcal{M}^{(2 j-2)},\left(Z^{j-1}, \zeta^{j}\right) \in \mathcal{M}\right\} \\
\mathcal{M}^{(2 j)} & :=\left\{\left(Z, \zeta^{1}, Z^{1}, \ldots, \zeta^{j}, Z^{j}\right) \in \mathbb{C}^{(2 j+1) N}:\left(Z, \zeta^{1}, Z^{1}, \ldots, \zeta^{j}\right) \in \mathcal{M}^{(2 j-1)},\left(Z^{j}, \zeta^{j}\right) \in \mathcal{M}\right\}
\end{aligned}
$$

Let us denote $\pi_{f}$ for the projection onto the first $N$ components and $\pi_{l}$ for the projection on the last $N$ components in some $\mathbb{C}^{l N}$. Then for $k$ even we have $\mathcal{S}_{p}^{k}=\pi_{f}\left(\pi_{l}^{-1}(p)\right)$ whenever $p \in \pi_{l}\left(\mathcal{M}^{(k)}\right)$ and if $k$ is odd we have $\mathcal{S}_{p}^{k}=\pi_{f}\left(\pi_{l}^{-1}(\bar{p})\right)$ whenever $\bar{p} \in \pi_{l}\left(\mathcal{M}^{(k)}\right)$. These observations provide a parametrization for the iterated Segre sets. It is also possible to give a description of $\mathcal{M}^{(k)}$ in terms of normal coordinates. ??? Write it down to make the connection to the manifold induced by the iterated flows in the next chapter ???
??? Fill in a diagram for the projections involved here ???
??? Different choices of coordinates for the Segre maps give parametrizations for Segre manifolds, which are biholomorphically equivalent, see section 4 on p. 10f of BER 2003 -Dynamics-article ???

Example 10. The question is if the Segre varieties contain open subsets of $\mathbb{C}^{N}$. If so, we basically could restrict to work with Segre varieties without losing any information, e.g., if we study mappings on $M$. To see what can happen, we give the following example:
The hypersurface $M=\left\{(z, w) \in \mathbb{C}^{2}: \operatorname{Im} w=|z|^{2}\right\}$ has the complexification $\mathcal{M}=\left\{(z, w, \chi, \tau) \in \mathbb{C}^{4}\right.$ : $w=\tau+2 \mathrm{i} z \chi=Q(z, \chi, \tau)\}$. Then $\mathcal{S}_{0}=\operatorname{image}\left(S^{1}(z ; 0)\right)=\operatorname{image}(z \mapsto(z, 0))=\{w=0\}$ and $\mathcal{S}_{0}^{2}=$ image $\left(S^{1}\left(z, \chi_{1} ; 0\right)\right)=\left(z, 2 \mathrm{i} z \chi_{1}\right)$. We observe that no neighborhood of 0 is contained in $\mathcal{S}_{0}^{2}$, since

$$
\left|\frac{\partial S^{2}\left(z, \chi_{1} ; 0\right)}{\partial\left(z, \chi_{1}\right)}\right|=\left|\begin{array}{cc}
1 & 0 \\
2 \chi_{1} & 2 \mathrm{i} z
\end{array}\right|=2 \mathrm{i} z
$$

therefore the variety, where $S^{2}$ is not of full rank, is given by $\{z=0\}$ and contains 0 . But if we go one step further and look at $\mathcal{S}_{0}^{3}=\operatorname{image}\left(S^{3}\left(z, \chi_{1}, z_{1} ; 0\right)\right)=\left(z, 2 \mathrm{i} \chi_{1}\left(z-z_{1}\right)\right)$, we obtain that $\mathcal{S}_{0}^{3}$ contains a neighborhood of 0 in $\mathbb{C}^{2}$, if we set $z_{1}:=z-z_{0}$, where $z_{0} \neq 0$ is independent of $z$ and $\chi_{1}$.

## 9 Finite Type

Definition 22. ??? We defined the following already in some previous section ??? Let $\mathcal{M}$ be the complexification of a generic and real-analytic submanifold $M$ of $\mathbb{C}^{N}$, then we define the space of $(1,0)$-vector fields

$$
\left.\mathcal{D}^{(1,0)}(p, \bar{p}):=\left\{X=\sum_{j=1}^{N} a_{j}(Z, \zeta) \frac{\partial}{\partial Z_{j}}: X \text { is tangent to } \mathcal{M} \text { near }(p, \bar{p}) \in \mathcal{M}\right)\right\}
$$

and the space of $(0,1)$-vector fields

$$
\mathcal{D}^{(0,1)}(p, \bar{p}):=\left\{Y=\sum_{j=1}^{N} b_{j}(Z, \zeta) \frac{\partial}{\partial \zeta_{j}}: Y \text { is tangent to } \mathcal{M} \text { near }(p, \bar{p}) \in \mathcal{M}\right\}
$$

Definition 23. ??? We already defined finite type in some previous section ??? Let $\mathcal{D}(p, \bar{p}):=$ $\mathcal{D}^{(1,0)}(p, \bar{p}) \oplus \mathcal{D}^{(0,1)}(p, \bar{p})$ be the space spanned by $(1,0)$ - and $(0,1)$-vector fields tangent to $\mathcal{M}$ near $(p, \bar{p}) \in \mathcal{M}$ and $\mathfrak{D}(p, \bar{p})$ the Lie algebra of $\mathcal{D}(p, \bar{p})$ evaluated at $(p, \bar{p}) \in \mathcal{M}$.
Then $M$ is of finite type at $p \in M$, if $\mathfrak{D}(p, \bar{p})=T_{(p, \bar{p})} \mathcal{M}$.
Remark 11. (i) An equivalent formulation for $p$ being a point of finite type is, that all $(1,0)-$ and $(0,1)-$ vector fields tangent to $M$ generate $\mathbb{C} T_{p} M$.
(ii) Since we have that $\operatorname{dim}_{\mathbb{C}}\left(\mathcal{D}^{(1,0)}(p, \bar{p})\right)=n=\operatorname{dim}_{\mathbb{C}}\left(\mathcal{D}^{(0,1)}(p, \bar{p})\right)$ and $\operatorname{dim}_{\mathbb{C}}\left(T_{(p, \bar{p})} \mathcal{M}\right)=2 N-d=$ $2(n+d)-d=2 n+d$ the commutators of all $(1,0)-$ and $(0,1)-$ vector fields tangent to $\mathcal{M}$ have to span $d$ additional directions, which are called missing directions.
Example 11. Let $M$ be a hypersurface given by $M=\{\operatorname{Im}(w)=\varphi(z, \bar{z}, \operatorname{Re}(w))\}$ and we assume w.l.o.g. that $\varphi(z, 0,0)=0=\varphi(0, \bar{z}, 0)$. Then $M$ is of finite type at 0 if and only if $\varphi(z, \bar{z}, 0) \not \equiv 0$. Moreover the length of the commutators, which span the missing direction, is the order of vanishing of $\varphi(z, \bar{z}, 0)$ at 0 . The length of these commutators is defined as the type of $p$. ??? Maybe fill in the computation here ???
??? The following Lemma is needed for the proof of Stanton's Theorem, but the Lemma actually follows from the proof of the Theorem concerning the Characterization of Segre sets; but an independent (and shorter) proof of this fact is also fine ???

Lemma 15. Let $M$ be a generic and real-analytic submanifold of $\mathbb{C}^{N}$ and $p \in M$ a point of finite type. Then there exists a proper, real-analytic subvariety $V \subset M$, such that any $q \in M \backslash V$ is a point of finite type.

Proof. ??? TBC, see BER 1999, Thm. 1.5.10, p. 19f ???
Theorem 7. Let $M$ be a generic and real-analytic submanifold of $\mathbb{C}^{N}$ and $p \in M$. Then the following statements are equivalent:
(i) $M$ is of finite type at $p \in M$.
(ii) There exists $j \in \mathbb{N}$ such that the iterated Segre map $S^{j}\left(x^{[j]} ; p\right)$ is generically of full rank.
(iii) There exists $j \in \mathbb{N}$ such that the iterated Segre set $\mathcal{S}_{p}^{j}$ contains an open set of $\mathbb{C}^{N}$.

Remark 12. (i) Theorem 7 (ii) means that for $x^{[j]}=\left(x^{j}, \ldots, x^{1}\right)$ with $x^{k}=\left(x_{1}^{k}, \ldots, x_{n}^{k}\right) \in \mathbb{C}^{n}$ for $1 \leq k \leq j$, there exist multiindices $k_{1}, \ldots, k_{N}$ with $1 \leq k_{r} \leq n$ and $j_{1}, \ldots, j_{N}$ with $1 \leq j_{s} \leq j$ for $1 \leq r, s \leq N$ such that

$$
\left|\frac{\partial S^{j}\left(x^{[j]} ; p\right)}{\partial\left(x_{k_{1}}^{j_{1}}, \ldots, x_{k_{N}}^{j_{N}}\right)}\right| \not \equiv 0 .
$$

(ii) The course of the proof of Theorem 7 (i) $\Leftrightarrow$ (ii) is the following: Starting with the next remark we forget about $M$ and actually show the statement for $p \in \mathbb{C}^{N}$. We start with a family of $\mathbb{K}$-analytic vector fields $Y$ in $\mathbb{C}^{N}$ and denote by $\mathfrak{Y}(p)$ its Lie algebra at $p \in \mathbb{K}^{N}$. The flow of elements in $\mathfrak{Y}$ consist of iterated flows of the elements in $Y$ and will be denoted by $\Phi$. Then we show that such $\Phi$ are generically of full rank $N$ at $p$ if and only if $\mathfrak{Y}(p)$ spans $T_{p} \mathbb{K}^{N}$. In a last step we identify the Segre mappings as flows of elements in $\mathcal{D}(p, \bar{p})$ and take into account that we are not in the flat case, but on a submanifold $M$.
Remark 13 (The setup). Let $\mathbb{K}=\{\mathbb{R}, \mathbb{C}\}$ and let $X=\left\{X_{1}, \ldots, X_{l}\right\}$ be a family of germs of $\mathbb{K}$-analytic vector fields in $\mathbb{K}^{N}$ near 0 . We write

$$
X_{j}=\sum_{k=1}^{N} a_{k}^{j}(x) \frac{\partial}{\partial x_{k}}
$$

where, for $1 \leq k \leq N$ and $1 \leq j \leq l, a_{k}^{j}: \mathbb{K}^{N} \rightarrow \mathbb{K}$ are germs of analytic functions near 0 . Further we denote $a^{j}=\left(a_{1}^{j}, \ldots, a_{N}^{j}\right)$. The flow of $X_{j} \in X$ at time $s \in \mathbb{K}$ at $x \in \mathbb{K}^{N}$ is denoted by $\varphi_{X_{j}}^{s}(x) \equiv \varphi^{j}(x, s)$ : $\mathbb{K}^{N} \times \mathbb{K} \rightarrow \mathbb{K}^{N}$, i.e., it is the unique solution of the following initial value problem for small $s$ :

$$
\begin{aligned}
\frac{\partial \varphi^{j}}{\partial s}(x, s) & =a^{j}\left(\varphi^{j}(x, s)\right) \\
\varphi^{j}(x, 0) & =x
\end{aligned}
$$

Next we start to flow with respect to all vector fields in $X$ : We let $t=\left(t_{1}, \ldots, t_{l}\right) \in \mathbb{K}^{l}$ and we write for $1 \leq k \leq l, y_{k}:=\varphi_{X_{k}}^{t_{k}} \circ \cdots \circ \varphi_{X_{1}}^{t_{1}}(x) \in \mathbb{K}^{N}$. If we temporarily denote for $1 \leq k \leq l, \psi_{k} \equiv \psi_{k}\left(x, t_{1}, \ldots, t_{k}\right):$ $\mathbb{K}^{N} \times \mathbb{K}^{k} \rightarrow \mathbb{K}^{N}$ the flow of the vector fields $X_{1}, \ldots, X_{k}$ at times $t_{1}, \ldots, t_{k}$. Then $\psi_{k}$ is the solution of

$$
\begin{aligned}
\frac{\partial \psi_{k}}{\partial t_{k}}\left(y_{k-1}, t_{k}\right) & =a^{k}\left(\psi_{k}\left(y_{k-1}, t_{k}\right)\right) \\
\psi_{k}\left(y_{k-1}, 0\right) & =y_{k-1}
\end{aligned}
$$

where $y_{0}:=x$, hence $\psi_{k}=\varphi_{X_{k}}^{t_{k}} \circ \cdots \circ \varphi_{X_{1}}^{t_{1}}(x)$ is a germ of an analytic mapping. ??? Where do we need the following notation ??? We introduce one more notation: We let $t^{[j]}=\left(t^{1}, \ldots, t^{j}\right)$ for each $t^{k} \in \mathbb{K}^{m}$ and set

$$
\begin{aligned}
\varphi_{X}^{t^{[1]}}(x) & :=\varphi_{X}^{t^{1}}(x) \\
\varphi_{X}^{t^{[j]}}(x) & :=\varphi_{X}^{t^{j}} \circ \varphi_{X}^{t^{[j-1]}}(x),
\end{aligned}
$$

and each $\varphi_{X}^{\dagger^{[j]}}(x) \equiv \varphi^{t^{[j]}}(x) \equiv \varphi\left(x, t^{[j]}\right): \mathbb{K}^{N} \times \mathbb{K}^{j m} \rightarrow \mathbb{K}^{N}$ is a germ of an analytic mapping. To sum up, for this definition we iterated the flows $\varphi_{X}^{t}(x)$, which are solutions of the appropriate initial value problem similar as above. The question to which vector fields the iterated flows correspond, will be addressed to Remark 16.
??? Is it useful to introduce the following ranks since we will prove directly that the dimension of the Lie algebra of $X$ is equal to the dimension of the tangent space of the orbits ??? We fix $x \in \mathbb{K}^{N}$ and denote by

$$
\operatorname{sk} X(x):=\max _{j \in \mathbb{N}} \operatorname{rk} \varphi\left(x, t^{[j]}\right)
$$

??? Separate definition for "generic rank" and "generically of full rank" (see below the statement of the previous theorem) somewhere in the intro ??? Here $\operatorname{rk} \varphi\left(x, t^{[j]}\right)$ denotes the generic rank of $\varphi\left(x, t^{[j]}\right)$, which means that we consider the rank of $\varphi\left(x, t^{[j]}\right)$ over the quotient field $\mathbb{K}\left\{t^{[j]}\right\}$. More precisely, $\operatorname{rk} \varphi\left(x, t^{[j]}\right)$ is the largest number $r \in \mathbb{N}$, such that the Jacobian of $\varphi\left(x, t^{[j]}\right)$ with respect to $t^{[j]}$ has a non vanishing minor of size $r$ near $0 \in \mathbb{K}^{j l}$.
Further we denote by

$$
\operatorname{rk} X(x):=\operatorname{dim}_{\mathbb{K}} \mathfrak{X}(x)
$$

the dimension of the Lie algebra of the family of vector fields $X$ evaluated at $x$.
For two vector fields $X, Y$ let us write $(\operatorname{ad} X)(Y):=[X, Y]$ and $(\operatorname{ad} X)^{j}(Y):=(\operatorname{ad} X)\left((\operatorname{ad} X)^{j-1}(Y)\right)$ for $j \geq 2$.

Theorem 8 (Vector fields vs. flows). Let $\mathbb{K}=\{\mathbb{R}, \mathbb{C}\}$ and let $X=\left\{X_{1}, \ldots, X_{l}\right\}$ be a family of germs of $\mathbb{K}$-analytic vector fields in $\mathbb{K}^{N}$ near 0 . Then we have for fixed $x \in \mathbb{K}^{N}$ sufficiently near 0

$$
\operatorname{rk} X(x)=\operatorname{sk} X(x)
$$

Remark 14. (i) It is clear that $1 \leq \operatorname{sk} X(x)$, rk $X(x) \leq N$. The advantage of considering sk $X(x)$ instead of rk $X(x)$ lies in the fact that $\left(\operatorname{rk} \varphi\left(x, t^{[j]}\right)\right)_{j \in \mathbb{N}}$ is a strictly increasing sequence ??? Refer to this aspect, when we prove this claim; we skipped the proof in the lecture, but BER 2003 Dynamics ..., Prop. 3.1 gives a formal argument ???. Hence, in order to compute the dimension of the Lie algebra of vector fields we are done after finitely many steps.
(ii) The following Lemma is the starting point of an induction process and describes the flows of the Lie bracket of two vector fields $X, Y$ in terms of one vector field $Y$ evaluated at the flow of $X$.
Lemma 16. Let $X=\sum_{k=1}^{N} a_{k}(x) \frac{\partial}{\partial x_{k}}$ and $Y=\sum_{l=1}^{N} b_{l}(x) \frac{\partial}{\partial x_{l}}$ be germs of $\mathbb{K}$-analytic vector fields near 0 and denote by $\varphi(x, t): \mathbb{K}^{N} \times \mathbb{K} \rightarrow \mathbb{K}^{N}$ the flow of $X$. Then

$$
W(x, t):=\varphi_{x}(\varphi(x, t),-t) Y(\varphi(x, t))
$$

uniquely solves

$$
\begin{align*}
\frac{\partial W}{\partial t}(x, t) & =\varphi_{x}(\varphi(x, t),-t)[X, Y](\varphi(x, t))  \tag{9.1}\\
W(x, 0) & =Y(x)
\end{align*}
$$

Moreover we have

$$
\varphi_{x}^{-1}(\varphi(x, t), t) Y(\varphi(x, t))=\sum_{j \in \mathbb{N}} \frac{1}{j!}(\operatorname{ad} X)^{j} Y(x) t^{j}
$$

where the sum converges absolutely and uniformly on compact subsets in a neighborhood of $0 \in \mathbb{K}$.
Proof. First we deduce some properties of the flow $\varphi$ : Since $\varphi(\varphi(x, t),-t)=x$ we obtain, if we take derivatives with respect to $x$, that

$$
\begin{equation*}
\varphi_{x}(\varphi(x, t),-t)=\varphi_{x}^{-1}(x, t) \tag{9.2}
\end{equation*}
$$

We write $a=\left(a_{1}, \ldots, a_{N}\right)$ and $b=\left(b_{1}, \ldots, b_{N}\right)$. Since $\varphi$ satisfies

$$
\begin{align*}
\frac{\partial \varphi}{\partial t}(x, t) & =a(\varphi(x, t))  \tag{9.3}\\
\varphi(x, 0) & =x
\end{align*}
$$

we get, if we differentiate (9.3) with respect to $x$ that

$$
\begin{equation*}
\varphi_{t, x}(x, t)=a_{x}(\varphi(x, t)) \varphi_{x}(x, t) \tag{9.4}
\end{equation*}
$$

Using (9.2) and $\varphi_{x}(\varphi(x, t),-t)=\varphi_{x}^{-1}(\varphi(x, t), t)$, we arrive at

$$
\begin{equation*}
\varphi_{x}(x, t) W(x, t)=Y(\varphi(x, t)) \tag{9.5}
\end{equation*}
$$

First we take derivatives with respect to $t$ of the left-hand side of (9.5) and stick in (9.4) and (9.5) to obtain

$$
\begin{equation*}
\frac{d}{d t}\left(\varphi_{x}(x, t) W(x, t)\right)=Y(X(\varphi(x, t)))+\varphi_{x}(x, t) \frac{d W}{d t}(x, t) \tag{9.6}
\end{equation*}
$$

Next we take derivatives with respect to $t$ of the right-hand side of (9.5) and obtain

$$
\begin{equation*}
\frac{d}{d t} Y(\varphi(x, t))=b_{x}(\varphi(x, t)) \varphi_{t}(x, t)=b_{x}(\varphi(x, t) X(\varphi(x, t))=X(Y(\varphi(x, t))) \tag{9.7}
\end{equation*}
$$

Finally, combining (9.6) and (9.7) we get the desired identity

$$
\begin{equation*}
\varphi_{x}(x, t) \frac{d W}{d t}(x, t)=X(Y(\varphi(x, t)))-Y(X(\varphi(x, t)))=[X, Y](\varphi(x, t)) \tag{9.8}
\end{equation*}
$$

To prove the expansion of $W(x, t)=\varphi_{x}^{-1}(\varphi(x, t), t) Y(\varphi(x, t))$, we write $W(x, t)$ as its Taylor series in the second variable. We differentiate (9.8) again with respect to $t$, evaluate at 0 and use (9.1), (9.2) and (9.4) to obtain the correct formula for $\frac{d^{2} W}{d t^{2}}(x, 0)$. The rest follows by induction and noting that the coefficients of the series are derivatives of a real-analytic mapping.

Remark 15. The following Lemma is an iteration of the previous Lemma 16 using the notation as in Remark 13 for iterated flows of a family of vector fields. In contrast to the previous Lemma 16, we get a formula for derivatives of $W$ with respect to $t$ at $t=0$ in terms of Lie brackets.
??? Adjust the notation for the iterated flows from the definition above ???
Lemma 17. Let $X_{1}, \ldots, X_{k}, Y$ be germs of $\mathbb{K}$-analytic vector fields near 0 and denote by $\varphi_{X_{j}}^{t}(x) \equiv \varphi^{j}(x, t)$ : $\mathbb{K}^{N} \times \mathbb{K} \rightarrow \mathbb{K}^{N}$ the flow of $X_{j}$. We write $t^{[j]}:=\left(t_{j}, \ldots, t_{1}\right) \in \mathbb{K}^{j}$ and by $\varphi\left(x, t^{[j]}\right)$ the iterated flow of the vector fields $X_{1}, \ldots, X_{j}$ at $x \in \mathbb{K}^{N}$ and time $t^{[j]} \in \mathbb{K}^{j}$. Then

$$
W\left(x, t^{[k]}\right):=\varphi_{x}\left(\varphi\left(x, t_{1}\right),-t_{1}\right) \cdots \varphi_{x}\left(\varphi\left(x, t^{[k-1]}\right),-t_{k-1}\right) \varphi_{x}\left(\varphi\left(x, t^{[k \mid}\right),-t_{k}\right) Y\left(\varphi\left(x, t^{[k]}\right)\right)
$$

satisfies

$$
\frac{\partial^{k} W}{\partial t_{1} \cdots \partial t_{k}}(x, 0)=\left[X_{1},\left[X_{2},\left[\cdots\left[X_{k}, Y\right] \cdots\right]\right]\right](x)
$$

Moreover, after setting $\Phi(x, t):=\varphi_{X_{k}}^{t_{k}} \circ \cdots \circ \varphi_{X_{1}}^{t_{1}}(x)$ we have

$$
\Phi_{x}^{-1}(\Phi(x, t), t) Y(\Phi(x, t))=\sum_{\alpha \in \mathbb{N}^{k}} \frac{1}{\alpha!}\left(\operatorname{ad} X_{1}\right)^{\alpha_{1}} \cdots\left(\operatorname{ad} X_{k}\right)^{\alpha_{k}} Y(x) t^{\alpha}
$$

where the sum converges absolutely and uniformly on compact subsets in a neighborhood of $0 \in \mathbb{K}^{k}$.

Proof. The proof is induction on $k \geq 1$. The induction hypothesis is Lemma 16 and the induction step is as in Lemma 16 with $t=0$.

Remark 16. Lemma 16 implies that the Lie bracket $[X, Y]$ is the derivative of the iterated flow $(s, t) \mapsto$ $\varphi_{X}^{-s} \circ \varphi_{Y}^{t} \circ \varphi_{X}^{s}$ with respect to $t$ at 0 . Let $\varphi(x, s)$ denote the flow of $X$ and $\psi(x, t)$ the flow of $Y$, then we get

$$
\left.\frac{d}{d t}\right|_{t=0} \varphi(\psi(\varphi(x, s), t),-s)=\left.\left.\varphi_{x}(\psi(\varphi(x, s), t),-s)\right|_{t=0} \psi_{t}(\varphi(x, s), t)\right|_{t=0}=\varphi_{x}(\varphi(x, s),-s) Y(\varphi(x, s))
$$

and by Lemma 16 the claim follows. For iterated flows of higher order we have to consider flows of the form

$$
\varphi\left(x, t_{J}\right):=\varphi^{j_{1},-t_{1}} \circ \cdots \circ \varphi^{j_{|J|-1},-t_{|J|-1}} \circ \varphi^{j_{|J|}, t_{|J|}} \circ \varphi^{j_{|J|-1}, t_{|J|-1}} \circ \cdots \circ \varphi^{j_{1}, t_{1}}(x),
$$

where $t_{J}=\left(t_{1}, \ldots, t_{|J|}\right)$ and $\varphi^{j_{k}, t_{k}}$ is the flow of the vector field $X_{k}$ for $1 \leq k \leq|J|-1$ and $Y$ for $k=|J|$. Then we take derivatives with respect to $t_{|J|}$ and apply Lemma 17 to get that the derivative of $\varphi\left(x, t_{J}\right)$ with respect to $t_{|J|}$ at $t_{|J|}=0$ is equal to $\left[X_{1},\left[X_{2},\left[\cdots\left[X_{k}, Y\right] \cdots\right]\right]\right](x)$.
Thus the flows of elements of the Lie algebra of a family of vector fields are iterated flows as we have introduced them in Remark 13.
??? One has to evaluate the flows at some fixed time $t_{0}$ ???
Definition 24. For $X$ a germ of a family of $\mathbb{K}$-analytic vector fields near $x \in \mathbb{K}^{N}$ we denote by $\mathfrak{X}$ its Lie algebra and by $\mathfrak{F}$ the collection of all flows of elements of $\mathfrak{X}$. We define by $\mathfrak{F}(x)$ the orbit of $X$ at $x$ as

$$
\mathfrak{F}(x):=\left\{y \in \mathbb{C}^{N}: \exists \varphi \in \mathfrak{F}: y=\varphi(x)\right\}
$$

Remark 17. Suppose there exists $Y \in \mathfrak{X}$ such that $Y(x) \neq 0$. Since the parametrizations of $\mathfrak{F}(x)$ consists of iterated flows $t^{[j]} \mapsto \varphi\left(x, t^{[j]}\right)$, there exists a $k \in \mathbb{N}$ such that $\left.\frac{d}{d t_{k}}\right|_{t=0} \varphi\left(x, t^{[j]}\right)=Y(x) \neq 0$, hence $\mathfrak{F}(x)$ is a submanifold of dimension $\operatorname{dim}(\operatorname{span}\{X(x): X \in \mathfrak{X}\})$ of $\mathbb{C}^{N}$. As for the Segre sets we can write $\mathfrak{F}$ as the projection of iterated submanifolds as follows:
Let $X=\left\{X_{1}, \ldots, X_{k}\right\}$ be a family of vector fields in $\mathbb{K}^{N}$. We set $j=\left(j_{1}, \ldots, j_{l}\right) \in \mathbb{N}^{l}$ with $1 \leq j_{m} \leq k$. Let $y \in \mathfrak{F}(x)$ be given by $y=\varphi\left(x, t^{[j]}\right)$. Then we define $M_{j_{m}}:=\left\{(x, y) \in \mathbb{K}^{N}: y=\varphi\left(x, t^{\left[j_{m}\right]}\right)\right\}$ and the iterated submanifold $M_{j}$ to be

$$
\left(x, y_{1}, \ldots, y_{l}\right) \in M_{j} \Leftrightarrow\left(x, y_{1}\right) \in M_{j_{1}},\left(y_{1}, y_{2}\right) \in M_{j_{2}}, \ldots,\left(y_{l-1}, y_{l}\right) \in M_{j_{l}}
$$

We get $\mathfrak{F}(x)=\pi_{l}\left(M_{j}\right)$ and after reordering the coordinates $\left(x, y_{1}, \ldots, y_{l}\right)$ we could have also written $\mathfrak{F}(x)=$ $\pi_{f}\left(M_{j}\right)$ to see the connection to the definition of the manifolds given in Remark 10 directly.

Proposition 12 (Nagano's Orbit Theorem light). Fix $X=\left\{X_{1}, \ldots, X_{k}\right\}$ a family of germs of $\mathbb{K}$-analytic vector fields near 0. Denote by $\mathfrak{X}(x)$ the Lie algebra of $X$ at $x \in \mathbb{K}^{N}$. Then we have for fixed $x \in \mathbb{K}^{N}$ sufficiently close to 0

$$
\operatorname{dim}_{\mathbb{K}} \mathfrak{X}(x)=\operatorname{dim}_{\mathbb{K}} \mathfrak{X}(y)=\operatorname{dim}_{\mathbb{K}} T_{y} \mathfrak{F}(x) \quad \forall y \in \mathfrak{F}(x)
$$

Proof. Let $y \in \mathfrak{F}(x)$ and $\Phi \in \mathfrak{F}$ be an iterated flow, such that $y=\Phi(x, t)$ for some $t$. Then $\operatorname{dim}_{\mathbb{K}} \mathfrak{X}(x) \leq$ $\operatorname{dim}_{\mathbb{K}} \mathfrak{X}(\Phi(x, t))$ holds for continuity reasons: Let $r:=\operatorname{dim}_{\mathbb{K}} \mathfrak{X}(x)$, choose a basis $Y_{1}(x), \ldots, Y_{r}(x) \in \mathfrak{X}(x)$ and write $Y_{j}=\left(Y_{j}^{1}, \ldots, Y_{j}^{N}\right)$, such that

$$
\left|\begin{array}{ccc}
Y_{1}^{1}(x) & \cdots & Y_{r}^{1}(x)  \tag{9.9}\\
\vdots & \ddots & \vdots \\
Y_{1}^{r}(x) & \cdots & Y_{r}^{r}(x)
\end{array}\right| \neq 0
$$

By continuity of the components of elements in $\mathfrak{X}(x)$ and their flows, we obtain

$$
\left|\begin{array}{ccc}
Y_{1}^{1}(\Phi(x, t)) & \cdots & Y_{r}^{1}(\Phi(x, t)) \\
\vdots & \ddots & \vdots \\
Y_{1}^{r}(\Phi(x, t)) & \cdots & Y_{r}^{r}(\Phi(x, t))
\end{array}\right| \neq 0
$$

which means that $Y_{1}(\Phi(x, t)), \ldots, Y_{r}(\Phi(x, t)) \in \mathfrak{X}(\Phi(x, t))$ is a set of $\mathbb{K}$-linearly independent vectors in $\mathfrak{X}(\Phi(x, t))$, hence $r \leq \operatorname{dim}_{\mathbb{K}} \mathfrak{X}(\Phi(x, t))$.
To show the converse inequality $\operatorname{dim}_{\mathbb{K}} \mathfrak{X}(x) \geq \operatorname{dim}_{\mathbb{K}} \mathfrak{X}(\Phi(x, t))$ we assume that for all choices of $r+1$ vectors $Z_{1}(x), \ldots, Z_{r+1}(x) \in \mathfrak{X}(x)$ each $r+1$-minor of the matrix $\left(Z_{1}(x), \ldots, Z_{r+1}(x)\right)$ vanishes. Then we have to conclude, that for each choice of $r+1$ vectors $V_{1}(\Phi(x, t)), \ldots, V_{r+1}(\Phi(x, t)) \in \mathfrak{X}(\Phi(x, t))$ every $r+1$-minor of the matrix $\left(V_{1}(\Phi(x, t)), \ldots, V_{r+1}(\Phi(x, t))\right)$ vanishes. As in Lemma 17 we write for $1 \leq j \leq r+1$

$$
\Phi_{x}^{-1}(\Phi(x, t), t) V_{j}(\Phi(x, t))=\sum_{\alpha \in \mathbb{N}^{k}} \frac{1}{\alpha!}\left(\operatorname{ad} X_{1}\right)^{\alpha_{1}} \cdots\left(\operatorname{ad} X_{k}\right)^{\alpha_{k}} V_{j}(x) t^{\alpha}
$$

where $Z_{\alpha, j}(x):=\left(\operatorname{ad} X_{1}\right)^{\alpha_{1}} \cdots\left(\operatorname{ad} X_{k}\right)^{\alpha_{k}} V_{j}(x) \in \mathfrak{X}(x)$. We set $Z_{\alpha, j}=\left(z_{\alpha, j}^{1}, \ldots, z_{\alpha, j}^{N}\right)$ and define $\widetilde{V}_{j}(x, t):=$ $\Phi_{x}^{-1}(\Phi(x, t), t) V_{j}(\Phi(x, t))$. Since $\Phi_{x}^{-1}(\Phi(x, t), t)$ is near the identity for small $t$, it is enough to show that each $r+1$-minor of the matrix $\widetilde{V}:=\left(\widetilde{V}_{1}(x, t), \ldots, \widetilde{V}_{r+1}(x, t)\right)$ vanishes. Let us take any $r+1$-minor of $\widetilde{V}$ and write $\widetilde{V}_{j}=\left(\widetilde{v}_{j}^{1}, \ldots, \widetilde{v}_{j}^{N}\right)$. Since all involved sums converge absolutely and uniformly, we can reorder in such a way that

$$
\left|\begin{array}{ccc}
\widetilde{v}_{\beta_{1}}^{\alpha_{1}} & \cdots & \widetilde{v}_{\beta_{r+1}}^{\alpha_{1}} \\
\vdots & \ddots & \vdots \\
\widetilde{v}_{\beta_{1}}^{\alpha_{r+1}} & \cdots & \widetilde{v}_{\beta_{r+1}}^{\alpha_{r+1}}
\end{array}\right|(x, t)=\sum_{\gamma \in \mathbb{K}^{k}} \sum_{\delta_{1}+\cdots+\delta_{r+1}=\gamma} \frac{1}{\delta_{1}!\cdots \delta_{r+1}!} \underbrace{\left|\begin{array}{ccc}
z_{\delta_{1}, \beta_{1}}^{\alpha_{1}} & \cdots & z_{\delta_{r+1}, \beta_{r+1}}^{\alpha_{1}} \\
\vdots & \ddots & \vdots \\
z_{\delta_{1}, \beta_{1}}^{\alpha_{r+1}} & \cdots & z_{\delta_{r+1}, \beta_{r+1}}^{\alpha_{r+1}}
\end{array}\right|}_{=0}(x) t^{\gamma}=0
$$

The second equality $\operatorname{dim}_{\mathbb{K}} \mathfrak{X}(y)=\operatorname{dim}_{\mathbb{K}} T_{y} \mathfrak{F}(x)$ is immediate, since we know $\mathfrak{F}(x)$ is a submanifold and consists of the flows of the elements in $\mathfrak{X}(y)$. ??? Is this clear? + identification of Lie algebra with tangent space ???

Remark 18. Note that (9.9) shows that the set, where $\operatorname{dim}_{\mathbb{K}} \mathfrak{X}(x)$ is not maximal, is a proper, $\mathbb{K}$-analytic variety of $\mathbb{K}^{N}$, which in particular means that $\operatorname{dim}_{\mathbb{K}} \mathfrak{X}(x)$ is generically locally constant. Moreover we have seen that formally the Lie algebras transform as follows:

$$
\mathfrak{X}(x)=\Phi_{x}^{-1}(\Phi(x, t), t) \mathfrak{X}(\Phi(x, t))
$$

Proof of Theorem 8. The iterated flows give a parametrization of $\mathfrak{F}(x)$, hence $\operatorname{dim}_{\mathbb{K}} \mathfrak{F}(x)=\operatorname{sk} X(x)$ and if we apply the previous Proposition 12 we are done.

Remark 19. To get Theorem 8 in the general case of generic submanifolds $M$, we have to take into account the codimension $d$ of $M$, when counting dimensions in Proposition 12.

Proof of Theorem 7. Since the equivalence (ii) $\Leftrightarrow$ (iii) is clear we are left by showing (i) $\Leftrightarrow$ (ii). This equivalence follows from Theorem 8 if we are able to identify the Segre maps as iterated flows of $(1,0)$ - and $(0,1)$-vector fields tangent to $\mathcal{M}$. For $\mathcal{M}$ given in normal coordinates near $p \in M$ we have for $z=\left(z_{1}, \ldots, z_{n}\right)$ and $w=\left(w_{1}, \ldots, w_{d}\right)$ that $w=Q(z, \chi, \tau)$, where $Q=\left(Q_{1}, \ldots, Q_{d}\right)$. Further we write $Z=(z, w)$ and $\zeta=(\chi, \tau)$. For $1 \leq j \leq d$ we set

$$
L_{j}:=\frac{\partial}{\partial z_{j}}+Q_{z_{j}}(z, \zeta) \frac{\partial}{\partial w} \quad \text { and } \quad \bar{L}_{j}:=\frac{\partial}{\partial \chi_{j}}+\bar{Q}_{\chi_{j}}(\chi, Z) \frac{\partial}{\partial \tau}
$$

Then $L=\left(L_{1}, \ldots, L_{n}\right)$ and $\bar{L}=\left(\bar{L}_{1}, \ldots, \bar{L}_{n}\right)$ are a basis of $\mathcal{D}^{(1,0)}(p, \bar{p})$ and $\mathcal{D}^{(0,1)}(p, \bar{p})$ respectively. We equip $T_{(p, \bar{p})} \mathcal{M}$ with coordinates $\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial w}, \frac{\partial}{\partial \chi}, \frac{\partial}{\partial \tau}\right)$. Then we see that the mappings

$$
\Phi_{L_{j}}^{t}(Z, \zeta):=\left(z+t e_{j}, Q\left(z+t e_{j}, \zeta\right), \zeta\right) \quad \text { and } \quad \Phi_{\bar{L}_{j}}^{t}(Z, \zeta):=\left(Z, \chi+t e_{j}, \bar{Q}\left(\chi+t e_{j}, Z\right)\right)
$$

are the flows of $L_{j}$ and $\bar{L}_{j}$, respectively. We denote by $\Phi_{L}^{t}$ and $\Phi_{\bar{L}}^{t}$ the iterated flow of $L$ and $\bar{L}$, respectively, at complex time $t=\left(t_{1}, \ldots, t_{n}\right) \in \mathbb{C}^{n}$. That means we have

$$
\begin{aligned}
& \Phi_{L}^{t}(Z, \zeta)=\left(z_{1}+t e_{1}, \ldots, z_{n}+t e_{n}, Q\left(z_{1}+t e_{1}, \ldots, z_{n}+t e_{n}, \zeta\right), \zeta\right) \quad \text { and } \\
& \Phi_{\bar{L}}^{t}(Z, \zeta)=\left(Z, \chi_{1}+t e_{1}, \ldots, \chi_{n}+t e_{n}, \bar{Q}\left(\chi_{1}+t e_{1}, \ldots, \chi_{n}+t e_{n}, Z\right)\right) .
\end{aligned}
$$

By Remark 9 using the notation of Remark 10 , we have that $S^{1}(t ; \zeta)$ is equal to

$$
t \mapsto \pi_{f}\left(\Phi_{L}^{t}(Z, \zeta)\right)
$$

We write for $k \geq 1 t^{k}=\left(t_{1}^{k}, \ldots, t_{n}^{k}\right) \in \mathbb{C}^{n}$. For higher iterated Segre maps we have, that if $j$ is odd, $S^{j}\left(t^{1}, \ldots, t^{j} ; \zeta\right)$ is equal to

$$
\left(t^{1}, \ldots, t^{j}\right) \mapsto \pi_{f}\left(\Phi_{L}^{t^{1}} \circ \Phi_{\bar{L}}^{t^{2}} \circ \cdots \circ \Phi_{L}^{t_{j}^{j}}(Z, \zeta)\right)
$$

and if $j$ is even, then $S^{j}\left(t^{1}, \ldots, t^{j} ; \zeta\right)$ is equal to

$$
\left(t^{1}, \ldots, t^{j}\right) \mapsto \pi_{f}\left(\Phi_{L}^{t^{1}} \circ \Phi_{\bar{L}}^{t^{2}} \circ \cdots \circ \Phi_{\bar{L}}^{t^{j}}(Z, \zeta)\right),
$$

if each $t^{k}$ stays in some small neighborhood around 0 .
In order to make the connection to sk $X$ of Theorem 8, we observe the following:
Since the above iterated flows $\Phi_{j}$ are given in their complexified form, the first component of $\Phi_{j}$ is a parametrization of the first component of submanifolds like $M_{j}$ from Remark 17. Also note that the iterated flow $\Psi_{j}$, we had in the definition of sk $X$, occurs in the first component of submanifolds like $M_{j}$. This means that the Segre maps are special cases of iterated flows of the form $\Psi_{j}$ and the submanifolds $\mathcal{M}^{(j)}$ are special cases of submanifolds like $M_{j}$. Hence the rank of the Segre map $S^{j}$ is the same as the rank of $\Psi_{j}$ and since $\operatorname{sk} X=\max _{j} \mathrm{rk} \Psi_{j}$, we have for large $k$, that sk $X=\operatorname{rk} S^{k}$.
??? There is still a part of the proof missing, we still need to show that the rank conditions $\mathrm{sk} X$ and rk $X$ coincide, see [dSJL 2011, Formal Theory of Segre Varieties] or can we work directly with Proposition 12 since if the rank of the iterated Segre map is full, this corresponds to the dim of the orbit-manifold ???

Corollary 4. Let $M$ be a generic and real-analytic submanifold of $\mathbb{C}^{N}, p \in M$ and $f \in \mathcal{O}_{p}$ with $f(M) \subset \mathbb{R}$. If $M$ is of finite type at $p$, then $f=f(p)$ is constant.
Proof. Let us assume $p=0$, then we have by the reality of $f$, that $f(Z)=\overline{f(Z)}$ for $Z \in M$. After complexifying we obtain $f(Z)=\bar{f}(\zeta)$ for $(Z, \zeta) \in \mathcal{M}$. Setting $\zeta=0$ we get $f(Z)=\bar{f}(0)$ for $Z \in \mathcal{S}_{0}$. If we let $Z \in \mathcal{S}_{0}^{2}=\bigcup_{\zeta \in \mathcal{S}_{0}} \mathcal{S}_{\zeta}$ we have $f(Z)=\bar{f}(\zeta)=f(0)$. Iterating this process we obtain that $f(Z)=f(0)$ for all $Z \in \mathcal{S}_{0}^{j}$ with $j \geq 1$. Hence by Theorem 7 and the Identity Principle we obtain $f(Z)=f(0)$ for all $Z \in\left(\mathbb{C}^{N}, 0\right)$.

## 10 Nondegeneracy Conditions

Motivation 2. If we start with a mapping $H$ sending $M$ to $M^{\prime}$, we can ask under which conditions on $M^{\prime}$ we are able to write $H(Z)=\Psi\left(\zeta, \bar{\partial}^{\alpha} \bar{H}(\zeta)\right)$ for $(Z, \zeta) \in \mathcal{M}$. Moreover we want $\Psi$ to be a "nice" mapping, which does not depend on $H$. If this should hold, we need to require that the "variable" $H(Z)$ occurs in the mapping equation or even more that the defining equation for $M^{\prime}$ depends on all variables. E.g. if we take $M^{\prime}=M=\left\{\operatorname{Im} w=\left|z_{1} z_{2}\right|^{2}\right\} \subset \mathbb{C}^{3}$, then we generically identify $M$ with $\left\{\operatorname{Im} w^{\prime}=\left|z^{\prime}\right|^{2}\right\} \in \mathbb{C}^{2}$ via $\left(z_{1}, z_{2}, w\right) \mapsto\left(z_{1} z_{2}, w\right)=:\left(z^{\prime}, w^{\prime}\right) \in \mathbb{C}^{2}$. This suggests that $M$ actually depends only on two variables. We could also argue as follows to see that such a representation of $H$ by $\Psi$ as above is not possible for $M$ : We let $H=\left(f_{1}, f_{2}, g\right)$ be a mapping from $M$ to $M$ and introduce $\widetilde{H}=\left(e^{\varphi} f_{1}, e^{-\varphi} f_{2}, g\right)$. If we write $\rho$ for the defining function of $M$, then $\rho \circ \widetilde{H}=\rho \circ H$, i.e., $\varphi$ cannot be determined by taking derivatives of the mapping equation.

Example 12. We start with an example which should illustrate how to construct a "parametrization" $\Psi$ as above in the case of the local automorphisms of $M=\left\{(z, w) \in \mathbb{C}^{2}: \operatorname{Im} w=|z|^{2}\right\}$ fixing 0 , denoted by Aut $_{0}(M, 0)$. ??? Copied from Survey: Jet embeddability of local automorphisms groups of r.-a. CR manifolds section 2.3 by Bernhard Lamel ??? We use coordinates $(z, w, \chi, \tau)$ on $\mathbb{C}^{4}$ to describe the complexification $\mathcal{M}$, which is given by

$$
w-\tau=2 i z \chi
$$

A map $H(z, w)=(f(z, w), g(z, w))$ is an automorphism of $(M, 0)$ fixing 0 if and only if

$$
\begin{equation*}
g(z, w)-\bar{g}(\chi, \tau)=2 i f(z, w) \bar{f}(\chi, \tau) \tag{10.1}
\end{equation*}
$$

when $w-\tau=2 i z \chi, f(0,0)=g(0,0)=0$ and if the matrix

$$
\left(\begin{array}{ll}
\frac{\partial f}{\partial z}(0) & \frac{\partial f}{\partial w}(0) \\
\frac{\partial g}{\partial z}(0) & \frac{\partial g}{\partial w}(0)
\end{array}\right)
$$

is invertible. We now evaluate (10.1) at $w=\chi=\tau=0$ to see that $g(z, 0)=0$; thus, the invertibility condition reduces to $f_{z}(0,0) g_{w}(0,0) \neq 0$.

Now substitute $w=\tau+2 i z \chi$ into (10.1) to see that

$$
\begin{equation*}
g(z, \tau+2 i z \chi)=\bar{g}(\chi, \tau)+2 i f(z, \tau+2 i z \chi) \bar{f}(\chi, \tau) \tag{10.2}
\end{equation*}
$$

an application of $\frac{\partial}{\partial z}$ to (10.2) leads to

$$
\begin{equation*}
g_{z}+g_{w} 2 i \chi=2 i\left(f_{z}+f_{w} 2 i \chi\right) f \tag{10.3}
\end{equation*}
$$

where we have dropped the arguments for better readability. Evaluation of (10.3) at $z=w=\tau=0$ gives

$$
\begin{equation*}
\bar{f}(\chi, 0)\left(f_{z}(0)+2 i \chi f_{w}(0)\right)=g_{w}(0) \chi \tag{10.4}
\end{equation*}
$$

??? From the above we get $g_{w}(0)=\left|f_{z}(0)\right|^{2}$ ??? If we apply $\frac{\partial}{\partial \tau}$ to (10.3) and evaluate again at $z=w=\tau=0$, we get a similar formula for $\bar{f}_{\tau}(\chi, 0)$, namely

$$
\begin{equation*}
2 i \bar{f}_{\tau}(\chi, 0)\left(f_{z}(0)+2 i \chi f_{w}(0)\right)=g_{z w}(0)+g_{w^{2}}(0) 2 i \chi-2 i \bar{f}(\chi, 0)\left(f_{z w}(0)+2 i \chi f_{w^{2}}(0)\right) \tag{10.5}
\end{equation*}
$$

and applying $\frac{\partial}{\partial \tau}$ to (10.2) also a formula for $\bar{g}_{\tau}(\chi, 0)$,

$$
\begin{equation*}
\bar{g}_{\tau}(\chi, 0)=g_{w}(0)-2 i f_{w}(0) \bar{f}(\chi, 0) \tag{10.6}
\end{equation*}
$$

Now we substitute $w=0, \tau=-2 i z \chi$ into (10.1) to obtain

$$
\begin{equation*}
-\bar{g}(\chi,-2 i z \chi)=2 i f(z, 0) \bar{f}(\chi,-2 i z \chi) \tag{10.7}
\end{equation*}
$$

and into (10.3), which leads to

$$
\begin{equation*}
g_{w}(z, 0) \chi=\left(f_{z}(z, 0)+f_{w}(z, 0) 2 i \chi\right) \bar{f}(\chi,-2 i z \chi) \tag{10.8}
\end{equation*}
$$

Substituting back the complex conjugates of (10.5) and (10.6) into (10.8) gives with $D=\bar{f}_{\chi}(0)-2 i z \bar{f}_{\tau}(0)$

$$
\begin{aligned}
\bar{f}(\chi,-2 i z \chi) & =\frac{g_{w}(z, 0) \chi}{f_{z}(z, 0)+2 i \chi f_{w}(z, 0)} \\
& =\frac{\chi\left(\bar{g}_{\tau}(0)+2 i \bar{f}_{\tau}(0) \frac{\bar{g}_{\tau}(0) z}{D}\right)}{\frac{\bar{f}_{\chi}(0) \bar{g}_{\tau}(0)}{D^{2}}+2 i \chi\left(\frac{-\bar{g}_{\chi \tau}(0)+2 i z \bar{g}_{\tau^{2}}(0)}{2 i D}-\frac{\bar{g}_{\tau}(0) z\left(\bar{f}_{\chi \tau}(0)-2 i z \bar{f}_{\tau^{2}}(0)\right)}{D^{2}}\right)} \\
& =\frac{\chi \bar{g}_{\tau}(0) D^{2}+2 i z \chi \bar{f}_{\tau}(0) \bar{g}_{\tau}(0) D}{\bar{f}_{\chi}(0) \bar{g}_{\tau}(0)+\chi D\left(-\bar{g}_{\chi \tau}(0)+2 i z \bar{g}_{\tau^{2}}(0)\right)-2 i z \chi \bar{g}_{\tau}(0)\left(\bar{f}_{\chi \tau}(0)-2 i z \bar{f}_{\tau^{2}}(0)\right)} \\
& =\frac{\chi \bar{f}_{\chi}(0)^{2} \bar{g}_{\tau}(0)-2 i z \chi \bar{f}_{\chi}(0) \bar{g}_{\tau}(0) \bar{f}_{\tau}(0)}{\bar{f}_{\chi}(0) \bar{g}_{\tau}(0)-\chi \bar{f}_{\chi}(0) \bar{g}_{\chi \tau}(0)-2 i z \chi\left(\bar{f}_{\chi \tau}(0) \bar{g}_{\tau}(0)-\bar{f}_{\chi}(0) \bar{g}_{\tau^{2}}(0)\right)+z^{2} \chi R} .
\end{aligned}
$$

Since the left hand side of this equation is a holomorphic function in $\chi$ and $-2 i z \chi, R$ has to vanish. So all in all, we arrive at

$$
\begin{equation*}
f(z, w)=\frac{f_{z}(0) z+f_{w}(0) w}{1-\frac{g_{z w}(0)}{g_{w}(0)} z+\left(\frac{f_{z z}(0)}{f_{z}(0)}-\frac{g_{w^{2}}(0)}{g_{w}(0)}\right) w} \tag{10.9}
\end{equation*}
$$

Using $g(z, 2 i z \chi)=2 i f(z, 2 i z \chi) \bar{f}(\chi, 0)$ and (10.4) we get a similar equation for $g$, namely

$$
\begin{equation*}
g(z, w)=\frac{g_{w}(0) w}{1-\frac{g_{z w}(0)}{g_{w}(0)} z+\left(\frac{f_{z w}(0)}{f_{z}(0)}-\frac{g_{w^{2}}(0)}{g_{w}(0)}\right) w} \tag{10.10}
\end{equation*}
$$

The right hand sides of (10.9) and (10.10) are parametrizations for $f$ and $g$, i.e. $H=\Psi\left(Z, j_{0}^{2} H\right)$. Of course, we would have to arrange things such that $j_{0}^{2} \Psi(Z, \Lambda)=\Lambda$; but computationally, it is preferable to use the additional information present in (10.9) and (10.10) in order to compute $j_{0}^{2}(\operatorname{Aut}(M, 0))$ : We thus check which map of the form

$$
\begin{equation*}
(z, w) \mapsto\left(\frac{\alpha z+\beta w}{1+\gamma z+\delta w}, \frac{\varepsilon w}{1+\gamma z+\delta w}\right) \tag{10.11}
\end{equation*}
$$

gives rise to an automorphism of $M$. Writing $D(z, w)=1+\gamma z+\delta w$, this amounts to finding all $\alpha, \beta, \gamma, \delta, \varepsilon$ such that

$$
\varepsilon(\tau+2 i z \chi) \bar{D}(\chi, \tau)-\bar{\varepsilon} \tau D(z, \tau+2 i z \chi)=(\alpha z+\beta(\tau+2 i z \chi))(\bar{\alpha} \chi+\bar{\beta} \tau)
$$

Comparing coefficients on both sides of this equation, we get

$$
\begin{array}{rr}
\varepsilon-\bar{\varepsilon}=0 & -2 i \beta \bar{\beta}+\varepsilon \bar{\delta}-\delta \bar{\varepsilon}=0 \\
\varepsilon-\alpha \bar{\alpha}=0 & 2 \beta \bar{\alpha}+i \varepsilon \bar{\gamma}=0 .
\end{array}
$$

This describes the image of $\operatorname{Aut}_{0}(M, p)$ defined by mappings of the form (10.11). An explicit parametrization of this image is given by

$$
\begin{equation*}
\alpha=r e^{i \theta}, \quad \beta=r e^{i \theta} a, \quad \gamma=-2 i \bar{a}, \quad \delta=t-i|a|^{2}, \quad \varepsilon=r^{2} \tag{10.12}
\end{equation*}
$$

where $\theta, t \in \mathbb{R}, r \in \mathbb{R}_{+}$and $a \in \mathbb{C}$, with which choices the group $\operatorname{Aut}_{0}(M, 0)$ can be written in the usual form as the mappings of the form

$$
(z, w) \mapsto\left(r e^{i \theta} \frac{z+a w}{1-2 i \bar{a} z+\left(t-i|a|^{2}\right) w}, \frac{r^{2} w}{1-2 i \bar{a} z+\left(t-i|a|^{2}\right) w}\right)
$$

Let us retrace our steps: We first determined $\bar{H}(\chi, 0)$ and $\bar{H}_{\tau}(\chi, 0)$ in (10.4), (10.5), and (10.6) in terms of $j_{0}^{2} H$. This was done using a so called "nondegeneracy condition" on ( $M, 0$ ), namely, that we could solve for $\bar{f}(\chi, 0)$ in (10.3) evaluated at $z=w=\tau=0$. Then we leveraged this knowledge to get a formula for $\bar{f}(\chi,-2 i z \chi)$ in terms of $j_{0}^{2} \bar{H}$, again using the fact that we could solve for it in (10.3) evaluated at $w=0$ in terms of $j_{(z, 0)}^{2} H$. This procedure led to a formula for $H$ since the Segre map $(z, \chi) \mapsto(\chi,-2 i z \chi)$ is generically of full rank.
Definition 25. ??? see Definition 2 ??? Let $p \in \mathbb{C}^{N}, h=\left(h_{1}, \ldots, h_{N^{\prime}}\right) \in \mathcal{O}_{p}^{N^{\prime}}$ and $D(k, N):=\binom{N+k}{N}$. Then the $k$-th jet mapping $j_{p}^{k}$ is defined as

$$
\begin{aligned}
j_{p}^{k} & : \mathcal{O}_{p}^{N^{\prime}} \rightarrow \mathbb{C}^{N^{\prime} D(k, N)} \\
j_{p}^{k} h & :=\left(\frac{\partial^{|\alpha|} h}{\partial^{\alpha} Z}(p):|\alpha| \leq k\right)
\end{aligned}
$$

Further if we write $j_{x}^{k} F(X, \Lambda)$, we take the $k$-jet of $F(X, \Lambda)$ with respect to the $X$-variable evaluated at $x$.

Definition 26. We call $F_{n}\left(\mathbb{C}^{N}\right)$ the (bundle of) germs of $n$-dimensional complex-analytic submanifolds of $\mathbb{C}^{N}$. Germs of fibres $F_{p} \in F_{n}\left(\mathbb{C}^{N}\right)$ are determined by $p \in F_{p}$, the $n$-dimensional subspace $T_{p} F_{p}$ and $d$ holomorphic functions $\varphi=\left(\varphi_{1}, \ldots, \varphi_{d}\right)$ such that $F_{p}=\left\{(x, \varphi(x)): x \in T_{p} F_{p}\right\} \subset T_{p} F_{p} \times T_{p} F_{p}^{\perp}$.
Remark 20. Since the defining functions $\varphi$ for $F_{p}$ are graphed over $T_{p} F_{p}$, they are uniquely determined. ??? argument, where did we do that ??? That means $F_{p} \in F_{n}\left(\mathbb{C}^{N}\right)$ can uniquely be represented by the triple $\left(p, T_{p} F_{p}, \varphi\right) \in \mathbb{C}^{N} \times G r(N, n) \times \mathcal{O}_{p}^{d}$, where $\operatorname{Gr}(N, n)$ stands for the manifold of $n$-dimensional subspaces of $\mathbb{C}^{N}$. It may seem that $F_{p}$ is an infinite dimensional object, but in fact the rank of $F_{p}$ is determined by a finite data. This can be realized by using the jet mapping.
Definition 27. We denote by $F_{n}^{k}\left(\mathbb{C}^{N}\right)$ the space of (bundles of) germs of $k$-jets of $n$-dimensional complexanalytic submanifolds of $\mathbb{C}^{N}$. A germ of a fibre $F_{p}^{k} \in F_{n}^{k}\left(\mathbb{C}^{N}\right)$ is given by the triple $\left(p, T_{p} F, j_{p}^{k} \varphi\right)$.
Remark 21. Here $j_{p}^{k}$ acts on $F_{p} \in F_{n}\left(\mathbb{C}^{N}\right)$ as follows: $j_{p}^{k}: F_{n}\left(\mathbb{C}^{N}\right) \rightarrow F_{n}^{k}\left(\mathbb{C}^{N}\right)$ is defined as $\left(p, T_{p} F_{p}, \varphi\right) \mapsto$ $\left(p, T_{p} F_{p}, j_{p}^{k} \varphi\right)$. Note that for $k=1$ we have $j_{p}^{1}\left(p, T_{p} F_{p}, \varphi\right)=\left(p, T_{p} F_{p}, T_{p} F_{p}\right)$.
Definition 28. Let $M$ be a generic and real-analytic submanifold of $\mathbb{C}^{N}$ of CR-dimension $n$ and $\mathcal{M}$ be its complexification. Then the mapping

$$
\begin{aligned}
\pi: \mathcal{M} & \longrightarrow F_{n}\left(\mathbb{C}^{N}\right), \\
(Z, \zeta) & \longmapsto\left(\mathcal{S}_{\bar{\zeta}}, Z\right),
\end{aligned}
$$

is called the reflection map $\pi$ of $\mathcal{M}$. The $k$-th jet mapping can be used to define

$$
\begin{aligned}
\pi^{k}: \mathcal{M} & \longrightarrow F_{n}^{k}\left(\mathbb{C}^{N}\right), \\
(Z, \zeta) & \longmapsto j_{Z}^{k} \mathcal{S}_{\bar{\zeta}},
\end{aligned}
$$

the reflection mapping $\pi^{k}$ of $\mathcal{M}$ of order $k$.
Definition 29. Let ( $M, p$ ) be a germ of a generic, real-analytic submanifold of $\mathbb{C}^{N}$. Then:
(i) $(M, p)$ is holomorphically nondegenerate if $\pi$ is generically of full rank.
(ii) $(M, p)$ is of class $\mathcal{C}$ at $p$ if $\left.\pi\right|_{p \times \mathcal{S}_{p}}$ is generically of full rank.
(iii) $(M, p)$ is essentially finite at $p$ if $\left.\pi\right|_{p \times \mathcal{S}_{p}}$ is a finite mapping.
(iv) $(M, p)$ is finitely nondegenerate at $p$ if $\left.\pi\right|_{p \times \mathcal{S}_{p}}$ is immersive.

Remark 22. (i) Each of the conditions in Definition 29 can be restated for $\pi^{k}$ if we replace $\pi$ or $\left.\pi\right|_{p \times \mathcal{S}_{p}}$ by the formulation " $\pi^{k}$ for sufficiently large $k \in \mathbb{N}$ " or " $\left.\pi^{k}\right|_{p \times \mathcal{S}_{p}}$ for sufficiently large $k \in \mathbb{N}$ " respectively.
(ii) A finite mapping $H: X \rightarrow$ image $H=: Y$ is a mapping whose fibre at $y \in Y$ consists of finitely many $x \in X$ for each $y \in Y$.
(iii) Note that the formulation for ( $M, p$ ) being holomorphically nondegenerate in Definition 29 does not depend on any particular point near $p$. We call $M$ which is not holomorphically nondegenerate at any point in ( $M, p$ ) holomorphically degenerate.
(iv) How do we use these properties on $\pi$ ? Throughout the computations in Example 12 we used an inversion of $\pi$ as can be illustrated as follows: If $H$ is a mapping (with no fibres, e.g., a biholomorphism or immersion) sending $M$ to $M^{\prime}$, then $\mathcal{H}(Z, \zeta)=(H(Z), \bar{H}(\zeta))$ sends $\mathcal{M}$ to $\mathcal{M}^{\prime}$ and we require that the following diagram commutes: ??? Maybe take the diagram from the lecture ???

(v) In the sequel we want to investigate the relations between the conditions in Definition 29 as well as to give descriptions of these nondegeneracy conditions in terms of normal coordinates.
Lemma 18. Let $M \subset \mathbb{C}^{N}$ be a generic and real-analytic submanifold of codimension $d$ with $n=N-d$. We choose normal coordinates $(z, w)$ centered at 0 so that $\mathcal{M}$ is given by $w=Q(z, \chi, \tau)$ and we write $Q(z, \chi, \tau)=\sum_{\alpha} Q_{\alpha}(\chi, \tau) z^{\alpha}$. Then the following holds:
(i) $M$ is holomorphically nondegenerate if and only if $\zeta \mapsto\left(Q_{\alpha}(\zeta)\right)_{\alpha}$ is generically of full rank $N$.
(ii) $M$ is of class $\mathcal{C}$ if and only if $\chi \mapsto\left(Q_{\alpha}(\chi, 0)\right)_{\alpha}$ is generically of full rank $n$.
(iii) $M$ is essentially finite if and only if $\chi \mapsto\left(Q_{\alpha}(\chi, 0)\right)_{\alpha}$ is finite at $\chi=0$.
(iv) $M$ is finitely nondegenerate if and only if $\chi \mapsto\left(Q_{\alpha}(\chi, 0)\right)_{\alpha}$ is immersive at $\chi=0$.

Proof. We write $Q(z, \chi, \tau)=\sum_{\alpha \in \mathbb{N}^{n}} Q_{z^{\alpha}}(0, \chi, \tau) z^{\alpha}=: \sum_{\alpha \in \mathbb{N}^{n}} Q_{\alpha}(\chi, \tau) z^{\alpha}$. Then for $\left(Z_{0}, \zeta_{0}\right)=\left(z_{0}, w_{0}, \chi_{0}, \tau_{0}\right) \in$ $\mathcal{M}$ we have $\mathcal{S}_{\bar{\zeta}_{0}}=\left\{(z, w) \in \mathbb{C}^{N}: w=Q\left(z, \chi_{0}, \tau_{0}\right)=\sum_{\alpha \in \mathbb{N}^{n}} Q_{\alpha}\left(\chi_{0}, \tau_{0}\right) z^{\alpha}\right\}$. Hence we identify $\pi\left(Z_{0}, \zeta_{0}\right)$ with $\left(\left(Q_{\alpha}\left(\zeta_{0}\right)\right)_{\alpha \in \mathbb{N}^{n}}, Z_{0}\right)$ and all the conditions imposed on $\pi$ are passed on to $\left(Q_{\alpha}(\zeta)\right)_{\alpha \in \mathbb{N}^{n} n}$. We immediately obtain (i) and since $\pi(0)=\left(\mathcal{S}_{0}, 0\right)=\{w=0\}$, we set $\tau=0$ to obtain (ii), (iii) and (iv).

Remark 23. The condition given in Lemma 18 (i) is equivalent to Stanton's criterion: We assume $p=0$ and write $Q=\left(Q^{1}, \ldots, Q^{d}\right)$ and $\zeta=(\chi, \tau) \in \mathbb{C}^{N}$. Since $Q(z, 0, \tau)=\tau$ we have that $Q_{\zeta}(0)=\left(0, I_{d \times d}\right) \in$ $\mathbb{C}^{d N}$. Then in Lemma 18 (i) the characterization of holomorphically nondegeneracy is that $\left(\left(Q_{\alpha}\right)_{\zeta}\left(\zeta_{0}\right)\right)_{\alpha}$ is generically of full rank $n$ for $\zeta_{0}$ near 0 which is equivalent to the statement that there exist indices $\alpha_{1}, \ldots, \alpha_{n}, \beta_{1}, \ldots, \beta_{n}$ where $1 \leq \alpha_{k}, \beta_{l} \leq n$ and $j_{1}, \ldots, j_{n}$, where $1 \leq j_{m} \leq d$ such that

$$
\left|\left(Q_{\alpha_{1}}^{j_{1}}\right)_{\chi_{\beta_{1}}} \cdots\left(Q_{\alpha_{n}}^{j_{n}}\right)_{\chi_{\beta_{n}}}\right| \not \equiv 0,
$$

as a power series in $\chi$ and $\tau$.
Lemma 19. Let $\left(M_{k}, 0\right) \subset \mathbb{C}^{N}$ for $1 \leq k \leq 4$ be germs of generic and real-analytic submanifolds. Then we have:
(i) If $M_{1}$ is finitely nondegenerate, then $M_{1}$ is essentially finite.
(ii) If $M_{2}$ is essentially finite, then $M_{2}$ is of class $\mathcal{C}$.
(iii) If $M_{3}$ is of class $\mathcal{C}$, then $M_{3}$ is holomorphically nondegenerate.
(iv) If $M_{4}$ is holomorphically nondegenerate, then there exist real-analytic subvarieties $V_{1} \subset V_{2} \subset V_{3} \subset M_{4}$, such that for all $p \in M_{4} \backslash V_{1}\left(M_{4}, p\right)$ is of class $\mathcal{C}$, for all $p \in M_{4} \backslash V_{2}\left(M_{4}, p\right)$ is essentially finite and for all $p \in M_{4} \backslash V_{3}\left(M_{4}, p\right)$ is finitely nondegenerate.

## Proof. ??? TBC ???

Example 13. To clarify that the inclusions in Lemma 19 are strict, we give the following examples:
(i) $M_{1}=\left\{(z, w) \in \mathbb{C}^{2}: \operatorname{Im} w=|z|^{4}\right\}$ is essentially finite, but not finitely nondegenerate, since $Q_{1}(z, \chi, \tau)=$ $\tau+2 \mathrm{i} z^{2} \chi^{2}$ and $\chi \mapsto\left(\left(Q_{1}\right)_{\alpha}(\chi, 0)\right)_{\alpha}=2 \mathrm{i} \chi^{2}$ is of rank 0 at $\chi=0$.
(ii) $M_{2}=\left\{\left(z_{1}, z_{2}, w\right) \in \mathbb{C}^{3}: \operatorname{Im} w=\left|z_{1} z_{2}\right|^{2}+\left|z_{1}\right|^{2}\right\}$ is of class $\mathcal{C}$, but not essentially finite, since we have $Q_{2}\left(z_{1}, z_{2}, \chi_{1}, \chi_{2}, \tau\right)=\tau+z_{1} \chi_{1}\left(1+z_{2} \chi_{2}\right)$ and $\left(\chi_{1}, \chi_{2}\right) \mapsto\left(\left(Q_{2}\right)_{\alpha}\left(\chi_{1}, \chi_{2}, 0\right)\right)_{\alpha}=\chi_{1}\left(1, \chi_{2}\right)$ sends $\left(\chi_{1}, \chi_{2}\right)=\left(0, \chi_{2}\right)$ to 0 .
(iii) $M_{3}=\left\{(z, w) \in \mathbb{C}^{2}: \operatorname{Im} w=\operatorname{Re} w|z|^{2}\right\}$ is holomorphically nondegenerate, but not of class $\mathcal{C}$, since we have $Q_{3}(z, \chi, \tau)=\tau \frac{1+\mathrm{i} z \chi}{1-\mathrm{i} z \chi}$ and $Q_{3}(z, \chi, 0) \equiv 0$.
Remark 24 (The situation in). In $\mathbb{C}^{2}$ we have that $M$ is essentially finite if and only if it is of class $\mathcal{C}$. If $M$ is of finite type in $\mathbb{C}^{2}$, then holomorphically nondegenerate is equivalent to class $\mathcal{C}$. A finite type hypersurface, which is holomorphically nondegenerate, but not of class $\mathcal{C}$ for Example 13 (iii) is given in $\mathbb{C}^{3}$. ??? see BER 99 article "Convergence and finite Determination of formal CR maps" at the very end ???

Lemma 20 (Geometric description of nondegeneracy conditions). Let ( $M, p$ ) be a germ of a connected, generic and real-analytic submanifold of $\mathbb{C}^{N}$.
(i) The following statements are equivalent:
(a) $M$ is holomorphically degenerate.
(b) There exists a nontrivial $X=\sum_{j=1}^{N} a_{j}(Z) \frac{\partial}{\partial Z_{j}}$ with $a_{j} \in \mathcal{O}_{q}$ for some $q \in(M, p)$ and $X$ is tangent to $M$ near $q$.
(c) Generically $M \cong \widehat{M} \times \mathbb{C}$ near $q \in(M, p)$, where $\widehat{M}$ is a real-analytic CR-submanifold of $\mathbb{C}^{N-1}$.
(ii) $M$ is essentially finite if and only if $\bigcap_{q \in \mathcal{S}_{p}} \mathcal{S}_{q}=\{p\}$.
(iii) ??? Geometric description of class $\mathcal{C}$ exists in terms of Segre sets ???
(iv) ??? geometric description of finite nondegeneracy or we take an equivalent description or the original definition respectively, with an ascending chain of subspaces $E_{k}$, ... , and see BER-book Def. 11.1.8, p. 317 as well as p. 319f ???

Proof. We start proving (i) and show the first equivalence: We write $Z=(z, w)$ and $\zeta=(\chi, \tau)$, such that $M$ is given in normal coordinates. $M$ is holomorphically degenerate if and only if there exist functions $f_{1}, \ldots, f_{N}$ in the quotient field of $\mathbb{C}\{\zeta\}$, which are not all equal to 0 , such that for all $\alpha$ and $\zeta$ near 0 ,

$$
\sum_{j=1}^{N} f_{j}(\zeta)\left(Q_{\alpha}\right)_{\zeta_{j}}(\zeta)=0
$$

By Taylor's Theorem we have that the previous equation is equivalent to

$$
0=\sum_{\alpha} \sum_{j=1}^{N} f_{j}(\zeta) \frac{\left(Q_{\alpha}\right)_{\zeta_{j}}(\zeta)}{\alpha!} z^{\alpha}=\sum_{j=1}^{N} f_{j}(\zeta) Q_{\zeta}(z, \zeta)
$$

which is equivalent to $X=\sum_{j=1}^{N} f_{j}(\zeta) \frac{\partial}{\partial \zeta_{j}}$ being tangent to $M$ and after conjugation we obtain $Y=$ $\sum_{j=1}^{N} \bar{f}_{j}(Z) \frac{\partial}{\partial Z_{j}}$ a holomorphic vector field tangent to $M$ considering $\tau=\bar{Q}(\chi, z, w)$ as defining function for $M$.
For the second equivalence in (i) the necessary direction is clear. For the sufficient direction suppose $X=$ $\sum_{j=1}^{N} a_{j}(Z) \frac{\partial}{\partial Z_{j}}$ a holomorphic vector field is tangent to $M$. Then $V:=\{p \in M: X(p)=0\}$ is a proper, real-analytic subset of $M$, since otherwise if $X$ vanishes on $M, X$ has to be identically 0 . Thus we can find a point $p_{0} \in M \backslash V$ such that $\{\operatorname{Re} X, \operatorname{Im} X\}$ is a set of linearly independent vector fields tangent to $M$ near $p_{0}$. Since they are deduced from a holomorphic vector field, we have $[\operatorname{Re} X, \operatorname{Im} X]=0$. By the (real) Frobenius Theorem we can find coordinates such that $X=\frac{1}{2}\left(\frac{\partial}{\partial x_{N}}+\mathrm{i} \frac{\partial}{\partial y_{N}}\right)=\frac{\partial}{\partial Z_{N}}$ is tangent to $M$ near $p_{0}$. If $q^{\prime} \in M$ is a point near $p_{0}$, then $\left\{q=\left(q_{1}, \ldots, q_{N}\right) \in \mathbb{C}^{N}: q_{N}=q_{N}^{\prime}\right\} \subset M$ near $q^{\prime}$, which means that generically $M \cong \widehat{M} \times \mathbb{C}$, where $\widehat{M}$ is as in the hypothesis.
??? TBC essential finite ???
Remark 25. One can show that if $M$ is generic and given in normal coordinates near $p$, then a holomorphic vector field $X$ is tangent to $M$ if and only if $X=\sum_{j=1}^{n} a_{j}(Z) \frac{\partial}{\partial z_{j}}$ for $a_{j}(Z) \in \mathcal{O}_{p}$. If we iterate the argument in the proof of (i) we obtain that $M \cong \widetilde{M} \times \mathbb{C}^{k}$, where $\widetilde{M}$ is a holomorphically nondegenerate and realanalytic CR-submanifold with $\operatorname{dim}_{\mathbb{C}} \widetilde{M}=N-k$ or if $M$ is holomorphically degenerate, we have $M \cong \mathbb{C}^{m}$ for $m \leq N$.
Example 14. For $M$ a hypersurface: $M$ is levi-nondegenerate if and only if $M$ is 1-nondegenerate. ??? introduce $k_{0}$-nondegeneracy, see BER-book, Prop. 11.1.12, p. 317 for a proof with differential forms; By Remark 11.1.15 this also works for submanifolds, but then the formulation is as follows: The Levi map is nondegenerate iff the submanifold is finitely nondegenerate ???

## 11 Automorphisms of Real-Analytic CR-Submanifolds

### 11.1 Excursus: Lie Groups and Lie Algebras

??? Very short section concerning the definition of Lie groups and Lie algebras plus basic properties of flows: Maybe we insert this section already when we handle the Lie bracket ???

The following object will be our prototype example of a Lie group.
Definition 30. We define

$$
G_{p}^{k}\left(\mathbb{C}^{N}\right):=\left\{j_{p}^{k} H: H:\left(\mathbb{C}^{N}, p\right) \rightarrow\left(\mathbb{C}^{N}, p\right), H \text { holomorphic, }\left|H^{\prime}(p)\right| \neq 0\right\}
$$

the jet group of order $k$ at $p$.

Remark 26. We can identify $G_{p}^{k}\left(\mathbb{C}^{N}\right)$ with the set of polynomial mappings $H: \mathbb{C}^{N} \rightarrow \mathbb{C}^{N}$ of degree at most $k$ with $H(p)=p$ and $\left|H^{\prime}(p)\right| \neq 0$.
Lemma 21. $G_{p}^{k}\left(\mathbb{C}^{N}\right)$ becomes a complex and algebraic Lie group if we define the operation $\cdot: G_{p}^{k}\left(\mathbb{C}^{N}\right) \times$ $G_{p}^{k}\left(\mathbb{C}^{N}\right) \rightarrow G_{p}^{k}\left(\mathbb{C}^{N}\right) b y$

$$
H \cdot G:=j_{p}^{k}(H \circ G)
$$

for every $H, G \in G_{p}^{k}\left(\mathbb{C}^{N}\right)$.
Proof. $G_{p}^{k}\left(\mathbb{C}^{N}\right)$ is a finite dimensional, complex submanifold with algebraic and immersive parametrization $\varphi=\operatorname{id}_{G_{p}^{k}\left(\mathbb{C}^{N}\right)}$ in the space of all germs of holomorphic mappings of $\left(\mathbb{C}^{N}, p\right)$. Hence all operations on $G_{p}^{k}\left(\mathbb{C}^{N}\right)$ are complex analytic and by the Implicit Function Theorem the group $G_{p}^{k}\left(\mathbb{C}^{N}\right)$ is closed under taking inverses.

### 11.2 Infinitesimal CR-Automorphisms vs. CR-Automorphisms

Motivation 3. ??? Why do we want (finite dimensional) Aut ( $M, p$ ): equivalence problem etc. ???
Definition 31. Let $(M, p)$ be a germ of a CR-submanifold in $\mathbb{C}^{N}$. The space of holomorphic vector fields, whose real part is tangent to $(M, p)$ is called the germ of infinitesimal CR-automorphisms $\mathfrak{h o l}(M, p)$ of ( $M, p$ ). In coordinates we have

$$
\mathfrak{h o l}(M, p):=\left\{X=\sum_{j=1}^{N} a_{j}(Z) \frac{\partial}{\partial Z_{j}}: a_{j} \in \mathcal{O}_{p}, \operatorname{Re} X \text { is tangent to } M \text { near } p\right\} .
$$

We define the subalgebra $\mathfrak{h o l}{ }_{0}(M, p) \subset \mathfrak{h o l}(M, p)$ as

$$
\mathfrak{h o l}_{0}(M, p):=\{X \in \mathfrak{h o l}(M, p): X(p)=0\}
$$

??? The Automorphism group as defined below is actually not a group, but a so called pseudogroup: we must be careful composing two elements, because the image of one automorphism has to be in the domain of the other ???
Definition 32. The automorphism group $\operatorname{Aut}(M, p)$ of $(M, p)$ is defined as

$$
\operatorname{Aut}(M, p)=\left\{H:\left(\mathbb{C}^{N}, p\right) \rightarrow \mathbb{C}^{N}: H(M) \subset M, H \text { holomorphic },\left|H^{\prime}(p)\right| \neq 0\right\}
$$

The isotropy group of $M$ at $p$ or stability group of $M$ at $p \operatorname{Aut}_{p}(M, p)$ is defined as

$$
\operatorname{Aut}_{p}(M, p):=\{H \in \operatorname{Aut}(M, p): H(p)=p\}
$$

Theorem 9. (i) The flows $\Phi_{X}^{t}(Z)$ of $X \in \mathfrak{h o l}(M, p)$ satisfy $\Phi_{X}^{t}(Z) \in \operatorname{Aut}(M, p)$ for $t \in \mathbb{R}$ in a small neighborhood of 0 .
(ii) Let $\operatorname{Aut}(M, p)$ be a finite dimensional Lie group. Then $\mathfrak{h o l}(M, p)$ is the Lie algebra of $\operatorname{Aut}(M, p)$.

Remark 27. (i) We point out that the conclusion of Theorem 9 (i) in particular holds for infinite dimensional Lie algebras $\mathfrak{h o l}(M, p)$.
(ii) Theorem 9 (ii) is also true in the case of $\operatorname{Aut}(M, p)$ being an infinite dimensional Lie group, but one has to use a different argument than below. ??? Is this true ???
(iii) If $\operatorname{Aut}(M, p)$ is a Lie group, then by (i) and (ii) of the above Theorem 9, it follows that there is a one-to-one correspondence of $\mathfrak{h o l}(M, p)$ and $\operatorname{Aut}(M, p)$, i.e., in this case we identify the flows $\Phi_{X}^{t}(Z)$ of $X \in \mathfrak{h o l}(M, p)$ with elements of $\operatorname{Aut}(M, p)$.

Proof. ??? see BER 1999, Prop. 12.4.26, p. 365 for holomorphic vector fields ??? We show the first statement: Let $\Phi(Z, t):=\Phi_{X}^{t}(Z)$ be the flow of $X=\sum_{j=1}^{N} a_{j}(Z) \frac{\partial}{\partial Z_{j}} \in \mathfrak{h o l}(M, p)$ and denote $A(Z)=\left(a_{1}(Z), \ldots, a_{N}(Z)\right)$. Then $\Phi(Z, t)$ solves

$$
\begin{aligned}
\frac{\partial \Phi}{\partial t}(Z, t) & =A(\Phi(Z, t)) \\
\Phi(Z, 0) & =Z
\end{aligned}
$$

Let $\rho$ be the defining function for $M$ and set $r(t):=\rho(\Phi(Z, t), \overline{\Phi(Z, t)})$. Then $r(0)=0$ and

$$
\begin{align*}
\frac{d r(t)}{d t} & =\rho_{Z}(\Phi(Z, t), \overline{\Phi(Z, t)}) \Phi_{t}(Z, t)+\rho_{\bar{Z}}(\Phi(Z, t), \overline{\Phi(Z, t)}) \overline{\Phi_{t}(Z, t)} \\
& =(X \rho)(\Phi(Z, t), \overline{\Phi(Z, t)})+(\bar{X} \rho)(\Phi(Z, t), \overline{\Phi(Z, t)}) \\
& =(2(\operatorname{Re} X) \rho)(\Phi(Z, t), \overline{\Phi(Z, t)}) \tag{11.1}
\end{align*}
$$

Since $\operatorname{Re} X$ is tangent to $M$, using a similar argument as in Example 8, we obtain that there exists a real matrix valued function $B$, such that $(2(\operatorname{Re} X) \rho)(Z, \bar{Z})=B(Z, \bar{Z}) \rho(Z, \bar{Z})$. If we use this fact in (11.1) and set $C(t):=B(\Phi(Z, t), \overline{\Phi(Z, t)})$, we obtain

$$
\frac{d r(t)}{d t}=C(t) r(t)
$$

The unique solution is given by $r \equiv 0$ near 0 . That means $\Phi(Z, t) \in M$ for small $t$ and since $\Phi_{Z}(Z, 0)=i d$, the mapping $Z \mapsto \Phi(Z, t)$ is a one-parameter family of automorphisms of $M$.
To prove (ii) of the Theorem we show that each element of $\operatorname{Aut}(M, p)$ is related to an element of $\mathfrak{h o l}(M, p)$. We let $\rho$ be a defining function for $M$ and since $\operatorname{Aut}(M, p)$ is finite dimensional, we parametrize $\operatorname{Aut}(M, p)$ via $s \mapsto H_{s} \in \operatorname{Aut}(M, p)$ belonging to a neighborhood of the identity, where $s \in \mathbb{R}^{K}$ near 0 for some $K \in \mathbb{N}$ and $H_{0}=i d$. Then for each $s$ there exists a matrix valued function $A_{s}$, such that

$$
\begin{equation*}
\rho\left(H_{s}(Z), \bar{H}_{s}(\bar{Z})\right)=A_{s}(Z, \bar{Z}) \rho(Z, \bar{Z}) \tag{11.2}
\end{equation*}
$$

Since $\operatorname{Aut}(M, p)$ is a Lie group, this equation depends smoothly on $s$. If we differentiate (11.2) with respect to $s$ and evaluate at 0 , we get

$$
\left.\rho_{Z}(Z, \bar{Z}) \frac{d}{d s}\right|_{s=0} H_{s}(Z)+\left.\rho_{\bar{Z}}(Z, \bar{Z}) \frac{d}{d s}\right|_{s=0} \bar{H}_{s}(\bar{Z})=\left.\frac{d}{d s}\right|_{s=0} A_{s}(Z, \bar{Z}) \rho(Z, \bar{Z})
$$

Hence $X=\left.\frac{d}{d s}\right|_{s=0} H_{s}(Z) \frac{\partial}{\partial Z} \in \mathfrak{h o l}(M, p)$ and we conclude that each automorphism gives rise to an infinitesimal automorphism.
??? The dimension of the group of automorphisms of $M$ which do not fix a point, is at most as large as the dimension of $M$ : see Remark 17 , that's why we only need to parametrize the isotropies ???

### 11.3 Jet Parametrization of the Stability Group of CR-Automorphisms

Motivation 4. This section is devoted to give characterizations for $\operatorname{Aut}_{p}(M, p)$ being a finite dimensional Lie group and to explicitly compute biholomorphisms of a large class of real-analytic submanifolds.
Remark 28. If $\operatorname{dim}_{\mathbb{R}} \mathfrak{h o l}_{0}(M, p)<\infty$, then by ??? Kobayashi, Transformation groups in Differential Geometry, p. 13 ??? we obtain that there is a unique topology for $\operatorname{Aut}_{p}(M, p)$ such that $\mathfrak{h o l}{ }_{0}(M, p)$ is the Lie algebra of $\operatorname{Aut}_{p}(M, p)$, but in general we do not know which topology for $\operatorname{Aut}_{p}(M, p)$ occurs. ??? What can happen, which topologies can occur ??? One possible topology for $\operatorname{Aut}_{p}(M, p)$ is the induced topology of the natural topology in the space of holomorphic mappings:

Definition 33. (Topology of uniform convergence on compact sets) Let $H,\left(H_{j}\right)_{j \in \mathbb{N}} \in \mathcal{O}_{p}^{N}$ be defined in a neighborhood $U$ of $p$. Then we say $H_{j}$ converges uniformly to $H$ on the compact neighborhood $K$ of $p$ if there exists a compact neighborhood $K$ of $p$, such that all $H_{j}$ are holomorphic in a neighborhood of $K$, and $H_{j}$ converges uniformly to $H$ on $K$. Further let $\left(K_{i}\right)_{i \in I}$ be a compact exhaustion of the neighborhood $U$ of $p$ and $\mathcal{H}_{i}$ be defined as the space of holomorphic mappings with respect to the uniform convergence on $K_{i}$. Then we define the topology with respect to uniform convergence on compact neighborhoods of $p$ as the inductive limit of the Frechét spaces $\mathcal{H}_{i}$. We equip $\operatorname{Aut}_{p}(M, p)$ with the induced topology of uniform convergence on compact neighborhoods of $p$ and refer to this topology as the natural topology of Aut ${ }_{p}(M, p)$. Definition 34. Let $V \subset G_{p}^{k}\left(\mathbb{C}^{N}\right)$ be a neighborhood of $j_{p}^{k} i d \in G_{p}^{k}\left(\mathbb{C}^{N}\right)$ and $U$ a neighborhood of $\{p\} \times V \subset$ $\mathbb{C}^{N} \times G_{p}^{k}\left(\mathbb{C}^{N}\right)$. Then a mapping $\Psi: U \rightarrow \mathbb{C}^{N}$, which is holomorphic in the first and real-analytic in the second variable, such that

$$
H(Z)=\Psi\left(Z, j_{p}^{k} H\right) \quad \text { and } \quad j_{p}^{l} H=j_{p}^{l} \Psi\left(Z, j_{p}^{k} H\right), \quad \forall l \leq k
$$

for all $H \in \operatorname{Aut}_{p}(M, p)$ with $j_{p}^{k} H \in V$, is called a jet parametrization $\Psi$ for $\operatorname{Aut}_{p}(M, p)$ of order $k$ near $i d \in \operatorname{Aut}_{p}(M, p)$.
Remark 29. (i) We write $\operatorname{Aut}_{p}^{k}(M, p):=\operatorname{image}\left(j_{p}^{k}\left(\operatorname{Aut}_{p}(M, p)\right)\right)$ and note that the jet parametrization is the continuous inverse of $j_{p}^{k}: \operatorname{Aut}_{p}\left(\mathbb{C}^{N}, p\right) \supset \operatorname{Aut}_{p}(M, p) \rightarrow \operatorname{Aut}_{p}^{k}(M, p) \subset G_{p}^{k}\left(\mathbb{C}^{N}\right)$ in a neighborhood $V \subset G_{p}^{k}\left(\mathbb{C}^{N}\right)$ of $j_{p}^{k} i d \in \operatorname{Aut}_{p}^{k}(M, p)$.
(ii) To get a Lie group structure on $\operatorname{Aut}_{p}(M, p)$, we need to transport the topology from $\operatorname{Aut}_{p}^{k}(M, p)$ to $\operatorname{Aut}_{p}(M, p)$. To conclude that $j_{p}^{k}$ is a homeomorphism from $\operatorname{Aut}_{p}(M, p)$ to $\operatorname{Aut}_{p}^{k}(M, p)$, we have the following characterization in terms of the jet parametrization.

Lemma 22. Let $(M, p)$ be a germ of a generic and real-analytic CR-submanifold in $\mathbb{C}^{N}$. The following statements are equivalent:
(i) Aut $_{p}^{k}(M, p)$ is a finite dimensional Lie group with the induced topology of $G_{p}^{k}\left(\mathbb{C}^{N}\right)$.
(ii) There exists a jet parametrization for $\operatorname{Aut}_{p}(M, p)$ of order $k$.
(iii) There exists an embedding $\iota: \operatorname{Aut}_{p}(M, p) \rightarrow G_{p}^{k}\left(\mathbb{C}^{N}\right)$, which is a homeomorphism on its image.

Proof. To verify (i) $\Leftrightarrow$ (ii), we need to identify Aut $_{p}^{k}(M, p)$ as a real-analytic subset of $G_{p}^{k}\left(\mathbb{C}^{N}\right)$. ??? TBC ???

Remark 30. (i) We point out, that if we assume a jet parametrization for $\operatorname{Aut}_{p}(M, p)$, we explicitly specify the real-analytic defining functions for $\operatorname{Aut}_{p}^{k}(M, p)$ in the previous proof.
(ii) In order to construct a jet parametrization for $\operatorname{Aut}_{p}(M, p)$ we need to require nondegeneracy conditions for $M$.

Theorem 10. For $(M, p)$ a germ of a finitely nondegenerate, generic and real-analytic submanifold of $\mathbb{C}^{N}$ and $p$ a point of finite type, there exists a jet parametrization for $\operatorname{Aut}_{p}(M, p)$ of order $l_{0}=l_{0}(M, p)$ near $i d \in \operatorname{Aut}_{p}(M, p)$.

Corollary 5. Let $M$ be a generic and real-analytic submanifold of $\mathbb{C}^{N}$ and assume $p \in M$ is a point of finite type and $M$ is finitely nondegenerate at $p$. Let $H_{1}, H_{2}:\left(\mathbb{C}^{N}, p\right) \rightarrow\left(\mathbb{C}^{N}, p\right)$ be germs of biholomorphisms with $H_{l}(M \cap U) \subset M$ for a neighborhood $U$ of $p$ and $l=1,2$. The following holds for $q \in U$ outside a proper, real-analytic subvariety of $M$ :
There exists $k_{0} \in \mathbb{N}$ such that, if we have

$$
\begin{equation*}
\frac{\partial H_{1}}{\partial Z^{\alpha}}(q)=\frac{\partial H_{2}}{\partial Z^{\alpha}}(q), \quad \forall|\alpha| \leq k_{0} \tag{11.3}
\end{equation*}
$$

then $H_{1} \equiv H_{2}$. Further $k_{0}$ only depends on $N$ and the codimension of $M$.
Proof. The proof of Corollary 5 is a direct consequence of Theorem 10 for $\operatorname{Aut}_{p}(M, p)$.

Remark 31. (i) The heart of the proof of Theorem 10 is Theorem 11, which already gives the jet parametrization along Segre sets. For this to be true, we only need to require the nondegeneracy condition. The main ingredient for the proof of Theorem 11 is the basic identity for biholomorphisms. In Theorem 12 we pass from the Segre sets to normal coordinates to complete the jet parametrization. Here we restrict ourselves to hypersurfaces in order to avoid technicalities. ??? We restrict ourselves to hypersurfaces ???
(ii) If $M$ is a hypersurface then $M$ is of finite type if and only if $M$ is finitely nondegenerate: From Example 11 we know that $M$ being of finite type is equivalent to $Q(z, \chi, 0) \not \equiv 0$, which says that the Segre map $S^{2}$ is generically of full rank.

Lemma 23 (Basic identity for biholomorphisms). Let $(M, p),\left(M^{\prime}, p^{\prime}\right) \in \mathbb{C}^{N}$ be generic and real-analytic submanifolds of real codimension $d, H:\left(\mathbb{C}^{N}, p\right) \rightarrow\left(\mathbb{C}^{N}, p^{\prime}\right)$ a germ of biholomorphisms with $H(M \cap U) \subset M^{\prime}$ for a neighborhood $U$ of $p$ and $H(p)=p^{\prime}$. Further let $(z, w)$ and $\left(z^{\prime}, w^{\prime}\right)$ be normal coordinates for $M$ and $M^{\prime}$ centered at $p$ and $p^{\prime}$ respectively, such that $M=\left\{(z, w) \in \mathbb{C}^{N}: w=Q(z, \chi, \tau)\right\}$ and $M^{\prime}=\left\{\left(z^{\prime}, w^{\prime}\right) \in\right.$ $\left.\mathbb{C}^{N}: w^{\prime}=Q^{\prime}\left(z^{\prime}, \chi^{\prime}, \tau^{\prime}\right)\right\}$. Set $H=(f, g)=\left(z^{\prime}, w^{\prime}\right) \in \mathbb{C}^{n} \times \mathbb{C}^{d}$.
Then there exist polynomials $\left(P_{\alpha}\right)_{\alpha \in \mathbb{N}^{n}}$, which only depend on $M$ and $M^{\prime}$, such that for $(z, w) \in M$ near $p$ we have

$$
\begin{equation*}
Q_{z^{\alpha}}^{\prime}(f(z, w), \bar{H}(\chi, \tau))=\frac{P_{\alpha}\left(j_{(z, w)}^{|\alpha|} H\right)}{D(f, Q)^{2|\alpha|-1}}, \tag{11.4}
\end{equation*}
$$

where $D(f, Q):=\operatorname{det}\left(f_{z}(z, w)+f_{w}(z, w) Q_{z}(z, \chi, \tau)\right)$. Further if we set $\Lambda:=j_{(z, w)}^{|\alpha|} H$ then $P_{\alpha}(\Lambda)$ has coefficients which are holomorphic in $M$.

Proof. W.l.o.g. we assume $p=0=p^{\prime}$. Then $H$ has to satisfy the following mapping equation if $H$ sends $M$ to $M^{\prime}$ :

$$
\begin{equation*}
g(z, Q(z, \chi, \tau))=Q^{\prime}(f(z, Q(z, \chi, \tau)), \bar{H}(\chi, \tau)), \quad \forall(z, \chi, \tau) \in\left(\mathbb{C}^{2 n+d}, 0\right) \tag{11.5}
\end{equation*}
$$

Setting $\tau=0=\chi$ we obtain $g(z, 0)=0$, thus $\left|H^{\prime}(0)\right|=\left|f_{z}(0)\right|\left|g_{w}(0)\right| \neq 0$, which in particular says that $f_{z}(z, w)$ is invertible near 0 . We write $(f, g)=\left(f^{1}, \ldots, f^{n}, g^{1}, \ldots, g^{d}\right), Q=\left(Q^{1}, \ldots, Q^{d}\right)$ and $Q^{\prime}=$ $\left(Q^{\prime 1}, \ldots, Q^{\prime d}\right)$ and differentiate the $m-t h$ component of (11.5) with respect to $z_{j}$ for $1 \leq j \leq n$. We obtain, if we skip the arguments,

$$
g_{z_{j}}^{m}+\sum_{k=1}^{d} g_{w_{k}}^{m} Q_{z_{j}}^{k}=\sum_{k=1}^{n} Q_{z_{k}^{\prime}}^{\prime m}(f, \bar{H})\left(f_{z_{j}}^{k}+\sum_{l=1}^{d} f_{w_{l}}^{k} Q_{z_{j}}^{l}\right)
$$

or - for short - in matrix notation

$$
\begin{equation*}
g_{z}+g_{w} Q_{z}=Q_{z^{\prime}}^{\prime}(f, \bar{H})\left(f_{z}+f_{w} Q_{z}\right) \tag{11.6}
\end{equation*}
$$

Since the $n \times n$-matrix $\left(f_{z}+f_{w} Q_{z}\right)$ is invertible near 0 , we apply Cramer's rule to (11.6) to obtain polynomials $P_{j}$ which satisfy

$$
Q_{z_{j}^{\prime}}^{\prime}(f(z, Q(z, \chi, \tau)), \bar{H}(\chi, \tau))=\frac{P_{j}\left(j_{(z, Q(z, \chi, \tau))}^{1} H\right)}{D(f, Q)}
$$

where $D(f, Q)=\operatorname{det}\left(f_{z}(z, w)+f_{w}(z, w) Q_{z}(z, \chi, \tau)\right)$. To obtain derivatives of $Q^{\prime}$ of order 2 and higher with respect to $z$, we write $\alpha=\beta+e_{k}$, i.e., $|\beta|=|\alpha|-1$. In (11.4) we replace $\alpha$ by $\beta$ and differentiate the result with respect to $z_{k}$ to obtain a polynomial $P_{\alpha}$ such that

$$
Q_{z^{\alpha}}^{\prime}(f, \bar{H}) D(f, Q)=\frac{\left(P_{\beta}\right)_{z_{k}} D(f, Q)^{2|\beta|-1}-(2|\beta|-1) D(f, Q)^{2|\beta|-2}(D(f, Q))_{z_{k}} P_{\beta}}{D(f, Q)^{4|\beta|-2}}=: \frac{P_{\alpha}}{D(f, Q)^{2|\beta|}}
$$

Solving this equation for $Q_{z^{\alpha}}^{\prime}(f, \bar{H})$ with Cramer's rule completes the induction.

Lemma 24. Let $A, b, h:\left(\mathbb{C}^{N}, 0\right) \rightarrow\left(\mathbb{C}^{N}, 0\right)$ be holomorphic and $\left|A^{\prime}(0)\right| \neq 0$. Write $x \in\left(\mathbb{C}^{p}, 0\right)$ and $t \in\left(\mathbb{C}^{q}, 0\right), p+q=N$.
Then for every $\delta \in \mathbb{N}^{q}$ there exists a mapping $p_{\delta}$, such that if

$$
\left.\frac{\partial}{\partial t^{\gamma}}\right|_{t=0} A(h(x, t))=\left.\frac{\partial}{\partial t^{\gamma}}\right|_{t=0} b(x, t), \quad|\gamma| \leq|\delta|,
$$

then

$$
\begin{equation*}
\left.\frac{\partial}{\partial t^{\delta}}\right|_{t=0} h(x, t)=p_{\delta}\left(\left.\frac{\partial}{\partial t^{\varepsilon}}\right|_{t=0} b(x, t),|\varepsilon| \leq|\delta|\right) . \tag{11.7}
\end{equation*}
$$

Further if we set $\Lambda_{\varepsilon}:=\left.\frac{\partial}{\partial t^{\varepsilon}}\right|_{t=0} b(x, t)$ we have that $p_{\delta}\left(\Lambda_{\varepsilon}\right)$ is polynomial for $|\varepsilon|>0$ and holomorphic for all $\varepsilon$.

Proof. First we let $\delta=0$ : We have to solve $A(h(x, 0))=b(x, 0)$ for $h$. By the Implicit Function Theorem there exists a holomorphic $B: \mathbb{C}^{N} \rightarrow \mathbb{C}^{N}$ such that $h(x, 0)=B(A(h(x, 0)))=B(b(x, 0))$. Since $h$ is only assumed to be holomorphic, $B$ at the right-hand side of the previous equation need not be polynomial in the variable $b(x, 0)$. Next let $|\delta|=1$ and write $t=\left(t_{1}, \ldots, t_{q}\right)$. For $1 \leq l \leq q$ we have

$$
\left.\frac{\partial}{\partial t_{l}}\right|_{t=0} A(h(x, t))=b_{t_{l}}(x, 0) \Leftrightarrow A^{\prime}(h(x, 0)) h_{t_{l}}(x, 0)=b_{t_{l}}(x, 0)
$$

which can be solved for $h_{t_{l}}(x, 0)$ by Cramer's rule. We obtain a holomorphic mapping $\widetilde{p}_{e_{l}}$ with

$$
h_{t_{l}}(x, 0)=\widetilde{p}_{e_{l}}\left(b_{t_{l}}(x, 0), h(x, 0)\right)=\widetilde{p}_{e_{l}}\left(b_{t_{l}}(x, 0), B(b(x, 0))\right)=: p_{e_{l}}\left(b_{t_{l}}(x, 0), b(x, 0)\right)
$$

and we have shown the claim for $|\delta|=1$. For $|\delta|>1$ we obtain a holomorphic mapping $F_{\delta}$ such that

$$
\left.\frac{\partial}{\partial t^{\delta}}\right|_{t=0} A(h(x, t))=A^{\prime}(h(x, 0)) h_{t^{\delta}}(x, 0)+F_{\delta}\left(h_{t^{\varepsilon}}(x, 0),|\varepsilon|<|\delta|\right)
$$

We apply the induction hypothesis to the arguments of $F_{\delta}$ and solve the resulting equation using Cramer's rule to obtain $p_{\delta}$ and the desired equation (11.7). Note that $F_{\delta}$ is polynomial in its arguments where $|\varepsilon|>0$.

Lemma 25. Let $k \in \mathbb{N}$ and a holomorphic mapping $H: \mathbb{C}^{N} \rightarrow \mathbb{C}^{N}$. Then there exist polynomials $P_{m}$, whose coefficients are analytic functions in $\left(x^{[k]} ; t\right) \in \mathbb{C}^{k n} \times \mathbb{C}^{N}$, such that

$$
j_{S^{k}\left(x^{[k]} ; t\right)}^{m} H=P_{m}\left(j_{\left(x^{[k]} ; t\right)}^{m}\left(H \circ S^{k}\right)\right), \quad \forall m \geq 0
$$

if $\left(x^{[k]} ; t\right)$ is in a small neighborhood of 0 .
Proof. Let $N=n+d$, then we write $t=\left(t_{1}, t_{2}\right) \in \mathbb{C}^{n} \times \mathbb{C}^{d}$ and $H=(f, g) \in \mathbb{C}^{n} \times \mathbb{C}^{d}$. Further we set $S^{k}\left(x^{[k]} ; t\right)=\left(x^{k}, V^{k}\left(x^{[k]} ; t\right)\right)$, then

$$
\begin{align*}
\frac{\partial\left(H \circ S^{k}\right)}{\partial\left(x^{k}, t_{2}\right)}\left(x^{[k]} ; t\right) & =\left(\begin{array}{cc}
f_{z}\left(S^{k}\right)+f_{w}\left(S^{k}\right) V_{x^{k}}^{k} & f_{w}\left(S^{k}\right) V_{t_{2}}^{k} \\
g_{z}\left(S^{k}\right)+g_{w}\left(S^{k}\right) V_{x^{k}}^{k} & g_{w}\left(S^{k}\right) V_{t_{2}}^{k}
\end{array}\right)\left(x^{[k]} ; t\right) \\
& =\frac{\partial H}{\partial(z, w)}\left(S^{k}\left(x^{[k]} ; t\right)\right)\left(\begin{array}{cc}
I_{n \times n} & 0 \\
V_{x^{k}}^{k} & V_{t_{2}}^{k}
\end{array}\right)\left(x^{[k]} ; t\right) \tag{11.8}
\end{align*}
$$

Since $Q\left(x^{1}, 0, t_{2}\right)=t_{2}$, we have that $V^{k}\left(0 ; 0, t_{2}\right)=t_{2}$, hence the $d \times d$-matrix $V_{t_{2}}^{k}\left(x^{[k]} ; t\right)$ is invertible for small $\left(x^{[k]} ; t\right)$. By Cramer's rule we can solve for the first order derivatives of $H$ evaluated at $S^{k}\left(x^{[k]} ; t\right)$ in terms of the required polynomial $P_{1}$ depending on first order derivatives of $H \circ S^{k}\left(x^{[k]} ; t\right)$. This computation covers the case $m=1$ in the hypothesis. For $m>1$ we proceed inductively: We take higher order derivatives of (11.8) and apply the induction hypothesis to the derivatives of $H$ at $S^{k}\left(x^{[k]} ; t\right)$, which are of order less than $m$. The expression, where we take derivatives of order $m$, is handled in the same manner as in the case for $m=1$.

Theorem 11. Let $M$ be a finitely nondegenerate, generic and real-analytic submanifold of $\mathbb{C}^{N}$.
Then for each $k \geq 1$ there exists $k_{0}=k_{0}(k) \in \mathbb{N}$ and a holomorphic mapping $\Psi_{k}: \mathbb{C}^{k n} \times G_{p}^{k_{0}}\left(\mathbb{C}^{N}\right) \rightarrow \mathbb{C}^{N}$, such that

$$
\begin{equation*}
H \circ S^{k}\left(x^{[k]} ; p\right)=\Psi_{k}\left(x^{[k]}, j_{p}^{k_{0}} H\right) \tag{11.9}
\end{equation*}
$$

for all $H \in \operatorname{Aut}_{p}(M, p)$ near $i d \in \operatorname{Aut}_{p}(M, p)$. We write $p=\left(p_{1}, p_{2}\right) \in \mathbb{C}^{n} \times \mathbb{C}^{d}$ and $p^{[k]}:=\left(p_{1}, 0, \ldots, 0 ; 0, p_{2}\right) \in$ $\mathbb{C}^{k n} \times \mathbb{C}^{N}$. Then for each $k \geq 1$ the mapping $\Psi_{k}$ satisfies the following condition:

$$
j_{p}^{l} H=j_{p^{[k]}}^{l} \Psi_{k}\left(x^{[k]}, j_{p}^{k_{0}} H\right), \quad \forall l \leq k_{0}
$$

Here we skip the conjugation of $p^{[k]}$ depending on $k$. ??? Here the RHS has to be understood as taking derivatives w.r.t. $x^{k}$ and $t_{2}$, as in (11.8) of Lemma 25 ???
In short we describe the conclusion of this Theorem as " $\operatorname{Aut}_{p}(M, p)$ has a jet parametrization along Segre sets".

Proof. W.l.o.g. we assume $p=0$. Further we choose normal coordinates for $\mathcal{M}$ and write $Z=(z, Q(z, \chi, \tau))$ and $\zeta=(\chi, \tau)$. We denote the right-hand side of (11.4) from Lemma 23 as $R_{\alpha}$, a $\mathbb{C}^{d}$-valued mapping and $Q=\left(Q^{1}, \ldots, Q^{d}\right)$ and $R_{\alpha}=\left(R_{\alpha}^{1}, \ldots, R_{\alpha}^{d}\right)$. We recall what it means for $M$ to be finitely nondegenerate: From Lemma 18 (iv) we know that $\chi \mapsto\left(Q_{z^{\alpha}}(0, \chi, 0)\right)_{\alpha}$ is immersive at $\chi=0$ if and only if there exist indices $\alpha^{1}, \ldots, \alpha^{n} \in \mathbb{N}$ and $j^{1}, \ldots, j^{n} \in \mathbb{N}$ such that

$$
\begin{equation*}
\left|Q_{z^{\alpha} \chi}(0)\right|:=\left|Q_{z^{\alpha_{1}} \chi}^{j^{1}}(0) \cdots Q_{z^{\alpha_{n}} \chi}^{j^{n}}(0)\right| \neq 0 \tag{11.10}
\end{equation*}
$$

We start by showing (11.9) for $k=1$. We choose the following parametrization for $(Z, \zeta)=(0, s, x, s) \in$ $\mathcal{M}$. Then we have from the mapping equation and (11.4) the following system for $\bar{H}(x, s)$ :

$$
\begin{aligned}
\bar{g}(x, s)-\bar{Q}(\bar{f}(x, s), H(0, s)) & =0 \\
Q_{z^{\alpha^{l}}}^{j^{l}}(f(0, s), \bar{f}(x, s), \bar{g}(x, s)) & =R_{\alpha^{l}}^{j^{l}}, \quad 1 \leq l \leq n
\end{aligned}
$$

We set $m_{0}:=\max _{1 \leq l \leq n} \alpha^{l}$ and denote the above system by $F^{1}(\bar{H}(x, s))=B^{1}$, where both sides are $\mathbb{C}^{N_{-}}$ valued and $B^{1}$ only depends on the $m_{0}$-jet of $H$ at $(0, s)$ and derivatives of $Q$. We write $y=\left(y_{1}, y_{2}\right):=$ $(\bar{f}(x, s), \bar{g}(x, s))$ and compute

$$
F_{y}^{1}(0)=\left(\begin{array}{cc}
0 & I_{d \times d} \\
Q_{z^{\alpha} \chi}(0) & *
\end{array}\right)
$$

which is invertible for $\bar{H}(x, s)$ near 0 since we assumed the nondegeneracy condition (11.10). We apply Lemma 24 to $F^{1}(\bar{H}(x, s))=B^{1}$ and obtain mappings $\Phi_{\beta}^{1}$ such that for all $\beta$

$$
\begin{equation*}
\left.\frac{\partial^{|\beta|}}{\partial s^{\beta}}\right|_{s=0} \bar{H}(x, s)=\Phi_{\beta}^{1}\left(x,\left.\frac{d^{|\gamma|}}{d s^{\gamma}}\right|_{s=0}\left(j_{(0, s)}^{m_{0}} H\right),|\gamma| \leq|\beta|\right)=: \Phi_{\beta}^{1}\left(x, j_{0}^{m_{0}+|\beta|} H\right) \tag{11.11}
\end{equation*}
$$

Note that we could have relied on Lemma 25 for $S^{0}(s)=(0, s)$, in the last equality, but this would be too much in this case. For $\beta=0$ and setting $s=Q(x, t)$ we have

$$
\bar{H} \circ S^{1}(x ; 0)=\bar{H}(x, 0)=\Phi_{0}^{1}\left(x, j_{0}^{m_{0}} H\right)
$$

and we get the desired formula for $k=1$. The above formula implies that $\Phi_{0}^{1}$ has to satisfy the compatibility condition.

We note that (11.11) gives $t$-derivatives of $\bar{H} \circ S^{1}(x ; t)$ at 0 in terms of derivatives of $H$ at 0 :

$$
\left.\frac{d^{|\beta|}}{d t^{\beta}}\right|_{t=0}\left(\bar{H} \circ S^{1}(x ; t)\right)=\left.\frac{d^{|\beta|}}{d t^{\beta}}\right|_{t=0} \bar{H}(x, Q(x, t))=\left.\frac{\partial^{|\beta|}}{\partial s^{\beta}}\right|_{s=0} \bar{H}(x, s)=\Phi_{\beta}^{1}\left(x, j_{0}^{m_{0}+|\beta|} H\right)
$$

If we take derivatives with respect to $x$ of the above formula we obtain that

$$
\begin{equation*}
j_{(x ; 0)}^{m}\left(\bar{H} \circ S^{1}\right)=\Psi_{m}^{1}\left(x, j_{0}^{m_{0}+m} H\right) \tag{11.12}
\end{equation*}
$$

To determine $H$ on higher iterates of the Segre maps we prove an induction on $k$, where a formula as in (11.12) serves as induction hypothesis:

$$
\begin{equation*}
j_{\left(x^{[k]} ; 0\right)}^{m}\left(H \circ S^{k}\right)=\Psi_{m}^{k}\left(x^{[k]}, j_{0}^{m_{0} k+m} H\right), \quad \forall m \geq 0 \tag{11.13}
\end{equation*}
$$

We skip the conjugation of $H$, which varies with $k$. Note that a formula as (11.13) for $k \geq 2$ proves the Theorem if we set $m=0$. We show (11.13) for $k$, assuming the formula for $k-1$.
We start by setting $k_{0}(k):=m_{0} k$ and noting that the Segre sets have the following property ??? Include this in the appropriate section ???

$$
\left(S^{k}\left(x^{[k]} ; t\right), \bar{S}^{k-1}\left(x^{[k-1]} ; t\right)\right) \in \mathcal{M} \quad \forall k \geq 1,\left(x^{[k]}, t\right) \in \mathbb{C}^{(k+1) N}
$$

From this fact it follows, as for $k=1$, that in the system $F^{k}\left(H \circ S^{k}\left(x^{[k]} ; t\right)\right)=B^{k}, B^{k}$ only depends on the $m_{0}$-jet of $H$ at $S^{k-1}\left(x^{[k-1]} ; t\right)$ and derivatives of $Q$. We write $y=\left(y_{1}, y_{2}\right):=\left(f \circ S^{k}\left(x^{[k]} ; t\right), g \circ S^{k}\left(x^{[k]} ; t\right)\right)$ and obtain as above that $F_{y}^{k}(0)$ is invertible assuming the nondegeneracy condition from (11.10). Thus we apply Lemma 24 to the system $F^{k}\left(H \circ S^{k}\left(x^{[k]} ; t\right)\right)=B^{k}$ to get holomorphic mappings $\Phi_{\beta}^{k}$ such that

$$
\begin{equation*}
\left.\frac{d^{|\beta|}}{d t^{\beta}}\right|_{t=0} H\left(S^{k}\left(x^{[k]} ; t\right)\right)=\Phi_{\beta}^{k}\left(x^{[k]},\left.\frac{d^{|\gamma|}}{d t^{\gamma}}\right|_{t=0}\left(j_{S^{k-1}\left(x^{[k-1]} ; t\right)}^{m_{0}} H\right),|\gamma| \leq|\beta|\right) . \tag{11.14}
\end{equation*}
$$

We want to write the right-hand side of (11.14) as a mapping depending on the $(k-1) m_{0}+|\beta|$-jet of $H \circ S^{k-1}\left(x^{[k-1]} ; t\right)$ evaluated at 0 . We use Lemma 25 in the first equality and the induction hypothesis (11.13) for the equality before the last equation to obtain

$$
\begin{aligned}
& \Phi_{\beta}^{k}\left(x^{[k]},\left.\frac{d^{|\gamma|}}{d t^{\gamma}}\right|_{t=0}\left(j_{S^{k-1}\left(x^{[k-1]} ; t\right)}^{m_{0}} H\right),|\gamma| \leq|\beta|\right) \\
= & \Phi_{\beta}^{k}\left(x^{[k]},\left.\frac{d^{|\gamma|}}{d t^{\gamma}}\right|_{t=0}\left(P_{m_{0}}\left(j_{\left(x^{[k-1]} ; t\right)}^{m_{0}} H \circ S^{k-1}\right)\right),|\gamma| \leq|\beta|\right) \\
= & : \widetilde{\Phi}_{\beta}^{k}\left(x^{[k]},\left.\frac{d^{|\gamma|}}{d t^{\gamma}}\right|_{t=0}\left(j_{\left(x^{[k-1]} ; t\right)}^{m_{0}} H \circ S^{k-1}\right),|\gamma| \leq|\beta|\right) \\
= & \widetilde{\Phi}_{\beta}^{k}\left(x^{[k]}, j_{\left(x^{[k-1]} ; 0\right)}^{m_{0}+|\gamma|} H \circ S^{k-1},|\gamma| \leq|\beta|\right) \\
= & \widetilde{\Phi}_{\beta}^{k}\left(x^{[k]}, \Psi_{m_{0}+|\gamma|}^{k-1}\left(x^{[k-1]}, j_{0}^{(k-1) m_{0}+m_{0}+|\gamma|} H\right),|\gamma| \leq|\beta|\right) \\
= & \left.: \widetilde{\Psi}_{\beta}^{k}\left(x^{[k]}, j_{0}^{m_{0} k+|\gamma|} H\right),|\gamma| \leq|\beta|\right) .
\end{aligned}
$$

Taking derivatives with respect to $x$ gives (11.13) for $k$. ??? Add a note how the compatibility follows from the compatibility of the jet parametrization along Segre sets ???

Remark 32. ??? The previous Theorem makes the picture of inverting the reflection map explicit ???

Theorem 12. Let $M$ be a real-analytic hypersurface of $\mathbb{C}^{N}$ of finite type at $p \in M$ and $\operatorname{Aut}_{p}(M, p)$ has a jet parametrization along Segre sets as in Theorem 11.
Then there exists a jet parametrization $\Psi$ for $\operatorname{Aut}_{p}(M, p)$ of order $k_{0}$ near id $\in \operatorname{Aut}_{p}(M, p)$, such that if we choose normal coordinates $(z, w) \in \mathbb{C}^{n} \times \mathbb{C}$ centered at $p \in M$, we have

$$
H(z, w)=\Psi\left(z, w, j_{0}^{k_{0}} H\right)
$$

for all $(z, w) \in M$ near $p$.

Remark 33. The proof of Theorem 12 consists of two parts. In the first part we invert the second Segre $\operatorname{map}(z, \chi) \mapsto(z, Q(z, \chi, 0))$ evaluated at $p=0$ "singularly", meaning that we write $(z, w)$ for $(z, Q(z, \chi, 0))$ with the consequence that the jet parametrization $\Psi_{k_{0}}$ along Segre sets has singularities. In the second step we "desingularise" the parametrization from the first step to obtain a holomorphic mapping $\Psi$, which is the desired jet parametrization for $\operatorname{Aut}_{p}(M, p)$.
Proof. Let $(z, w)$ denote normal coordinates for $M$. We write $(z, \chi)=\left(z_{1}, \ldots, z_{n}, \chi_{1}, \ldots, \chi_{n}\right)$ and since $M$ is a hypersurface of finite type, $M$ is finitely nondegenerate. This means by Lemma 18 (iv), that there exist multiindices $\alpha_{1}, \ldots, \alpha_{n} \in \mathbb{N}^{n}$, such that

$$
\left|\begin{array}{ccc}
Q_{z^{\alpha_{1}} \chi_{1}}(0) & \cdots & Q_{z^{\alpha_{1}} \chi_{n}}(0) \\
\vdots & & \vdots \\
Q_{z^{\alpha_{n}} \chi_{1}}(0) & \cdots & Q_{z^{\alpha_{n}} \chi_{n}}(0)
\end{array}\right| \neq 0 .
$$

We take the lexicographical ordering in $\mathbb{N}^{n}$ and write

$$
Q_{k}(z):=Q_{z^{\alpha_{1}} \chi_{k}}(0) z^{\alpha_{1}}+\ldots+Q_{z^{\alpha_{n}} \chi_{k}}(0) z^{\alpha_{n}}
$$

where for all $1 \leq k \leq n$ we have $Q_{k} \not \equiv 0$. Then we expand

$$
Q(z, \chi, 0)=\sum_{|\gamma| \geq 1} a_{\gamma}(z) \chi^{\gamma}
$$

and note that $a_{\gamma}(0)=0$ for all $\gamma$. We have $A_{k}(z):=a_{e_{k}}(z)=Q_{k}(z)+B_{k}(z)$ for some holomorphic function $B_{k}$, where in $B_{k}$ only monomial in $z$ occurs which are of higher order with respect to the lexicographical ordering. Thus, taking a small neighborhood of 0 , we can assume $A_{k} \not \equiv 0$ for all $1 \leq k \leq n$. We want to solve for $\chi$ in

$$
\begin{equation*}
w=Q(z, \chi, 0)=\sum_{k=1}^{n} A_{k} \chi_{k}+\sum_{|\gamma| \geq 2} a_{\gamma}(z) \chi^{\gamma} \tag{11.15}
\end{equation*}
$$

We set $\chi_{2}=\ldots=\chi_{n}=0$ and divide (11.15) by $A_{1}^{2}$ to obtain

$$
\frac{w}{A_{1}^{2}}=\frac{\chi_{1}}{A_{1}}+\sum_{j \geq 2} b_{j}^{1}(z) \frac{\chi_{1}^{j}}{A_{1}^{2}}
$$

and set $t_{1}:=\frac{w}{A_{1}^{2}}$ and $u_{1}:=\frac{\chi_{1}}{A_{1}}$. We solve the so obtained equation

$$
t_{1}=u_{1}+\sum_{j \geq 2} b_{j}^{1}(z) A_{1}^{j-2} u_{1}^{j}
$$

for $u_{1}$ applying the Implicit Function Theorem. The solution is given by

$$
u_{1}=t_{1}+\sum_{j \geq 2} v_{j}^{1}(z) t_{1}^{j}=: \varphi_{1}\left(z, t_{1}\right)
$$

where $v_{j}^{1}(0)=0$ for $j \geq 2$. Thus

$$
\chi_{1}=A_{1} u_{1}=A_{1} \varphi_{1}\left(z, t_{1}\right)=A_{1} \varphi_{1}\left(z, \frac{w}{A_{1}^{2}}\right)
$$

solves $w=Q\left(z, \chi_{1}, 0,0\right)$, i.e., $w=Q\left(z, A_{1} \varphi_{1}\left(z, \frac{w}{A_{1}^{2}}\right), 0,0\right)$, if $t_{1}=\frac{w}{A_{1}^{2}}$ stays in in a small neighborhood of 0 according to the neighborhood given by the Implicit Function Theorem. Assume we have functions $\varphi_{l}$ such
that for $1 \leq l \leq k$ we have $\chi_{l}=A_{l} \varphi_{l}\left(z, w / A_{1}^{2}, \ldots, w / A_{l}^{2}\right)$ satisfying $w=Q\left(z, \chi_{1}, \ldots, \chi_{k}, 0,0\right)$. We want to find $\chi_{k+1}$, depending on $z, A_{k+1}$ and $\chi_{1}, \ldots, \chi_{k}$ such that $w=Q\left(z, \chi_{1}, \ldots, \chi_{k+1}, 0,0\right)$.
For $1 \leq k \leq n-2$ we set $\chi_{k+2}=\ldots=\chi_{n}=0$ and $\chi^{\prime}=\left(\chi_{1}, \ldots, \chi_{k}\right)$ in (11.15), which becomes

$$
\begin{equation*}
w=\widetilde{A}_{k+1}\left(z, \chi^{\prime}\right)+A_{k+1}(z) \chi_{k+1}+\sum_{j \geq 2} b_{j}^{k+1}\left(z, \chi^{\prime}\right) \chi_{k+1}^{j} \tag{11.16}
\end{equation*}
$$

where $\widetilde{A}_{k+1}$ is a function only depending on $z$ and linearly on $\chi^{\prime}$. Then we divide the previous equation (11.16) by $A_{k+1}^{2}$ to obtain

$$
\frac{w}{A_{k+1}^{2}}=\frac{\widetilde{A}_{k+1}}{A_{k+1}^{2}}+\frac{\chi_{k+1}}{A_{k+1}}+\sum_{j \geq 2} b_{j}^{k+1} \frac{\chi_{k+1}^{j}}{A_{k+1}^{2}}
$$

and set $t_{k+1}:=\frac{w}{A_{k+1}^{2}}$ and $u_{k+1}:=\frac{\chi_{k+1}}{A_{k+1}}$. We solve the so obtained equation for $u_{k+1}$

$$
t_{k+1}=C_{k+1}\left(z, \chi^{\prime}, t_{k+1}\right)+u_{k+1}+\sum_{j \geq 2} b_{j}^{k+1} A_{k+1}^{j-2} u_{k+1}^{j}
$$

where $C_{k+1}$ does not depend on $u_{k+1}$, applying the Implicit Function Theorem. The solution is denoted by $u_{k+1}=\varphi_{k+1}\left(z, \chi^{\prime}, t_{k+1}\right)$. Thus

$$
\chi_{k+1}=A_{k+1} u_{k+1}=A_{k+1} \varphi_{k+1}\left(z, \chi^{\prime}, t_{k+1}\right)=A_{k+1} \varphi_{k+1}\left(z, \chi^{\prime}, \frac{w}{A_{k+1}^{2}}\right)=: A_{k+1} \widetilde{\varphi}_{k+1}\left(z, \frac{w}{A_{1}^{2}}, \ldots, \frac{w}{A_{k+1}^{2}}\right)
$$

solves $w=Q\left(z, \chi^{\prime}, \chi_{k+1}, 0,0\right)$, i.e.,

$$
w=Q\left(z, A_{1} \varphi_{1}\left(z, w / A_{1}^{2}\right), A_{2} \widetilde{\varphi}_{2}\left(z, w / A_{1}^{2}, w / A_{2}^{2}\right), \ldots, A_{k+1} \widetilde{\varphi}_{k+1}\left(z, w / A_{1}^{2}, \ldots, w / A_{k+1}^{2}\right), 0,0\right)
$$

for $1 \leq k \leq n-1$, if $t_{1}, \ldots, t_{k+1}$ and $z$ stay in a small neighborhood of 0 according to the neighborhoods given by the Implicit Function Theorem. The case $k=n-1$ works in the same way as the induction step from $k$ to $k+1$.
We end up with a holomorphic map $\widetilde{\varphi}:=\left(\varphi_{1}, \widetilde{\varphi}_{2}, \ldots, \widetilde{\varphi}_{n}\right)$, depending on $z$ and $w / A_{1}^{2}, \ldots, w / A_{n}^{2}$, satisfying $w=Q\left(z, \widetilde{\varphi}\left(z, w / A_{1}^{2}, \ldots, w / A_{n}^{2}\right)\right)$. We set $A:=A_{1} \cdots A_{n}$ and consider $\varphi\left(z, w / A^{2}\right):=\widetilde{\varphi}\left(z, w / A_{1}^{2}, \ldots, w / A_{n}^{2}\right)$. If we set $\chi=\varphi$ we obtain

$$
\begin{align*}
H(z, w) & =H(z, Q(z, \chi, 0))=H \circ S^{2}(z, \chi ; 0) \\
& =\Psi\left(z, \chi, j_{0}^{k_{0}} H\right)=\Psi\left(z, \varphi\left(z, \frac{w}{A^{2}}\right), j_{0}^{k_{0}} H\right)=\sum_{j \geq 1} c_{j}\left(z, j_{0}^{k_{0}} H\right)\left(\frac{w}{A^{2}}\right)^{j} \tag{11.17}
\end{align*}
$$

After a linear change of coordinates in $z$ we assume that $A$ is $m$-regular for some $m \in \mathbb{N}$. Note that a linear change of normal coordinates $(z, w)$ preserves the normality condition. Applying the Weierstrass Division Theorem there exist functions $q_{j}$ and $r_{j}$, both holomorphic in the first variable, such that

$$
c_{j}\left(z, j_{0}^{k_{0}} H\right)=A^{2 j}(z) q_{j}\left(z, j_{0}^{k_{0}} H\right)+r_{j}\left(z, j_{0}^{k_{0}} H\right)
$$

Since the left-hand side of (11.17) is required to be holomorphic with respect to $(z, w)$, we must have that $r_{j} \equiv 0$ for all $j \geq 1$. Hence

$$
H(z, w)=\sum_{j \geq 1} q_{j}\left(z, j_{0}^{k_{0}} H\right) w^{j}
$$

which is formally a power series expansion without singularities in $(z, w)$ at 0 . Since the $c_{j}$ are the coefficients of a holomorphic mapping $\Phi(t):=H(z, Q(z, \varphi(z, t), 0))$, the convergence of $H$ is guaranteed by the estimates given in the Weierstrass Division Theorem, which provide Cauchy estimates for $q_{j}$ and the convergence of $H$.
??? Discuss how to get the jet parametrization in the general case of a submanifold instead of a hypersurface ???
??? Maybe add a note how to get a jet parametrization for $\operatorname{Aut}(M, p)$ ???

### 11.4 Infinitesimal CR-Automorphisms

??? Stanton's Theorem would follow for $\mathfrak{h o l}{ }_{0}(M)$ directly from the jet parametrization Theorem if we have the jet parametrization for holomorphically nondegenerate submanifolds of finite type, since then we know that the stability group of the automorphism group is a finite dimensional Lie group, which is of the same dimension as its Lie Algebra - the infinitesimal automorphisms sending 0 to 0 . The transitive part of the automorphisms is finite dimensional automatically. ??? But for the proof of Stanton's Theorem one just needs the jet determination (a consequence of the jet parametrization) for finitely nondegenerate submanifolds. Till now, we only have the jet parametrization for finitely nondegenerate hypersurfaces ???
Remark 34. We denote the space $\mathfrak{h t}(M, p)$ of tangent holomorphic vector fields on $(M, p)$ by

$$
\mathfrak{h t}(M, p):=\left\{X=\sum_{j=1}^{N} a_{j}(Z) \frac{\partial}{\partial Z_{j}}: a_{j}(Z) \in \mathcal{O}_{p}, X \text { is tangent to } M \text { near } p\right\}
$$

Then $\mathfrak{h t}(M, p) \subset \mathfrak{h o l}(M, p)$ as algebras and even more is true:

$$
\begin{aligned}
\mathfrak{h o l}(M, p) \cap \mathfrak{i h o l}(M, p) & =\{X \in \mathfrak{h o l}(M, p): \operatorname{Re} X \text { tangent to } M\} \cap\{\mathrm{i} X \in \mathfrak{h o l}(M, p): \operatorname{Re}(\mathrm{i} X) \text { tangent to } M\} \\
& =\{X \in \mathfrak{h o l}(M, p): X \text { tangent to } M\}=\mathfrak{h t}(M, p) .
\end{aligned}
$$

That means $\mathfrak{h t}(M, p)$ is the maximal complex subspace of $\mathfrak{h o l}(M, p)$ or seen differently $\mathfrak{h o l}(M, p)$ is totally real if and only if $M$ is holomorphically nondegenerate. If $M$ is holomorphically degenerate we have $\operatorname{dim}_{\mathbb{R}} \mathfrak{h o l}(M, p)=\infty$, since if there exists a holomorphic vector field $X \neq 0$ tangent to $M$, then also $f \cdot X \in \mathfrak{h t}(M, p)$ for any $f \in \mathcal{O}_{p}$. Now one can ask for a sufficient condition for $\operatorname{dim}_{\mathbb{R}} \mathfrak{h o l}(M, p)<\infty$. Note that if $\operatorname{dim}_{\mathbb{R}} \mathfrak{h o l}(M, p)<\infty$ for one $p \in M$, then this also holds for all $q$ in the connected component of $p$.
Theorem 13 (Stanton's Theorem). Let $M$ be a connected, generic and real-analytic submanifold of $\mathbb{C}^{N}$ and $p \in M$ a point of finite type. Then

$$
\exists q \in(M, p): \operatorname{dim}_{\mathbb{R}} \mathfrak{h o l}(M, q)<\infty \Longleftrightarrow M \text { is holomorphically nondegenerate. }
$$

Remark 35. (i) It is a fact from the theory of Lie groups, that the dimension of the Lie group agrees with the dimension of its Lie algebra ??? Actually we prove this for the characterization of finite type ???. If $M$ satisfies the hypothesis of Theorem 13, i.e., if $\operatorname{dim}_{\mathbb{R}} \mathfrak{h o l}(M, p)<\infty$, then, if $\operatorname{Aut}(M, p)$ is a Lie group, we have $\operatorname{dim}_{\mathbb{R}} \operatorname{Aut}(M, p)<\infty$. The question under which conditions on $M$ we can guarantee that $\operatorname{Aut}(M, p)$ is a Lie group, is answered in Section 11.3.
(ii) Note that the existence of a point of finite type is crucial: Consider $M=\left\{(z, w) \in \mathbb{C}^{2}: \operatorname{Im} w=0\right\}$. Then all points in $M$ are not of finite type and $X=\frac{\partial}{\partial z}$ is tangent to $M$, hence $\operatorname{dim}_{\mathbb{R}} \mathfrak{h o l}(M, p)=\infty$ for $p \in M$.
(iii) We have already discussed the necessary condition for $\mathfrak{h o l}(M, p)$ being finite dimensional. In order to proof Theorem 13 we use the jet determination for $\operatorname{Aut}_{p}(M, p)$.
Example 15. ??? Example in infinite type case (definition!) in $\mathbb{C}^{3}$ : Does there exist $M$ of infinite type at 0 and holomorphically nondegenerate, with $\operatorname{dim}_{\mathbb{R}} \mathfrak{h o l}(M, 0)=\infty$ ? (for $\mathbb{C}^{2}$ : Stanton 1995, Thm. 4.3: finite type implies holomorphic nondegeneracy, easy to show with our characterizations), for higher dimensions: [BER98, Theorem 3 (ii)]: "CR Automorphisms of real-analytic manifolds", one has either the possibility the the dimension is $\infty$ or 0 if it is nowhere minimal. ???
Lemma 26. Let us consider $y=\left(y_{1}, \ldots, y_{l}\right) \in \mathbb{R}^{l}, x=\left(x_{1}, \ldots, x_{m}\right) \in \mathbb{R}^{m}$ and $X_{k}:=\sum_{j=1}^{m} a_{j}^{k}(x) \frac{\partial}{\partial x_{j}}$ for $1 \leq k \leq l$. We require that $\left\{X_{1}, \ldots, X_{l}\right\}$ is a set of real-linearly independent, real-analytic vector fields for all $x$ near 0 . Further we define $S(y):=\sum_{j=1}^{l} y_{j} X_{j}$ and $F(x, y):=\Phi_{S(y)}^{1}(x)$, the flow of the vector field $S(y)$ at time $t=1$ and $x$ near 0 .
If $F\left(x, y^{\prime}\right)=F\left(x, y^{\prime \prime}\right)$, for small $y^{\prime}, y^{\prime \prime} \in \mathbb{R}^{l}$ and $x$ near 0 , then $y^{\prime}=y^{\prime \prime}$.

Proof. ??? Proof from BER99, Proof of Thm. 12.5.3, p. 367f ??? Let us denote

$$
A(x):=\left(A^{1}(x), \ldots, A^{l}(x)\right):=\left(\begin{array}{ccc}
a_{1}^{1}(x) & \ldots & a_{1}^{l}(x) \\
\vdots & \ddots & \vdots \\
a_{m}^{1}(x) & \ldots & a_{m}^{l}(x)
\end{array}\right)
$$

Since $\left\{X_{1}, \ldots, X_{l}\right\}$ are linearly independent, we have that the operator norm $\|A(x)\|=\sup _{y \in \mathbb{R}^{l}}\left|\frac{A(x) y}{\|y\|}\right|$ of $A(x)$ is not zero for all $x$ near 0 . Let us assume that there exists $C_{1}>0$ such that $\|A(x)\| \geq C_{1}$, hence $\|A(x) y\| \geq C_{1}\|y\|$ for all $y \in \mathbb{R}^{m}$. The flow $\Phi(t, x, y):=\Phi_{S(y)}^{t}(x)$ of the vector field $S(y)$ satisfies

$$
\begin{align*}
\frac{\partial \Phi}{\partial t}(t, x, y) & =\sum_{k=1}^{l} y_{k} A^{k}(\Phi(t, x, y))  \tag{11.18}\\
\Phi(0, x, y) & =x
\end{align*}
$$

for all $(t, x, y) \in \mathbb{R} \times \mathbb{R}^{m} \times \mathbb{R}^{l}$ near 0 . By the Fundamental Theorem of ODEs we obtain that $\Phi(t, x, y)$ is real-analytic in $(t, x)$ in a neighborhood of $0 \in \mathbb{R} \times \mathbb{R}^{m}$ ??? KN1 p.267 in smooth case ???. There is a certain invariance encoded in (11.18): If we let $s \in \mathbb{R}$, then for small $y \in \mathbb{R}^{l}, \Phi(x, t, s y)$ solves an equation similar to (11.18), i.e.,

$$
\begin{aligned}
\frac{\partial \Phi}{\partial t}(t, x, s y) & =\sum_{k=1}^{l} s y_{k} A^{k}(\Phi(t, x, s y)) \\
\Phi(0, x, s y) & =x
\end{aligned}
$$

Now we define $\varphi(t, x, y):=\Phi(s t, x, y)$, set $\widetilde{t}:=s t$ and use (11.18) to obtain

$$
\begin{aligned}
\frac{\partial \varphi}{\partial t}(t, x, y) & =\frac{d \Phi}{d t}(s t, x, y)=s \frac{\partial \Phi}{\partial \widetilde{t}}(\widetilde{t}, x, y)=s \sum_{k=1}^{l} y_{k} A^{k}(\Phi(\widetilde{t}, x, y))=\sum_{k=1}^{l} s y_{k} A^{k}(\varphi(t, x, y)) \\
\varphi(0, x, y) & =x
\end{aligned}
$$

That means both $\Phi(s t, x, y)$ and $\Phi(t, x, s y)$ solve the following initial value problem:

$$
\begin{align*}
\frac{\partial \Psi}{\partial t}(t, x, y) & =\sum_{k=1}^{l} s y_{k} A^{k}(\Psi(t, x, y))  \tag{11.19}\\
\Psi(0, x, y) & =x
\end{align*}
$$

Since the solution of (11.19) is unique, we obtain

$$
\begin{equation*}
\Phi(s t, x, y)=\Phi(t, x, s y), \quad s \in \mathbb{R} \tag{11.20}
\end{equation*}
$$

which implies that $\Phi$ is real-analytic in a neighborhood of $0 \in \mathbb{R} \times \mathbb{R}^{m} \times \mathbb{R}^{l}$. If we use the identity of (11.20) and the second and then the first equation of (11.18), we get

$$
\frac{\partial F}{\partial y_{i}}(x, 0)=\frac{\partial \Phi}{\partial y_{i}}(1, x, 0)=\left.\frac{d}{d s}\right|_{s=0}\left(\Phi\left(1, x, s e_{i}\right)\right)=\left.\frac{d}{d s}\right|_{s=0}\left(\Phi\left(s, x, e_{i}\right)\right)=A^{i}(x)
$$

Finally we use Taylor's Theorem in the second variable and write

$$
F(x, y)=F(x, 0)+F_{y}(x, 0) y+\sum_{|\alpha| \geq 2} b_{\alpha}(x) y^{\alpha}
$$

Then we have

$$
\begin{aligned}
\left\|F\left(x, y^{\prime}\right)-F\left(x, y^{\prime \prime}\right)\right\| & =\left\|F(x, 0)+F_{y}(x, 0) y^{\prime}+\sum_{|\alpha| \geq 2} b_{\alpha}(x) y^{\prime \alpha}-F(x, 0)-F_{y}(x, 0) y^{\prime \prime}-\sum_{|\beta| \geq 2} b_{\beta}(x) y^{\prime \prime \beta}\right\| \\
& \geq\left\|F_{y}(x, 0)\left(y^{\prime}-y^{\prime \prime}\right)\right\|-\left\|\sum_{|\gamma| \geq 2} b_{\gamma}(x)\left(y^{\prime \gamma}-y^{\prime \prime \gamma}\right)\right\| \\
& \geq C_{1}\left\|y^{\prime}-y^{\prime \prime}\right\|-C_{2}\left\|y^{\prime \prime}-y^{\prime}\right\|^{2},
\end{aligned}
$$

where $C_{2}>0$, for all $x, y$ near 0 . The last inequality shows the claim.
Proof of Theorem 13. $(\Leftarrow)$ ??? see BER 1999, p. 366 ff , Thm. 12.5.3 bzw. Lemma 12.5.10, what we need here and Thm 12.3.1 and Thm. 1.5.10 and the previous Lemmas, respectively ??? By Lemma 15 we can find $p_{0} \in M$ such that in a neighborhood $U$ of $p_{0}, M$ is of finite type and holomorphically nondegenerate. From Lemma 19 (iv) we obtain that there exists a point $q_{0} \in M$, such that near $q_{0}$ every point $q \in M$ is of finite type and finitely nondegenerate. W.l.o.g. assume $q=0$. Let $\left\{X_{1}, \ldots, X_{l}\right\} \in$ $\mathfrak{h o l}(M, 0)$ be linearly independent vector fields and consider for $y=\left(y_{1}, \ldots, y_{l}\right) \in \mathbb{R}^{l}, S(y):=\sum_{j=1}^{l} y_{j} X_{j}$ and $F(Z, y):=\Phi_{S(y)}^{1}(Z)$, the flow of the vector field $S(y)$ at time $t=1$ and $Z \in M$ near 0 . Since $S(0)=0$, we have by Theorem $9, F(Z, y) \in \operatorname{Aut}_{0}(M, 0)$. Let $k_{0} \in \mathbb{N}$ be chosen according to Corollary 5. Lemma 26 implies that, $y \mapsto j_{0}^{k_{0}} F(Z, y)$ is injective, where $j_{0}^{k_{0}}$ acts on $F(Z, y)$ in the $Z$-component at 0 . Hence we obtain

$$
\operatorname{dim}_{\mathbb{R}} \mathfrak{h o l}(M, 0)=l \leq \operatorname{dim}_{\mathbb{R}} F_{n}^{k_{0}}\left(\mathbb{C}^{N}\right)<\infty
$$

Remark 36. We remark that it is not enough to know that $j_{p}^{k}$ is injective, to conclude that Aut $_{p}^{k}(M, p)$ is a Lie group. ??? example: torus + lines with irrational slope ??? As a consequence we cannot use Stanton's Theorem to get a Lie group $\operatorname{Aut}_{p}(M, p)$, instead we need to construct a jet parametrization, as shown in Section 11.3.
Example 16. We compute $\mathfrak{h o l}(M, p)$ for $M=\left\{(z, w) \in \mathbb{C}^{2}: \operatorname{Im} w=|z|^{2}\right\}$ and $p=0$. Since we know that $\operatorname{Aut}_{0}(M, 0)$ is a finite dimensional Lie group, we get $\operatorname{Aut}_{0}(M, 0)$ from $\mathfrak{h o l}_{0}(M, 0)$. We work with the complexification $\mathcal{M}$ given by $\rho(z, w, \bar{z}, \bar{w})=w-\bar{w}-2 \mathrm{i} z \bar{z}$.
Similar as we have shown in Example 8 we obtain that a vector field $Y$ is tangent to $M$ if and only if there exists a real-analytic, real-valued function $A(z, w, \bar{z}, \bar{w})$ such that $Y \rho(z, w, \bar{z}, \bar{w})=A(z, w, \bar{z}, \bar{w}) \rho(z, w, \bar{z}, \bar{w})$ for all $(z, w, \bar{z}, \bar{w}) \in \mathbb{C}^{4}$ near 0 .
To compute $\mathfrak{h o l}(M, 0)$ we consider homogeneous parts of the previous equation as follows:
We endow $z$ with weight 1 and $w$ with weight 2 and say that a real-analytic function $F(z, w, \bar{z}, \bar{w})$ is of weight $k$ if $F\left(t z, t^{2} w, t \bar{z}, t^{2} \bar{w}\right)=t^{k} F(z, w, \bar{z}, \bar{w})$ for $t \in \mathbb{R}$. This concept translates to - in our case - holomorphic vector fields $Z=\sum_{j} a_{j}(z, w) \frac{\partial}{\partial z_{j}}+b_{j}(z, w) \frac{\partial}{\partial w_{j}}$ and we have that $Z$ is of weight $k$ if and only if $a_{j}$ is of weight $k+1$ and $b_{j}$ is of weight $k+2$, i.e., we endow $\frac{\partial}{\partial z_{j}}$ with weight -1 and $\frac{\partial}{\partial w_{j}}$ with weight -2 . If we start with any vector field $T$ tangent to $M$ we write $T=\sum_{k} T_{k}$, where each $T_{k}$ is of weight $k$ and tangent to $M$. In the case of $X \in \mathfrak{h o l}(M, 0)$ the homogeneous expansion starts with $k \geq-2$.
Now we compute $X_{k}$ for $-2 \leq k \leq 2$ whose real parts are tangent to $M$ and their flows $\Phi_{X_{k}}^{t}$, which give us the elements of $\operatorname{Aut}(M, 0)$.
We write $X_{-2}=a \frac{\partial}{\partial w}$ with $a \in \mathbb{C}$. Then $\operatorname{Re} X_{-2}$ is tangent to $M$ if and only if there exists a real-analytic, real-valued function $A_{-2}$ with

$$
\begin{equation*}
\operatorname{Re} X_{-2} \rho(z, w, \bar{z}, \bar{w})=A_{-2}(z, w, \bar{z}, \bar{w}) \rho(z, w, \bar{z}, \bar{w}) \tag{11.21}
\end{equation*}
$$

Since the left-hand side of (11.21) is of weight 0 , this implies $A_{-2} \equiv 0$ and we have to solve

$$
\left(a \frac{\partial}{\partial w}+\bar{a} \frac{\partial}{\partial \bar{w}}\right)(w-\bar{w}-2 \mathrm{i} z \bar{z})=0
$$

which can only happen if and only if $a \in \mathbb{R}$. The flow $\Phi_{X_{-2}}^{t}(0)=:(z(t), w(t))$ is the solution of $\{\dot{z}=0, \dot{w}=a\}$ and is equal to $(z, w)=\left(z^{\prime}\right.$,at $\left.+w^{\prime}\right)$, where $\left(z^{\prime}, w^{\prime}\right) \in \mathbb{C}^{2}$ are constants with respect to $t \in \mathbb{R}$. Since the flow of $X_{-2}$ has its values in $M$ for all $t$, we plug $(z, w)=\left(z^{\prime}, a t+w^{\prime}\right)$ into the defining function for $M$ and obtain $\left(z^{\prime}, w^{\prime}\right) \in M$. In total we obtain the automorphism of $M$ given by $\left(z^{\prime}, w^{\prime}\right) \mapsto\left(z^{\prime}, w^{\prime}+r\right)$ for $r:=a t \in \mathbb{R}$. We denote $X_{-1}=a \frac{\partial}{\partial z}+b z \frac{\partial}{\partial w}$, where $a, b \in \mathbb{C}$ and we have that $\operatorname{Re} X_{-1}$ is tangent to $M$ if and only if

$$
\left(a \frac{\partial}{\partial z}+b z \frac{\partial}{\partial w}+\bar{a} \frac{\partial}{\partial \bar{z}}+\bar{b} \bar{z} \frac{\partial}{\partial \bar{w}}\right)(w-\bar{w}-2 \mathrm{i} z \bar{z})=0
$$

Comparing coefficients we get that $b=2 \mathrm{i} \bar{a}$. The flow of $X_{-1}$ is the solution of $\{\dot{z}=a, \dot{w}=2 \mathrm{i} \bar{a} z\}$ and is equal to $(z(t), w(t))=\left(a t+z^{\prime}, \mathrm{i}|a|^{2} t^{2}+2 \mathrm{i} \bar{a} z^{\prime} t+w^{\prime}\right)$. Again if we plug in $(z(t), w(t))$ into the defining function of $M$ we obtain that $\left(z^{\prime}, w^{\prime}\right) \in M$ and the automorphism $\left(z^{\prime}, w^{\prime}\right) \mapsto\left(z^{\prime}+b, w^{\prime}+2 \mathrm{i} \bar{b} z^{\prime}+\mathrm{i}|b|^{2}\right)$ for $b:=a t \in \mathbb{C}$. Note that the automorphisms we got so far form a real 3-dimensional subgroup of the automorphism group of $M$, the so called group of translations which is given by all mappings of the form $(z, w) \mapsto\left(z+z_{0}, w+\right.$ $\left.w_{0}+2 \mathrm{i} z \bar{z}_{0}\right)$ with $\left(z_{0}, w_{0}\right) \in M$. ??? In general true: vector fields with negative weight give translations, since the weight of the coefficients of the vector fields must be less than the weight of the coordinates. E.g. for levi-nondegenerate only the weights $-2,-1,0,1,2$ can occur ???
Next, $X_{0}=a z \frac{\partial}{\partial z}+\left(b z^{2}+c w\right) \frac{\partial}{\partial w}$ for $a, b, c \in \mathbb{C}$ is tangent to $M$ if and only if there exists $A_{0}(z, w, \bar{z}, \bar{w})=$ $A \in \mathbb{R}$ such that

$$
\begin{equation*}
\frac{1}{2}\left(a z \frac{\partial}{\partial z}+\left(b z^{2}+c w\right) \frac{\partial}{\partial w}+\bar{a} \bar{z} \frac{\partial}{\partial \bar{z}}+\left(\bar{b} \bar{z}^{2}+\bar{c} \bar{w}\right) \frac{\partial}{\partial \bar{w}}\right)(w-\bar{w}-2 \mathrm{i} z \bar{z})=A(w-\bar{w}-2 \mathrm{i} z \bar{z}) \tag{11.22}
\end{equation*}
$$

Comparing coefficients in (11.22) we obtain $c=2 A, \operatorname{Re} a=A$ and $b=0$. In fact we conclude that no $z^{l}$-terms for $l \geq 2$ can occur as coefficients of $\frac{\partial}{\partial w}$ in $X_{k}$ for $k \geq 0$, since there is no possibility that such pure terms occur on the right-hand side of equations as (11.22) for $k \geq 0$.
We end up with a familiy of vector fields $X_{0}$ with parameters $(A, \operatorname{Im} a) \in \mathbb{R}^{2}$. Taking the standard basis of $\mathbb{R}^{2}$ for the parameter space $(A, \operatorname{Im} a)$, we obtain that $X_{0}^{1}=z \frac{\partial}{\partial z}+2 w \frac{\partial}{\partial w}$ and $X_{0}^{2}=\mathrm{i} z \frac{\partial}{\partial z}$ generate all vector fields belonging to $X_{0}$.
The flow of $X_{0}^{1}$ is the solution of $\{\dot{z}=z, \dot{w}=2 w\}$ and is equal to $(z(t), w(t))=\left(e^{t} z^{\prime}, e^{2 t} w^{\prime}\right)$. Again the condition for the flow to stay in $M$ is satisfied if and only if $\left(z^{\prime}, w^{\prime}\right) \in M$ and we end up with the automorphism $\left(z^{\prime}, w^{\prime}\right) \mapsto\left(\lambda z^{\prime}, \lambda^{2} w^{\prime}\right)$ for $\lambda:=e^{t}>0$.
Similar for $X_{0}^{2}$ : the flow is the solution of $\{\dot{z}=\mathrm{i} z, \dot{w}=0\}$ and can be written as $(z(t), w(t))=\left(e^{\mathrm{it}} z^{\prime}, w^{\prime}\right)$ which corresponds to the automorphism $\left(z^{\prime}, w^{\prime}\right) \mapsto\left(u z^{\prime}, w^{\prime}\right)$ for $u:=e^{\mathrm{it}} \in \mathbb{C}$ if and only if $\left(z^{\prime}, w^{\prime}\right) \in M$. Considering weight $k=1$, we write $X_{1}=\left(a z^{2}+b w\right) \frac{\partial}{\partial z}+c z w \frac{\partial}{\partial w}$ and $A_{1}$ has to be of the form $A_{1}(z, w, \bar{z}, \bar{w})=$ $A z+\bar{A} \bar{z}$ for $A \in \mathbb{C}$. Then we have

$$
\begin{equation*}
\frac{1}{2}\left(\left(a z^{2}+b w\right) \frac{\partial}{\partial z}+c z w \frac{\partial}{\partial w}+\left(\bar{a} \bar{z}^{2}+\bar{b} \bar{w}\right) \frac{\partial}{\partial \bar{z}}+\bar{c} \bar{z} \bar{w} \frac{\partial}{\partial \bar{w}}\right)(w-\bar{w}-2 \mathrm{i} z \bar{z})=(A z+\bar{A} \bar{z})(w-\bar{w}-2 \mathrm{i} z \bar{z}) \tag{11.23}
\end{equation*}
$$

Comparing coefficients in (11.23) we obtain that $a=2 A, b=\mathrm{i} \bar{A}$ and $c=2 A$. The flow is obtained by solving $\left\{\dot{z}=2 A z^{2}+\mathrm{i} \bar{A} w, \dot{w}=2 A z w\right\}$. The second equation gives $z=\frac{\dot{w}}{2 A w}$ and together with the first one we obtain

$$
\frac{1}{w}\left(\left(\frac{\dot{w}}{w}\right)^{\cdot}-\frac{\dot{w}^{2}}{w^{2}}\right)=2 \mathrm{i}|A|^{2} \Leftrightarrow\left(\frac{1}{w}\right)^{. \cdot}=-2 \mathrm{i}|A|^{2}
$$

If we take $z^{\prime}, w^{\prime}$ as constants not depending on $t$, the solution is given by

$$
(z(t), w(t))=\frac{\left(2 \mathrm{i}|A|^{2} t-z^{\prime}, 2 A\right)}{2 A\left(-\mathrm{i}|A|^{2} t^{2}+z^{\prime} t+w^{\prime}\right)}
$$

After writting $z^{\prime}=2 \mathrm{i} A z^{\prime \prime}$ and plugging in the flow into $\rho=0$ for $M$, we get that $\left(z^{\prime \prime}, w^{\prime}\right) \in M$. Defining $c=\mathrm{i} \bar{A} t \in \mathbb{C}$ and $(\hat{z}, \hat{w}) \in M$ via $z^{\prime \prime}=\mathrm{i} \hat{z} / \hat{w}$ and $w^{\prime}=1 / \hat{w}$, we obtain the automorphism

$$
(\hat{z}, \hat{w}) \mapsto \frac{(\hat{z}+c \hat{w}, \hat{w})}{1-2 \mathrm{i} \bar{c} \hat{z}-\mathrm{i}|c|^{2} \hat{w}}
$$

Finally $X_{2}=\left(a z^{3}+b z w\right) \frac{\partial}{\partial z}+\left(c z^{2} w+d w^{2}\right) \frac{\partial}{\partial w}$ and $A_{2}(z, w, \bar{z}, \bar{w})=A w+\bar{A} \bar{w}+B z^{2}+\bar{B} \bar{z}^{2}$ for $A, B \in \mathbb{C}$. Then

$$
\begin{align*}
& \frac{1}{2}\left(\left(a z^{3}+b z w\right) \frac{\partial}{\partial z}+\left(c z^{2} w+d w^{2}\right) \frac{\partial}{\partial w}+\left(\bar{a} \bar{z}^{3}+\bar{b} \bar{z} \bar{w}\right) \frac{\partial}{\partial \bar{z}}+\left(\bar{c} \bar{z}^{2} \bar{w}+\bar{d} \bar{w}^{2}\right) \frac{\partial}{\partial \bar{w}}\right)(w-\bar{w}-2 \mathrm{i} z \bar{z}) \\
= & \left(A w+\bar{A} \bar{w}+B z^{2}+\bar{B} \bar{z}^{2}\right)(w-\bar{w}-2 \mathrm{i} z \bar{z}) . \tag{11.24}
\end{align*}
$$

We immediately obtain $B=0$ and $A=\bar{A}$. Further comparison of coefficients in (11.24) show $a=0=c$ and $b=2 A=d$ and we end up with real multiples of $X_{3}=z w \frac{\partial}{\partial z}+w^{2} \frac{\partial}{\partial w}$. The flow has to satisfy $\left\{\dot{z}=z w, \dot{w}=w^{2}\right\}$ and is equal to $(z(t), w(t))=\frac{\left(z^{\prime},-1\right)}{t+w^{\prime}}$. Again we obtain $\left(z^{\prime}, w^{\prime}\right) \in M$ and after setting $(\hat{z}, \hat{w})=-\left(z^{\prime} / w^{\prime}, 1 / w^{\prime}\right) \in M$ and $s=-t \in \mathbb{R}$, the automorphism $(\hat{z}, \hat{w}) \mapsto \frac{(\hat{z}, \hat{w})}{1+s \hat{w}}$.
Comparing with Example 12, the automorphisms coming from $X_{0}, X_{1}$ and $X_{2}$ form the real 5 -dimensional group of isotropies or stability group of the automorphism group of $M$.
To handle the case $X_{k}$ for $k \geq 3$ we write $X_{k}=f_{k+1} \frac{\partial}{\partial z}+g_{k+2} \frac{\partial}{\partial w}$, where $f_{k+1}$ and $g_{k+2}$ are polynomials of weight $k+1$ and $k+2$ respectively. We want to conclude that $f_{k+1}=0=g_{k+2}$ for $k \geq 3$. In general if a monomial $z^{l} w^{m}$ is of weight $k$, then necessarily $k / 2 \geq m$. Thus if $k$ is even, then the monomial with lowest degree in $X_{k}$ is $w^{k / 2}$ and if $k$ is odd the monomial with lowest degree in $X_{k}$ is $z w^{\frac{k-1}{2}}$. Hence the only monomial of degree two occurring as coefficient of $X_{k}$ is $w^{2}$ in $f_{4}$. Let us fix $k \geq 3$, then $\operatorname{Re} X_{k}$ is tangent to $M$ if

$$
\begin{aligned}
\left.\operatorname{Re} X_{k} \rho\right|_{M} \equiv 0 & \left.\Leftrightarrow\left(-2 \mathrm{i} \bar{z} f_{k+1}+g_{k+2}-2 \mathrm{i} z \bar{f}_{k+1}+\bar{g}_{k+2}\right)\right|_{M} \equiv 0 \\
& \left.\Leftrightarrow \operatorname{Re}\left(i g_{k+2}+2 \bar{z} f_{k+1}\right)\right|_{M} \equiv 0
\end{aligned}
$$

We write $H=(f, g)$ for a mapping of $M, H_{k}=\left(f_{k+1}, g_{k+2}\right)$ and denote $L(f, g):=\left.\operatorname{Re}(i g+2 \bar{z} f)\right|_{M}$, the so called Chern-Moser operator for $M$. Under the hypothesis that $k \geq 3$ we show that the kernel of $L$ applied to $H_{k}$ is trivial, i.e., $X_{k}=0$ for $k \geq 3$. More precisely we show the following Lemma:

Lemma 27 (Chern-Moser). Let $H=(f, g)$ be a mapping from $\mathbb{C}^{2}$ to $\mathbb{C}^{2}$. If $f(0)=g(0)=f_{z}(0)=g_{z}(0)=$ $f_{w}(0)=g_{w}(0)=g_{z^{2}}(0)=\operatorname{Re} g_{w^{2}}(0)=0$, then $L(f, g) \equiv 0$ has the unique solution $(f, g) \equiv 0$.

As discussed above, we have for $k \geq 3$, that $f_{k+1}$ is of degree two or higher and $g_{k+2}$ is of degree three or higher, thus we apply Lemma 27 and our claim is proved.

## ??? Make link to section about normal forms ???

