Convergence of the Chern-Moser-Beloshapka normal forms

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Abstract

In this article, we first describe a normal form of real-analytic, Levi-nondegenerate submanifolds of $\mathbb{C}^N$ of codimension $d \geq 1$ under the action of formal biholomorphisms, that is, of perturbations of Levi-nondegenerate hyperquadrics. We give a sufficient condition on the formal normal form that ensures that the normalizing transformation to this normal form is holomorphic. We show that our techniques can be adapted in the case $d = 1$ in order to obtain a new and direct proof of Chern-Moser normal form theorem.

1 Introduction

In this paper, we study normal forms for real-analytic, Levi-nondegenerate manifolds of $\mathbb{C}^N$. A real submanifold $M \subset \mathbb{C}^N$ (of real codimension d) is given, locally at a point $p \in M$, in suitable coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}^d = \mathbb{C}^N$, by a defining function of the form

$$\operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w),$$

where $\varphi\colon \mathbb{C}^n \times \mathbb{R}^d \to \mathbb{R}^d$ is a germ of a real analytic map satisfying $\varphi(0,0,0) = 0$, and $\nabla\varphi(0,0,0) = 0$. Its natural second order invariant is its Levi form $\mathcal{L}_p$: This is a natural Hermitian vector-valued form, defined on $\mathcal{V}_p = \mathbb{C}T_pM \cap \mathbb{C}T_p^{(0,1)}\mathbb{C}^N$ as

$$\mathcal{L}_p(X_p, Y_p) = [X_p, \bar{Y}_p] \mod \mathcal{V}_p \oplus \bar{\mathcal{V}}_p \quad \in {}^{\mathbb{C}T_pM}\big/_{\mathcal{V}_p \oplus \bar{\mathcal{V}}_p}.$$

We say that M is Levi-nondegenerate (at p) if the Levi-form $\mathcal{L}_p$ is a nondegenerate, vector-valued Hermitian form, and is of full rank.

Let us recall that we say that $\mathcal{L}_p$ is *nondegenerate* if it satisfies that $\mathcal{L}_p(X_p, Y_p) = 0$ for all $Y_p \in \mathcal{V}_p$ implies $X_p = 0$ and that we say that $\mathcal{L}_p$ is of full rank, if $\theta(\mathcal{L}_p(X_p, Y_p)) = 0$ for all $X_p, Y_p \in \mathcal{V}_p$ and for $\theta \in T_p^0M = \mathcal{V}_p^\perp \cap \bar{\mathcal{V}}_p^\perp$ (where $\mathcal{V}_p^\perp \subset \mathbb{C}T^*M$ is the holomorphic cotangent bundle) implies $\theta = 0$.

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The typical model for this situation is a *hyperquadric*, that is, a manifold of the form

$$\operatorname{Im} w = Q(z,\bar z) = \begin{pmatrix} Q_1(z,\bar z) \\ \vdots \\ Q_d(z,\bar z) \end{pmatrix} = \begin{pmatrix} \bar z^t J_1 z \\ \vdots \\ \bar z^t J_d z \end{pmatrix},$$

where each J_k is a Hermitian $n \times n$ matrix, and the conditions of nondegeneracy and full rank are expressed by

$$\bigcap_{k=1}^{d} \ker J_k = \{0\}, \quad \sum_{k=1}^{d} \lambda_k J_k = 0 \Rightarrow \lambda_k = 0, \quad k = 1,\dots,d. \tag{1}$$

The defining equation of the hyperquadric becomes *quasihomogeneous* of degree 1, if we endow z with the weight 1 and w with the weight 2, which we shall do from now on. A Levi-nondegenerate manifold can thus, at each point, be thought of as a "higher order deformation" of a hyperquadric, that is, their defining functions $\operatorname{Im} w = \varphi(z,\bar z)$ can be rewritten as

$$\operatorname{Im} w = Q(z,\bar z) + \Phi_{\geq 3}(z,\bar z,\operatorname{Re} w),$$

where $\Phi_{\geq 3}$ only contains quasihomogeneous terms of order at least 3.

We are going to classify germs of such real analytic manifolds under the action of the group of germs of biholomorphisms of $\mathbb{C}^N$. The classification problem for Levi-nondegenerate manifolds has a long history, especially in the case of hypersurfaces ($d = 1$). It was first studied (and solved) for hypersurfaces in $\mathbb{C}^2$ by Elie Cartan in a series of papers [Car33, Car32] in the early 1930s, using his theory of moving frames. Later on, Tanaka [Tan62] and Chern and Moser [CM74] solved the problem for Levi-nondegenerate hypersurfaces in $\mathbb{C}^n$. They used differential-geometric approaches, but also, in the case of Chern-Moser an approach coming from the theory of dynamical systems: finding a normal form for the defining function, or equivalently, finding a special coordinate system for the manifold. We refer to the papers by Vitushkin [Vit85b, Vit85a], the book by Jacobowitz [Jac90], the survey by Huang [Hua04] and the survey by Beals, Fefferman, and Grossman [BFG83] in which the geometric and analytic significance of the Chern-Moser normal form are discussed.

Our paper takes up a very classical problem with a new tool, and gives a formal normal form for Levi-nondegenerat real analytic manifolds which under a rather simple condition (see (85)) can be shown to be convergent. Recent advances in normal forms for real submanifolds of complex spaces with respect to holomorphic transformations have been significant: We would like to cite in this context the recent works of Huang and Yin [HY09, HY16, HY17], the second author and Gong [GS16], and Gong and Lebl [GL15].

We will discuss our construction and the difficulties involved with it by contrasting it to the Chern-Moser case. Before we describe the Chern-Moser normal form, let us comment shortly on why the differential geometric approach taken by Tanaka and Chern-Moser works in the case of hypersurfaces. The reason for this is that actually locally, the geometric information induced by the (now scalar-valued!) Levi-form can be reduced to its signature and therefore stays, in a certain sense "constant". This makes it possible to study the structure using tools which are nowadays formalized under the umbrella of *parabolic geometry*–for further information, we refer the reader to the book of Cap and Slovak [CS09]. In particular, every Levi-nondegenerate hypersurface can be endowed with a structure bundle carrying a Cartan connection and an associated intrinsic curvature. However, in the case of Levi-nondegenerate manifolds of higher codimension, our basic second order invariant, the vector-valued Levi form $\mathcal{L}_p$, has more invariants than just the simple integer-valued signature of a scalar-valued form, and its behaviour thus can (and in general will) change dramatically with p. Of course, if it is nondegenerate at the given point 0, it stays so in neighbourhood of it. There have thus been rather few circumstances in which the geometric approach has been successfully applied to Levi-nondegenerate manifolds of higher codimension, such as in the work of Schmalz, Ezhov, Cap, and others (see [SS06] and references therein).

In our paper, we take the different (dynamical systems inspired) approach taken by Chern-Moser, who introduced a *convergent normal form* for the problem. They prescribe a space of *normal forms* $\mathcal{N}_{CM} \subset \mathbb{C}[\![z, \bar{z}, s]\!]$ such that for each element of the infinitesimal automorphism algebra of the model hyperquadric $\operatorname{Im} w = \bar{z}^t J z$, one obtains a unique formal choice (z, w) of coordinates in $\mathbb{C}^N = \mathbb{C}^n \times \mathbb{C}$ in which the defining equation takes the form

$$\operatorname{Im} w = \bar{z}^t J z + \Phi(z, \bar{z}, \operatorname{Re} w),$$

with $\Phi \in \mathcal{N}_{CM}$. It turns out (after the fact) that the coordinates are actually *holomorphic coordinates*, not only formal ones, which is the reason why we say that the Chern-Moser normal form is *convergent*. Let us shortly note that the dependence on the infinitesimal automorphism algebra is actually necessary; after all, some of the hypersurfaces studied have a normal form which still carries some symmetries (in particular, the normal form of the model quadric will be the model quadric itself).

The normal form space of Chern and Moser is described as follows. One needs to introduce the *trace operator*

$$T = \left(\frac{\partial}{\partial \bar{z}}\right)^t J \left(\frac{\partial}{\partial z}\right)$$

and the homogeneous parts in z and $\bar{z}$ of a series $\Phi(z, \bar{z}, u) = \sum_{j,k} \Phi_{j,k}(z, \bar{z}, u)$, where $\Phi_{j,k}(tz, s\bar{z}, u) = t^j s^k \Phi_{j,k}(z, \bar{z}, u)$; $\Phi_{j,k}$ is said to be of *type* (j, k).

We then say that $\Phi \in \mathcal{N}_{CM}$ if it satisfies the following (Chern-Moser) normal form conditions:

$$\begin{aligned} \Phi_{j,0} = \Phi_{0,j} &= 0, && j \geq 0; \\ \Phi_{j,1} = \Phi_{1,j} &= 0, && j \geq 1; \\ T\Phi_{2,2} = T^2\Phi_{2,3} &= T^3\Phi_{3,3} = 0. \end{aligned}$$

There are a number of aspects particular to the case $d = 1$ which allow Chern and Moser to construct, based on these conditions (which arise rather naturally from a linearization of the problem with respect to the ordering by type), a convergent choice of coordinates. In particular, Chern and Moser are able to restate much of their problem in terms of ODEs, which comes from the fact that there is only one transverse variable when $d = 1$; in particular, existence and regularity of solutions is guaranteed. In higher codimension, this changes dramatically, and we obtain systems of PDEs; neither do we a priori know that those are solvable nor do we know anything about the regularity of their solutions (should they exist). Our normal form has to take this into account.

Another aspect of the problem, which really changes dramatically from the case $d = 1$ to $d > 1$, is the second line of the normal form conditions above: We cannot impose that

$\Phi_{1,j} = \Phi_{j,1} = 0$ for $j \geq 1$, as those terms - it turns out - *actually carry invariant information.* We shall however present a rather simple normal form, defined by equations which one can write down.

We should note at this point that some parts of the problem associated to a formal normal form have already been studied by Beloshapka [Bel90]. In there, a linearization of the problem is given, and a formal normal form construction (with a completely arbitrary normal form space) is discussed. However, for applications, a choice of a normal form space which actually gives rise to a convergent normal form is of paramount importance, and only in very special circumstances (codimension 2 in $\mathbb{C}^4$) there have been resolutions to this problem.

The failure of a simple normalization of the terms of type $(1, j)$ and $(j, 1)$ in the higher codimension case has more and subtle consequences which destroy much of the structure which allows one to succeed in the case $d = 1$. We are able to overcome some of these problems by using a new technique from dynamical systems introduced by the second author [Sto16]. In that paper, one can already find an illustration of a kind of "higher codimension Chern-Moser failure" in a quite different but easier problem. It concerns normal forms of singularities of holomorphic functions. If the singularity is isolated, then usual proofs (Arnold-Tougeron) of the locally holomorphic conjugacy to a normal form reduces to the existence of holomorphic solutions of ODE's depending on a parameter (issued from "la méthode des chemins"). If the singularity is not isolated, there is no way to obtain such an ODE but the main result of [Sto16](Big denominator theorem) allows to solve the problem directly.

In this paper we shall first discuss the convergent solution of a "restricted" (yet still infinite-dimensional) normalization problem: Given a Levi-nondegenerate hyperquadric $\operatorname{Im} w = Q(z, \bar{z})$, for perturbations of the form

$$\operatorname{Im} w = Q(z, \bar{z}) + \Phi_{\geq 3}(z, \bar{z}, \operatorname{Re} w),$$

find a *formal* normal form. Our first main result can therefore be thought of as a concrete realization of Beloshapka's construction of an abstract normal form in this setting:

Theorem 1. *Fix a nondegenerate form of full rank $Q(z, \bar{z})$ on $\mathbb{C}^n$ with values in $\mathbb{C}^d$, i.e. a map of the form $Q(z, \bar{z}) = (\bar{z}^t J_1 z, \ldots, \bar{z}^t J_d z)$ with the J_k satisfying* (1). *Then there exists a subspace $\hat{\mathcal{N}}_f \subset \mathbb{C}[\![z, \bar{z}, \operatorname{Re} w]\!]$ (explicitly given in* (16) *below) such that the following holds. Let M be given near $0 \in \mathbb{C}^N$ by an equation of the form*

$$\operatorname{Im} w' = Q(z', \bar{z}',) + \tilde{\Phi}_{\geq 3}(z', \bar{z}', \operatorname{Re} w'),$$

with $\tilde{\Phi} \in \mathbb{C}[\![z, \bar{z}, \operatorname{Re} w]\!]$. Then there exists a unique formal biholomorphic map of the form $H(z, w) = (z + f_{\geq 2}, w + g_{\geq 3})$ such that in the new (formal) coordinates $(z, w) = H^{-1}(z', w')$ the manifold M is given by an equation of the form

$$\operatorname{Im} w = Q(z, \bar{z}) + \Phi_{\geq 3}(z, \bar{z}, \operatorname{Re} w)$$

with $\Phi_{\geq 3} \in \hat{\mathcal{N}}_f$.

The solution of the *analytic* normal form problem, however, runs into all of the difficulties described above. However, there is a partial, "weak" normalization problem, described by a normal form space $\hat{\mathcal{N}}_f^w \supset \hat{\mathcal{N}}_f$ (again defined below in (16)), which in practice does not try to normalize the $(3,2)$ and the $(2,3)$-terms and therefore treats the transversal d-manifold $z = f_0(w)$ as a parameter. This fact is somewhat of independent interest, so we state it as a theorem:

Theorem 2. *Fix a nondegenerate form of full rank $Q(z,\bar z)$ on $\mathbb{C}^n$ with values in $\mathbb{C}^d$, i.e. a map of the form $Q(z,\bar z) = (\bar z^t J_1 z, \dots, \bar z^t J_d z)$ with the J_k satisfying* (1)*. Then for the subspace $\mathcal{N}^w = \hat{\mathcal{N}}_f^w \cap \mathbb{C}\{z,\bar z, \operatorname{Re} w\}$ defined below in* (16) *the following holds. Let M be given near $0 \in \mathbb{C}^N$ by an equation of the form*

$$\operatorname{Im} w' = Q(z', \bar z',) + \tilde\Phi_{\geq 3}(z', \bar z', \operatorname{Re} w').$$

Then for any $f_0 \in (w)\mathbb{C}\{w\}$ there exists a unique biholomorphic map of the form $H(z,w) = (z + f_0 + f_{\geq 2}, w + g_{\geq 3})$ with $f_{\geq 2}(0,w) = 0$ such that in the new coordinates $(z,w) = H^{-1}(z',w')$ the manifold M is given by an equation of the form

$$\operatorname{Im} w = Q(z,\bar z) + \Phi_{\geq 3}(z,\bar z, \operatorname{Re} w)$$

with $\Phi_{\geq 3} \in \mathcal{N}^w$.

Let us note that (as is apparent from the construction of the convergent solution) the corresponding formal problem also has a solution.

Geometrically speaking, the convergent normal form given here provides for a unique convergent "framing" of the complex tangent spaces along and parametrization for any germ of a real manifold $N \subset M$ transverse to $T_0^c M$, i.e. a map $\gamma\colon \mathbb{R}^d \to M$ parametrizing N and for each $t \in \mathbb{R}^d$, a basis of $T^c_{\gamma(t)} M$.

The *analytic* choice of such a transverse manifold satisfying the additional restrictions to be in $\hat{\mathcal{N}}_f$ is actually quite more involved than the choice of a transverse curve in the case of a hypersurface, as the "resonant terms" already alluded to above provide for an intricate coupling of the PDEs which we will derive in their nonlinear terms. It is with that in mind that one has to put some additional constraint in order to provide for a complete normalization. We note, however, that we obtain a complete solution to the formal normalization problem.

As already stated, in this generality we cannot guarantee convergence of the normal form. However, there are some *purely algebraic* conditions describing a subset of formal normal forms, for which the transformation to the normal form (and therefore also the normal form) can be shown to be convergent if the data is.

Theorem 3. *Fix a nondegenerate form of full rank $Q(z,\bar z)$ on $\mathbb{C}^n$ with values in $\mathbb{C}^d$, i.e. a map of the form $Q(z,\bar z) = (\bar z^t J_1 z, \dots, \bar z^t J_d z)$ with the J_k satisfying* (1)*. Let M be given near $0 \in \mathbb{C}^N$ by an equation of the form*

$$\operatorname{Im} w' = Q(z', \bar z',) + \tilde\Phi_{\geq 3}(z', \bar z', \operatorname{Re} w'),$$

with $\tilde\Phi \in \mathbb{C}\{z,\bar z, \operatorname{Re} w\}$. Then any formal biholomorphic map into the normal form from Theorem 1 is convergent if the (formal) normal form

$$\operatorname{Im} w = Q(z,\bar z) + \Phi_{\geq 3}(z,\bar z, \operatorname{Re} w)$$

satisfies

$$\Phi'_{1,1}\Phi_{1,2} - \Phi'_{1,2}(Q + \Phi_{1,1}) = 0. \tag{2}$$

It is a natural question to ask how our normal form relates to the Chern-Moser normal form. In fact, our normalization procedure in Theorem 3 is a bit different from the Chern-Moser procedure. Let us emphasize that in the hypersurface case ($d = 1$) the normal form in Theorem 1, even though necessarily different from the Chern-Moser normal form, is automatically convergent. Indeed, in this case, (2) on the formal normal form is *automatically satisfied* since $\Phi_{1,1} = \Phi_{1,2} = 0$.

The construction of our normal form is different than the Chern-Moser construction, since it is geared towards higher codimensional manifolds. However, we can adapt it in such a way that in codimension one, we obtain a completely new proof of the convergence of the Chern-Moser normal form, which relies completely on the inductive procedure used to construct it. We shall discuss this in detail in section 8.

2 Framework

We first gather some notational and technical preliminaries, which are going to be used in the sequel without further mentioning.

2.1 Initial quadric

Let $\tilde{M}$ be a germ of a real analytic manifold at the origin of $\mathbb{C}^{n+d}$ defined by an equation of the form

$$v' = Q(z', \bar{z}') + \tilde{\Phi}_{\geq 3}(z', \bar{z}', u') \tag{3}$$

where $w' := u' + iv' \in \mathbb{C}^d$, $u' = \operatorname{Re} w' \in \mathbb{R}^d$, $v' = \operatorname{Im} w' \in \mathbb{R}^d$ and $z' \in \mathbb{C}^n$. Here, $Q(z', \bar{z}')$ is a quadratic polynomial map with values in $\mathbb{R}^d$ and $\tilde{\Phi}_{\geq 3}(z', \bar{z}', u')$ an analytic map germ at 0. We endow the variables $z', \bar{z}', w'$ with weights: z' and $\bar{z}'$ get endowed with weights $p_1 = p_2 = 1$ and w' (and also u and v) with $p_3 = 2$ respectively. Hence, the defining equation of the *model quadric* $\operatorname{Im} w = Q(z, \bar{z})$ is quasihomogeneous (q-h) of quasi-degree (q-d) 2. We assume that the higher order deformation $\tilde{\Phi}_{\geq 3}(z, \bar{z}, u)$ has quasi-order (q-o) ≥ 3, that is

$$\tilde{\Phi}_{\geq 3}(z', \bar{z}', u') = \sum_{p \geq 3} \tilde{\Phi}_p(z', \bar{z}', u'),$$

with $\tilde{\Phi}_p(z', \bar{z}', u')$ q-h of degree p. Hence, $\tilde{M}$ is a higher order perturbation of the quadric defined by the homogeneous equation $v' = Q(z', \bar{z}')$. We assume that the quadratic polynomial Q is a Hermitian form on $\mathbb{C}^n$, valued in $\mathbb{R}^d$, meaning it is of the form

$$Q(z, \bar{z}) = \begin{pmatrix} Q_1(z, \bar{z}) \\ \vdots \\ Q_d(z, \bar{z}) \end{pmatrix},$$

where each $Q_k(z, \bar{z}) = \bar{z}^t J_k z$ is a Hermitian form on $\mathbb{C}^n$ defined by a Hermitian $n \times n$-matrix J_k. In particular, we observe that $\overline{Q(a, \bar{b})} = Q(b, \bar{a})$, for any $a, b \in \mathbb{C}^n$.

We assume that $Q(z, \bar{z})$ is *nondegenerate*, if $Q(v, e) = 0$ for all $v \in \mathbb{C}^n$ implies $e = 0$, or equivalently,

$$\bigcap_{k=1}^{d} \ker J_k = \{0\}.$$

We also assume that the forms J_k are *linearly independent,* which translates to the fact that if $\sum_k \lambda_k J_k = 0$ for scalars λ_k, then $\lambda_k = 0$, $k = 1, \dots, d$.

In terms of the usual nondegeneracy conditions of CR geometry (see e.g. [BER99]) these conditions can be stated equivalently by requiring that the model quadric $v = Q(z, \bar{z})$ is 1-nondegenerate and of finite type at the origin.

2.2 Complex defining equations

We will also have use for the complex defining equations for the real-analytic (or formal) manifold M. If M is given by

$$\operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w),$$

where φ is at least quadratic, an application of the implicit function theorem (solving for w) shows that one can give an equivalent equation

$$w = \theta(z, \bar{z}, \bar{w}).$$

Such an equation comes from the defining equation of a real hypersurface if and only if $\theta(z, \bar{z}, \bar{\theta}(\bar{z}, z, w)) = w$.

We say that the coordinates (z, w) are normal if $\varphi(z, 0, u) = \varphi(0, \bar{z}, u) = 0$, or equivalently, if $\theta(z, 0, \bar{w}) = \theta(0, \bar{z}, \bar{w})$. The following fact is useful:

Lemma 4. *Let $\varrho(z, \bar{z}, w, \bar{w})$ be a defining function for a germ of a real-analytic submanifold $M \subset \mathbb{C}^n_z \times \mathbb{C}^d_w$. Then (z, w) are normal coordinates for M if and only if $\varrho(z, 0, w, w) = \varrho(0, \bar{z}, w, w) = 0$.*

For a proof, we refer to [BER99].

2.3 Fischer inner product

Let V be a finite dimensional vector space (over $\mathbb{C}$ or $\mathbb{R}$), endowed with an inner product $\langle \cdot, \cdot \rangle$. We denote by $u = (u_1, \dots, u_d)$ a (formal) variable, and write $V[\![u]\!]$ for the space of formal power series in u with values in V. A typical element $f \in V[\![u]\!]$ will be written as

$$f(u) = \sum_{\alpha \in \mathbb{N}^d} f_\alpha u^\alpha, \quad f_\alpha \in V. \tag{4}$$

We define an extension of this inner product to $V[\![u]\!]$ by

$$\left\langle f_\alpha u^\alpha, g_\beta u^\beta \right\rangle = \begin{cases} \alpha! \, \langle f_\alpha, g_\alpha \rangle & \alpha = \beta \\ 0 & \alpha \neq \beta. \end{cases} \tag{5}$$

The inner product $\langle f, g \rangle$ is not defined on all of $V[\![u]\!]$, but is only defined whenever at most finitely many of the products $f_\alpha g_\alpha$ are nonzero. In particular, $\langle f, g \rangle$ is defined whenever $g \in F[u]$. This inner product is called the Fischer inner product [Fis17, Bel79]. If $T \colon \mathbb{C}[\![F_1]\!]u \to \mathbb{C}[\![F_2]\!]u$ is a linear map, we say that T has a formal adjoint if there exists a map $T^* \colon : \mathbb{C}[\![F_2]\!]u \to \mathbb{C}[\![F_1]\!]u$ such that

$$\langle Tf, g \rangle_2 = \langle f, T^* g \rangle_1$$

whenever both sides are defined.

Lemma 5. *A linear map T as above has a formal adjoint if $T(F_1[u]) \subset F_2[u]$.*

Proof. Let $T(f_\alpha u^\alpha) =: g^\alpha = \sum_\beta g^\alpha_\beta u^\beta$, and set $T^*(h_\beta u^\beta) = s^\beta(u) = \sum_\alpha s^\beta_\alpha u^\alpha$. We need that

$$\begin{aligned}\left\langle T(f_\alpha u^\alpha), h_\beta u^\beta \right\rangle_2 &= \beta! \left\langle g^\alpha_\beta, h_\beta \right\rangle_2 \\ &= \left\langle f, T^*(h_\beta u^\beta) \right\rangle_1 \\ &= \alpha! \left\langle f_\alpha, s^\beta_\alpha \right\rangle_1,\end{aligned}$$

which has to hold for all α, β, and arbitrary $f_\alpha \in F_1, h_\beta \in F_2$. This condition determines s^β_α uniquely: Fix h_β and consider the linear form $F_1 \ni f_\alpha \mapsto \langle T f_\alpha u^\alpha, h_\beta \rangle$. Since $\langle \cdot, \cdot \rangle_1$ is non-degenerate, there exists a uniquely determined $s^\beta_\alpha \in F_1$ such that $\left\langle g^\alpha_\beta, h_\beta \right\rangle_2 = \frac{\alpha!}{\beta!} \left\langle f_\alpha, s^\beta_\alpha \right\rangle_1$.

We now only need to ensure that the series T^*h is well-defined for $h = \sum_\beta h_\beta u^\beta$. It would be given by

$$T^*h = \sum_\alpha \left(\sum_\beta s^\beta_\alpha \right) u^\alpha,$$

which is a well-defined expression under the condition that $T(f_\alpha u^\alpha)$ is a polynomial. $\square$

We are now quickly going to review some of the facts and constructions which we are going to need.

The map $D_\alpha \colon F[\![u]\!] \to F[\![u]\!]$,

$$D_\gamma f(u) = \frac{\partial^{|\gamma|} f}{\partial u^\gamma} = \sum_\beta \binom{\alpha!}{\gamma!} \gamma! f_\alpha u^{\alpha-\gamma}$$

has the formal adjoint

$$M_\gamma g(u) = u^\gamma g(u).$$

Indeed,

$$\left\langle D_\gamma f_\alpha u^\alpha, g_\beta u^\beta \right\rangle = \begin{cases} \binom{\alpha}{\gamma} \gamma! (\alpha-\gamma)! \left\langle f_\alpha, g_{\alpha-\gamma} \right\rangle = \left\langle f_\alpha u^\alpha, g_\beta u^{\beta+\gamma} \right\rangle & \beta = \alpha - \gamma \\ 0 & \beta \neq \alpha - \gamma \end{cases}.$$

If $L \colon F_1 \to F_2$ is a linear operator, then the induced operator $T_L \colon F_1[\![u]\!] \to F_2[\![u]\!]$ defined by

$$T_L \left(\sum_\alpha f_\alpha u^\alpha \right) = \sum_\alpha L f_\alpha u^\alpha$$

has the formal adjoint $T_L^* = T_{L^*}$, since

$$\left\langle T_L f_\alpha u^\alpha, g_\beta u^\beta \right\rangle_2 = \begin{cases} \alpha! \left\langle L f_\alpha, g_\beta \right\rangle_2 = \alpha! \left\langle f_\alpha, L^* g_\beta \right\rangle_1 = \left\langle f_\alpha u^\alpha, T_{L^*} g_\beta u^\beta \right\rangle & \alpha = \beta \\ 0 & \text{else.} \end{cases}$$

Let $L_j \colon F[\![u]\!] \to F_j[\![u]\!]$ be linear operators, $j = 1, \dots, n$, each of which possesses a formal adjoint L_j^*. Then the operator

$$L = (L_1, \dots, L_n) \colon F[\![u]\!] \to \oplus_j F_j[\![u]\!],$$

where $\oplus_j F_j$ is considered as an orthogonal sum, has the formal adjoint $L^* = \sum_j L_j^*$.

More generally, it is often convenient to gather all derivatives together: consider the map $D_k\colon F[\![u]\!] \to \operatorname{Sym}^k F[\![u]\!]$, where $\operatorname{Sym}^k F$ is the space of symmetric k-tensors on $\mathbb{C}^d$ (respectively $\mathbb{R}^d$) with values in F, defined by

$$D_k f(u) = (D_\alpha f(u))_{\substack{\alpha\in\mathbb{N}^d\\|\alpha|=k}}$$

has the formal adjoint $D_k^* = M_k$ given by

$$M_k g(u) = \sum_{\substack{\gamma\in\mathbb{N}^d\\|\gamma|=k}} g_\gamma(u) u^\gamma.$$

Here we realize the space $\operatorname{Sym}^k F$ as the space of homogeneous polynomials of degree k in d variables $(u_1, \ldots, u_d)$, i.e.

$$\operatorname{Sym}^k F = \bigoplus_{j=1}^{\binom{k+d-1}{d-1}} F,$$

with the induced norm as an orthogonal sum (which is the usual induced norm on that space).

If $L_1\colon F[\![u]\!] \to F_1[\![u]\!]$ and $L_2\colon F_1[\![u]\!] \to F_2[\![u]\!]$ are linear maps each of which possesses a formal adjoint, then $L = L_2 \circ L_1$ has the formal adjoint $L^* = L_1^* \circ L_2^*$.

It is often convenient to use the *normalized* Fischer product [LS10], which is defined by

$$\left\langle f_\alpha u^\alpha, g_\beta u^\beta \right\rangle = \begin{cases} \frac{\alpha!}{|\alpha|!} \langle f_\alpha, g_\alpha \rangle & \alpha = \beta \\ 0 & \alpha \neq \beta. \end{cases} \tag{6}$$

While the adjoints with respect to the normalized and the standard Fischer inner product differ by constant factors for terms of the same homogeneity, the existence of adjoints and their kernels agree. Thus, it is not necessary to distinguish between the normalized and the standard Fischer product when looking at kernels of adjoints. The normalized version of the inner product is far more suitable when dealing convergence issues and also better for nonlinear problems [LS10][proposition 3.6-3.7].

Our coefficient spaces F_1 and F_2 are often going to be spaces of polynomials (in z and $\bar{z}$) of certain homogeneities, themselves equipped with the Fischer norm. Let $\mathcal{H}_{n,m}$ be the space of homogeneous polynomials of degree m in $z \in \mathbb{C}^n$. We shall omit to write dependance on the dimension n if the context permits. Our definition of the (normalized) Fischer inner product $\langle \cdot, \cdot \rangle$, means that on monomials

$$\left\langle z^\alpha, z^\beta \right\rangle = \begin{cases} \frac{\alpha!}{|\alpha|!} & \alpha = \beta, \\ 0 & \alpha \neq \beta, \end{cases} \tag{7}$$

and the inner product on $(\mathcal{H}_{n,m})^\ell$ is induced by declaring that the components are orthogonal with each other : if $f = (f^1, \ldots, f^\ell) \in (\mathcal{H}_{n,m})^\ell$, then $\langle f, g \rangle = \sum_{j=1}^\ell \left\langle f^j, g^j \right\rangle$.

Let $\mathcal{R}_{m,k}$ be the space of polynomials in z and $\bar{z}$, valued in $\mathbb{C}^d$, which are homogeneous of degree m (resp. k) in z (resp. $\bar{z}$). Also this space will be equipped with the Fischer inner product $\langle\cdot,\cdot\rangle_{d,k}$, where the components are declared to be orthogonal as well. That is, the inner product of a polynomial $P = (P_1,\dots,P_d)^t \in \mathcal{R}_{m,k}$ with a polynomial $Q = (Q_1,\dots,Q_d)^t \in \mathcal{R}_{m,k}$ is defined by $\langle P,Q\rangle = \sum_\ell \langle P_\ell, Q_\ell\rangle$, and the latter inner products are given on the basis monomials by

$$\left\langle z^{\alpha_1}\bar{z}^{\alpha_2}, z^{\beta_1}\bar{z}^{\beta_2}\right\rangle = \begin{cases} \frac{\alpha_1!\alpha_2!}{(|\alpha_1|+|\alpha_2|)!} & \alpha_1=\beta_1,\ \alpha_2=\beta_2 \\ 0 & \alpha_1\neq\beta_1 \text{ or } \alpha_2\neq\beta_2. \end{cases} \tag{8}$$

2.4 The normalization conditions

In this section, we shall discuss some of the operators which we are going to encounter and discuss the normalization conditions used in Theorem 1, Theorem 2, and Theorem 3. The first normalization conditions on the $(p,0)$ and $(0,p)$ terms of a power series $\Phi(z,\bar{z},u) \in \mathbb{C}[\![z,\bar{z},u]\!]$, decomposed as

$$\Phi(z,\bar{z},u) = \sum_{j,k=0}^{\infty} \Phi_{j,k}(z,\bar{z},u),$$

is that

$$\Phi_{p,0} = \Phi_{0,p} = 0, \quad p\geq 0. \tag{9}$$

With the potential to confuse the notions, we note that this corresponds to the requirement that (z,w) are "normal" coordinates in the sense of Baouendi, Ebenfelt, and Rothschild (see e.g. [BER99]) (it is also equivalent to the requirement that Φ "does not contain harmonic terms"). We write

$$\mathcal{N}^0 := \{\Phi \in \mathbb{C}[\![z,\bar{z},u]\!] \colon \Phi(z,0,u) = \Phi(0,\bar{z},u) = 0\}. \tag{10}$$

The first important operator, $\mathcal{K}$, is defined on formal power series in z and u (or w), and maps them to power series in $z,\bar{z},u$, linear in $\bar{z}$, by

$$\mathcal{K}\colon \mathbb{C}[\![z,u]\!]^d \to \mathbb{C}[\![z,\bar{z},u]\!]^d/(\bar{z}^2), \quad \mathcal{K}(\varphi(z,u)) = Q(\varphi(z,u),\bar{z}) = \begin{pmatrix} \bar{z}^t J_1(\varphi(z,u)) \\ \vdots \\ \bar{z}^t J_d(\varphi(z,u)) \end{pmatrix}.$$

We can also consider $\bar{\mathcal{K}}$, defined by

$$\bar{\mathcal{K}}\colon \mathbb{C}[\![\bar{z},u]\!]^d \to \mathbb{C}[\![z,\bar{z},u]\!]^d/(z^2), \quad \mathcal{K}(\varphi(\bar{z},u)) = Q(z,\varphi(\bar{z},u)) = \begin{pmatrix} (\varphi(\bar{z},u))^t J_1 z \\ \vdots \\ (\varphi(\bar{z},u))^t J_d z \end{pmatrix}.$$

The important distinction for these operators to the case $d=1$, is that for $d>1$, they are not of full range. They are still injective, as we'll show later in Lemma 7. We will also construct a rather natural complementary space for their range, namely the kernels of

$$\mathcal{K}^*\colon \mathbb{C}[\![z,\bar{z},u]\!]^d/(\bar{z}^2) \to \mathbb{C}[\![z,u]\!]^d, \quad \mathcal{K}^*\begin{pmatrix} b_1(z,\bar{z},u) \\ \vdots \\ b_d(z,\bar{z},u) \end{pmatrix} = \sum_{j=1}^{d} \left(J_j \begin{pmatrix} \frac{\partial}{\partial \bar{z}_1}\big|_0 \\ \vdots \\ \frac{\partial}{\partial \bar{z}_n}\big|_0 \end{pmatrix} \right) b_j$$

and of $(\bar{\mathcal{K}})^*$, respectively. These operators are needed for the normalization of the $(p,1)$ and $(1,p)$ terms for $p>1$ and constitute our first set of normalization conditions different from the Chern-Moser conditions:

$$\mathcal{K}^*\Phi_{p,1} = \bar{\mathcal{K}}^*\Phi_{1,p} = 0, \quad p > 1. \tag{11}$$

We set the corresponding normal form space

$$\mathcal{N}^1_{\leq k} = \left\{\Phi \in \mathbb{C}[\![z,\bar{z},u]\!] \colon \mathcal{K}^*\Phi_{p,1} = \bar{\mathcal{K}}^*\Phi_{1,p} = 0, 1 < p \leq k\right\}. \tag{12}$$

For our other normalization conditions, in addition the operator $\mathcal{K}$, we shall need the operator Δ, introduced by Beloshapka in [Bel90]. It is defined for a power series map in $(z,\bar{z},u)$ (valued in an arbitrary space) by

$$(\Delta\varphi)(z,\bar{z},u)) = \sum_{j=1}^{d} \varphi_{u_j}(z,\bar{z},u) Q_j(z,\bar{z}).$$

Its adjoint with respect to the Fischer inner product is going to play a prominent role: It is defined, again for an arbitrary power series map φ, by

$$\Delta^*\varphi = \sum_{j=1}^{d} u_j Q_j\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right)\varphi.$$

The operator Δ^* is the equivalent to the trace operator which we are going to use. The possible appearance of "unremovable" terms in $\Phi_{1,1}$ makes it a bit harder to formulate the corresponding trace conditions, as not only the obviously invariant Q plays a role, but rather all the invariant parts of $\Phi_{j,j}$ for $j \leq 3$. Furthermore, in the general setting, we do not have a "polar decomposition" for $\Phi_{1,1}$, making it hard to decide which terms to "remove" and which to "keep" when normalizing the diagonal tems. We opt for a balanced approach in our second set of normalization conditions, involving the diagonal terms $(1,1)$, $(2,2)$, and $(3,3)$:

$$\begin{gathered} -6\Delta^*\Phi_{1,1} + (\Delta^*)^3\Phi_{3,3} = 0 \\ \mathcal{K}^*(\Phi_{1,1} - i\Delta^*\Phi_{2,2} - (\Delta^*)^2\Phi_{3,3}) = 0. \end{gathered} \tag{13}$$

We define the set of power series $\Phi \in \mathbb{C}[\![z,\bar{z},u]\!]$ satisfying these normalization conditions as $\mathcal{N}^d$ ("d" stands for "diagonal terms"). Let us note that in the case $d=1$, these conditions are different from the Chern-Moser conditions.

The last set of normalization conditions deals with the $(2,3)$ and the $(3,2)$ terms; those possess terms which are not present in the Chern-Moser setting, but which simply disappear in the case $d=1$, reverting to the Chern-Moser conditions:

$$\mathcal{K}^*(\Delta^*)^2\left(\Phi_{2,3} + i\Delta\Phi_{1,2}\right) = \bar{\mathcal{K}}^*(\Delta^*)^2\left(\Phi_{3,2} - i\Delta\Phi_{2,1}\right) = 0. \tag{14}$$

The space of the power series which satisfy this condition will be denoted by

$$\mathcal{N}^{\text{off}} = \left\{\Phi \in \mathbb{C}[\![z,\bar{z},u]\!] \colon \mathcal{K}^*(\Delta^*)^2\left(\Phi_{2,3} + i\Delta\Phi_{1,2}\right) = \bar{\mathcal{K}}^*(\Delta^*)^2\left(\Phi_{3,2} - i\Delta\Phi_{2,1}\right) = 0\right\}. \tag{15}$$

This is the normal forms space of "off-diagonal terms". Let us note that in the case $d = 1$, because in our choice of normalization we have that $\Phi_{1,1} \neq 0$ in general, even though our normalization condition for the $(3, 2)$ term reverts to the same differential equation as the differential equation for a chain, our full normal form will not necessarily produce chains. We discuss this issue later in section 8.

We can now define the spaces $\hat{\mathcal{N}}_f \subset \hat{\mathcal{N}}_f^w$ of normal forms:

$$\hat{\mathcal{N}}_f := \mathcal{N}^0 \cap \mathcal{N}^1_{\leq\infty} \cap \mathcal{N}^d \cap \mathcal{N}^{\text{off}} \qquad \hat{\mathcal{N}}_f^w := \mathcal{N}^0 \cap \mathcal{N}^1_{\leq\infty} \cap \mathcal{N}^d \tag{16}$$

3 Transformation of a perturbation of the initial quadric

We consider a formal holomorphic change of coordinates of the form

$$z' = Cz + f_{\geq 2}(z, w) =: f(z, w), \quad w' = sw + g_{\geq 3}(z, w) =: g(z, w) \tag{17}$$

where the invertible $n \times n$ matrix C and the invertible real $d \times d$ matrix s satisfy

$$Q(Cz, \bar{C}\bar{z}) = sQ(z, \bar{z}).$$

In these new coordinates, equation (3) reads

$$v = Q(z, \bar{z}) + \Phi_{\geq 3}(z, \bar{z}, u). \tag{18}$$

This is the new equation of the manifold M (in the coordinates (z, w)). We need to find the expression of $\Phi_{\geq 3}$. We have the following *conjugacy equation*:

$$\begin{aligned} sv + \operatorname{Im}(g_{\geq 3}(z, w)) \quad = \quad & Q\left(Cz + f_{\geq 2}(z, w), \bar{C}\bar{z} + \bar{f}_{\geq 2}(\bar{z}, \bar{w})\right) \\ & + \tilde{\Phi}_{\geq 3}\left(Cz + f_{\geq 2}(z, w), \bar{C}\bar{z} + \bar{f}_{\geq 2}(\bar{z}, \bar{w}), su + \operatorname{Re}(g_{\geq 3}(z, w))\right). \end{aligned}$$

Let us set as notation $f := f(z, u+iv)$ and $\bar{f} := \bar{f}(\bar{z}, u-iv)$ with $v := Q(z, \bar{z}) + \Phi_{\geq 3}(z, \bar{z}, u)$. We shall write Q for $Q(z, \bar{z})$. The conjugacy equation reads

$$\frac{1}{2i}(g - \bar{g}) = Q\left(f, \bar{f}\right) + \tilde{\Phi}_{\geq 3}\left(f, \bar{f}, \frac{g + \bar{g}}{2}\right). \tag{19}$$

As above, we set $f_{\geq 2} := f_{\geq 2}(z, u + iv)$ and $\bar{f}_{\geq 2} := \bar{f}_{\geq 2}(\bar{z}, u - iv)$. We have

$$\begin{aligned} \frac{1}{2i}(s(u + iv) - s(u - iv)) &= sQ(z, \bar{z}) + s\Phi_{\geq 3}(z, \bar{z}, v) \\ Q\left(f, \bar{f}\right) &= Q\left(Cz + f_{\geq 2}, \bar{C}\bar{z} + \bar{f}_{\geq 2}\right) \\ &= Q\left(Cz, \bar{f}_{\geq 2}\right) + Q\left(f_{\geq 2}, \bar{C}\bar{z}\right) + Q\left(Cz, \bar{C}\bar{z}\right) + Q\left(f_{\geq 2}, \bar{f}_{\geq 2}\right) \\ \tilde{\Phi}_{\geq 3}\left(f, \bar{f}, \frac{1}{2}[g + \bar{g}]\right) &= \tilde{\Phi}_{\geq 3}\left(Cz, \bar{C}\bar{z}, su\right) \\ &\quad + \left(\tilde{\Phi}_{\geq 3}\left(f, \bar{f}, \frac{1}{2}[g + \bar{g}]\right) - \tilde{\Phi}_{\geq 3}\left(Cz, \bar{C}\bar{z}, su\right)\right) \end{aligned} \tag{20}$$

Therefore, we can rewrite (19) in the following way:

$$
\begin{aligned}
\frac{1}{2i}[g_{\geq 3}(z,u+iQ) - \bar{g}_{\geq 3}(\bar{z},u-iQ)] - \left(Q\left(Cz,\bar{f}_{\geq 2}(\bar{z},u-iQ)\right) + Q\left(f_{\geq 2}(z,u+iQ),\bar{C}\bar{z}\right)\right)& \\
= Q\left(f_{\geq 2},\bar{f}_{\geq 2}\right)& \\
+ \tilde{\Phi}_{\geq 3}\left(Cz,\bar{C}\bar{z},su\right) - s\Phi_{\geq 3}(z,\bar{z},u)& \\
+ \left(\tilde{\Phi}_{\geq 3}\left(f,\bar{f},\frac{1}{2}(g+\bar{g})\right) - \tilde{\Phi}_{\geq 3}\left(Cz,\bar{C}\bar{z},su\right)\right)& \\
+ \frac{1}{2i}\left(g_{\geq 3}(z,u+iQ) - g_{\geq 3}\right) - \frac{1}{2i}\left(\bar{g}_{\geq 3}(\bar{z},u-iQ) - \bar{g}_{\geq 3}\right)& \\
+ \left(Q\left(Cz,\bar{f}_{\geq 2}\right) - Q\left(Cz,\bar{f}_{\geq 2}(z,u-iQ)\right)\right)& \\
+ \left(Q\left(f_{\geq 2},\bar{C}\bar{z}\right) - Q\left(f_{\geq 2}(z,u+iQ),\bar{C}\bar{z}\right)\right)&
\end{aligned}
$$

Let us set $C = \mathrm{id}$ and $s = 1$. We shall write this equation as

$$
\mathcal{L}(f_{\geq 2}, g_{\geq 3}) = \mathcal{T}(z,\bar{z},u; f_{\geq 2}, g_{\geq 3}, \Phi) - \Phi \tag{21}
$$

where $\mathcal{L}(f_{\geq 2}, g_{\geq 3})$ (resp. $\mathcal{T}(z,\bar{z},u;, f_{\geq 2}, g_{\geq 3}, \Phi)$) denotes the linear (resp. nonlinear) operator defined on the left (resp. right) hand side of (21). The linear operator $\mathcal{L}$ maps the space the space of quasihomogeneous holomorphic vector fields QH_{k-2} of quasi degree $k-2 \geq 1$, that is, of expressions of the form

$$
f_{k-1}(z,w)\frac{\partial}{\partial z} + g_k(z,w)\frac{\partial}{\partial w} = f_{k-1}(z,w)\cdot\begin{pmatrix}\frac{\partial}{\partial z_1}\\ \vdots \\ \frac{\partial}{\partial z_n}\end{pmatrix} + g_k(z,w)\cdot\begin{pmatrix}\frac{\partial}{\partial w_1}\\ \vdots \\ \frac{\partial}{\partial w_d}\end{pmatrix},
$$

where f_{k-1} and g_k are quasi-homogeneous polynomials taking values in $\mathbb{C}^n$ and $\mathbb{C}^d$, respectively to the space of quasi-homogeneous polynomials of degree $k \geq 3$ with values in $\mathbb{C}^d$. We shall denote the restriction of $\mathcal{L}$ to QH_{k-2} by $\mathcal{L}_k$.

By expanding into quasihomogeneous component, equation (21) reads

$$
\mathcal{L}(f_{k-1}, g_k) = \{\mathcal{T}(z,\bar{z},u; f_{\geq 2}, g_{\geq 3}, \Phi)\}_k - \Phi_k = \{\mathcal{T}(z,\bar{z},u; f_{\geq 2}^{<k-1}, g_{\geq 3}^{<k}), \Phi_{<k}\}_k - \Phi_k. \tag{22}
$$

Here, $\{\mathcal{T}(z,\bar{z},u; f_{\geq 2}, g_{\geq 3}), \Phi\}_k$ (resp. $f_{\geq 2}^{<k-1}$) denotes the quasi-homogeneous term of degree k (resp. $< k-1$) of the Taylor expansion of $\mathcal{T}(z,\bar{z},u; f_{\geq 2}, g_{\geq 3}, \Phi)$ (resp. $f_{\geq 2}$) at the origin.

It is well-known (see e.g. [BER98]) that the operator $\mathcal{L}$, considered as an operator on the space of (formal) holomorphic vector fields, under our assumptions of linear independence and nondegeneracy of the form Q, has a finite-dimensional (as a real vector space) kernel, which coincides with the space of infinitesimal CR automorphisms of the model quadric $\operatorname{Im} w = Q(z,\bar{z})$ fixing the origin. It follows that, for any $k \geq 3$, any complementary subspace $\mathcal{N}_k$ to the image of $\mathcal{L}_k$ gives rise to a *formal normal form* of degree k. By induction on k, we prove that there exists a (f_{k-1}, g_k) and a $\Phi_k \in \mathcal{N}_k$ such that equation (22) is solved. A a consequence, up to elements of the space of infinitesimal automorphisms of the model quadric, there exists a unique formal holomorphic change of coordinates such that the "new" defining function lies in the space of normal form $\mathcal{N} := \bigoplus_{k\geq 3}\mathcal{N}_k$.

In order to find a way to choose $\mathcal{N}$ with the additional property that for analytic defining functions, the change of coordinates is also analytic, we shall pursue a path which tries to rewrite the important components of $\mathcal{L}$ as partial differential operators.

From now on, we write $\{h\}_{p,q}$ for the term in the Taylor expansion of h which is homogeneous of degree p in z and of degree q in $\bar{z}$. For a map $h = h(z, \bar{z}, u)$, we have $\{h\}_{p,q} = h_{p,q}(u)$ for some map $h_{p,q}(u)$ taking values in the space of polynomials homogeneous of degree p in z and of degree q in $\bar{z}$ (with values in the same space as h), which is analytic in a fixed domain of u independent of p and q (provided that h is analytic). We also will from now on write $f_k(z, u)$ for the homogeneous polynomial of degree k (in z) in the Taylor expansion of f. Even though this conflicts with our previous use of the subscript, no problems shall arise from the dual use.

In what follows our notation can be considered as an abuse of notation: in an expression such as $D_u^k g(z, u)(Q + \Phi)^k$, we write as if $Q + \Phi$ was a scalar. This is harmless since we are only interested in a lower bound of the vanishing order of some fix set of monomials in $z, \bar{z}$. However, if one decides to consider $D_u^k g$ as a symmetric multilinear form and considers powers as appropriate "filling" of these forms by arguments, one can also consider the equations as actual equalities.

We have

$$g_{\geq 3}(z, u + iQ) - g_{\geq 3}(z, u + iQ + i\Phi) = \sum_{k\geq 1} \frac{i^k}{k!} D_u^k g_{\geq 3}(z, u) \left(Q^k - (Q + \Phi)^k\right), \tag{23}$$

and

$$Q\left(f_{\geq 2} - f_{\geq 2}(z, u + iQ), \bar{C}\bar{z}\right) = Q\left(\sum_{k\geq 1} \frac{i^k}{k!} D_u^k f_{\geq 2}(z, u) \left(Q^k - (Q + \Phi)^k\right), \bar{C}\bar{z}\right),$$

and therefore

$$\left\{D_u^k g(z, u) \left(Q^k - (Q + \Phi)^k\right)\right\}_{p,q} = \sum_{l=0}^{p} D_u^k g_l(z, u) \left\{Q^k - (Q + \Phi)^k\right\}_{p-l,q}$$

and

$$\begin{aligned} \left\{Q\left(f_{\geq 2} - f_{\geq 2}(z, u + iQ), \bar{C}\bar{z}\right)\right\}_{p,q} &= Q\left(\{f_{\geq 2} - f_{\geq 2}(z, u + iQ)\}_{p,q-1}, \bar{C}\bar{z}\right) \\ &= \sum_{l=0}^{p} \sum_{k\geq 1} \frac{i^k}{k!} Q\left(D_u^k f_l(z, u) \left\{Q^k - (Q + \Phi)^k\right\}_{p-l,q-1}, \bar{C}\bar{z}\right). \end{aligned} \tag{24}$$

4 Equations for the (p, q)-term of the conjugacy equation

For any non negative integers p, q, let us set

$$T_{p,q} := \left\{\tilde{\Phi}_{\geq 3}\left(f, \bar{f}, \frac{1}{2}(g + \bar{g})\right) - \tilde{\Phi}_{\geq 3}\left(Cz, \bar{C}\bar{z}, su\right)\right\}_{p,q}.$$

4.1 $(p,0)$-terms

According to (95), (101),(105) , the $(p,0)$-term of the conjugacy equation (19), for $p \geq 2$, is

$$\frac{1}{2i} g_p = Q(f_p, \bar{f}_0) + T_{p,0} + \tilde{\Phi}_{p,0}\left(Cz, \bar{C}\bar{z}, su\right) - s\Phi_{p,0}(z,\bar{z},u) =: F_{p,0}. \tag{25}$$

For $p = 1$, the linear map $\mathcal{L}$ gives a new term $-Q(Cz, \bar{f}_0)$ to the previous one. Hence, we have

$$\frac{1}{2i} g_1 - Q(Cz, \bar{f}_0) = Q(f_1, \bar{f}_0) + T_{1,0} + \tilde{\Phi}_{1,0}\left(Cz, \bar{C}\bar{z}, su\right) - s\Phi_{1,0}(z,\bar{z},u) =: F_{1,0}. \tag{26}$$

For $p = 0$, we have

$$\mathrm{Im}(g_0) = Q(f_0, \bar{f}_0) + T_{0,0} + \tilde{\Phi}_{0,0}\left(Cz, \bar{C}\bar{z}, su\right) - s\Phi_{0,0}(z,\bar{z},u) =: F_{0,0} \tag{27}$$

4.2 $(p,1)$-terms

According to (96), (101),(106) , the $(p,1)$-term of the conjugacy equation (19), for $p \geq 3$, is

$$\begin{aligned} \frac{1}{2} D_u g_{p-1} Q - Q(f_p, \bar{C}\bar{z}) \quad = \quad & \mathrm{Im}\left(iD_u g_{p-2}(u)\Phi_{2,1} + iD_u g_{p-1}(u)\Phi_{1,1}\right) + Q(f_p, \bar{f}_1) \\ & + iQ(Df_{p-1}(Q + \Phi_{1,1}), \bar{f}_0) - iQ(f_{p-1}, D_u\bar{f}_0(Q + \Phi_{1,1})) \\ & + \tilde{\Phi}_{p,1}(Cz, su) - s\Phi_{p,1}(z,u) + T_{p,1} =: F_{p,1}. \end{aligned} \tag{28}$$

For $p = 2$, we get the same expression on the right hand side, but the linear part gains the term $iQ(Cz, D_u\bar{f}_0 Q)$. Hence, we have

$$\frac{1}{2} D_u g_1 Q - Q(f_2, \bar{C}\bar{z}) + iQ(Cz, D_u\bar{f}_0 Q) = F_{2,1}. \tag{29}$$

For $p = 1$, we have

$$D_u \,\mathrm{Re}(g_0(u)) \cdot Q - Q(Cz, \bar{f}_1(\bar{z}, u)) - Q(f_1(z,u), \bar{C}\bar{z}) = F_{1,1} \tag{30}$$

4.3 $(3,2)$

For the $(3,2)$-terms, we obtain

$$\begin{aligned} -\frac{1}{4i} D_u^2 g_1(z,u) Q^2 + \frac{1}{2} Q(Cz, D_u^2 \bar{f}_0(u) Q^2) - iQ(D_u f_2(z,u)Q, \bar{C}\bar{z}) = (109) + \frac{1}{2i}(99) + (104) \\ + \tilde{\Phi}_{3,2}(Cz, \bar{C}\bar{z}, su) - s\Phi_{3,2}(z,\bar{z},u) - \frac{1}{2i}\overline{(99)} + \overline{(104)} + (110)_{3,2}. \end{aligned} \tag{31}$$

where $(110)_{3,2}$ denotes the $(3,2)$-component of (110), $\overline{(99)}$ (resp. $\overline{(104)}$) denotes the $(3,2)$-component of $(\bar{g}_{\geq 3}(\bar{z}, u - iQ) - \bar{g}_{\geq 3})$ (resp. $\left(Q\left(Cz, \bar{f}_{\geq 2}\right) - Q\left(Cz, \bar{f}_{\geq 2}(z, u - iQ)\right)\right)$).

4.4 $(2,2)$-terms

For the $(2,2)$ term, we have

$$-\frac{1}{2}D_u^2\operatorname{Im}(g_0)\cdot Q^2+iQ(Cz,D_u\bar f_1(\bar z,u)\cdot Q)-iQ(D_uf_1(z,u)\cdot Q,\bar C\bar z)=(107)+\frac{1}{2i}(97)+(102)$$
$$+\tilde\Phi_{2,2}(Cz,\bar C\bar z,su)-s\Phi_{2,2}(z,\bar z,u)-\frac{1}{2i}\overline{(97)}+\overline{(102)}+(110)_{2,2}=:F_{2,2}. \tag{32}$$

4.5 $(3,3)$-terms

For the $(3,3)$ term, we have

$$-\frac{1}{6}D_u^3\operatorname{Re}(g_0)\cdot Q^3+Q(Cz,D_u^2\bar f_1(\bar z,u)\cdot Q^2)+Q(D_u^2f_1(z,u)\cdot Q^2,\bar C\bar z)=(108)+\frac{1}{2i}(98)+(103)$$
$$+\tilde\Phi_{3,3}(Cz,\bar C\bar z,su)-s\Phi_{3,3}(z,\bar z,u)-\frac{1}{2i}\overline{(98)}+\overline{(103)}+(110)_{3,3}=:F_{3,3}. \tag{33}$$

5 A full formal normal form: Proof of Theorem 1

We recall that we have used above the following notation for the grading of the transformation : we consider transformations of the form

$$z^*=z+\sum_{k\geq 0}f_k,\, w^*=w+\sum_{k\geq 0}g_k$$

where $f_k(z,w)$ and $g_k(z,w)$ are homogeneous of degree k in z; f_k and g_k can also be considered as power series maps in w valued in the space of holomorphic polynomials in z of degree k taking values in $\mathbb{C}^n$ and $\mathbb{C}^d$, respectively. We then collect from the equations computed in Section 4: Using (27), (25) and (28), we have

$$\begin{aligned}\operatorname{Im}(g_0)&=F_{0,0}\\ \frac{1}{2i}g_p&=F_{p,0}\\ \frac{1}{2}D_ug_pQ-Q(f_{p+1},\bar z)&=F_{p+1,1}\end{aligned}$$

Using (29) and (31), we have

$$\begin{aligned}\frac{1}{2}D_ug_1Q-Q(f_2,\bar z)+iQ(z,D_u\bar f_0Q)&=F_{2,1}\\ -\frac{1}{4i}D_u^2g_1(z,u)Q^2+\frac{1}{2}Q(z,D^2\bar f_0(u)Q^2)-iQ(D_uf_2(z,u)Q,\bar z)&=F_{3,2}\end{aligned}$$

Using (30),(32) and (33), we have $\operatorname{Im}(g_0)=F_{0,0}$

$$\begin{aligned}D_u\operatorname{Re}(g_0(u))\cdot Q-Q(z,\bar f_1(\bar z,u))-Q(f_1(z,u),\bar z)&=F_{1,1}\\ -\frac{1}{2}D_u^2\operatorname{Im}(g_0)\cdot Q^2+iQ(z,D_u\bar f_1(\bar z,u)\cdot Q)-iQ(D_uf_1(z,u)\cdot Q,\bar z)&=F_{2,2}\\ -\frac{1}{6}D_u^3\operatorname{Re}(g_0)\cdot Q^3+Q(z,D_u^2\bar f_1(\bar z,u)\cdot Q^2)+Q(D_u^2f_1(z,u)\cdot Q^2,\bar z)&=F_{3,3}\end{aligned}$$

In order to obtain an operator $\mathcal{L}$ acting on the space of maps, and taking values in the space of formal power series in $\mathbb{C}[\![z,\bar{z},u]\!]^d$ endowed with Hermitian product 8, we simplify a bit the left hand sides, express the linear occurence of the terms $\Phi_{p,q}$ of the "new" manifold, and change the right hand side accordingly:

$$\begin{aligned}
\operatorname{Im} g_0 &= \Phi_{0,0} + \tilde{F}_{0,0} \\
\frac{1}{2i} g_p &= \Phi_{p,0} + \tilde{F}_{p,0} \\
-Q(f_{p+1},\bar{z}) &= \Phi_{p+1,1} + \tilde{F}_{p+1,1} \\
-Q(f_2,\bar{z}) + iQ(z, D_u\bar{f}_0 Q) &= \Phi_{2,1} + \tilde{F}_{2,1} \\
\frac{1}{2}Q(z, D^2\bar{f}_0(u)Q^2) - iQ(D_u f_2(z,u)Q,\bar{z}) &= \Phi_{3,2} + \tilde{F}_{3,2} \\
D_u \operatorname{Re}(g_0(u))\cdot Q - Q(z,\bar{f}_1(\bar{z},u)) - Q(f_1(z,u),\bar{z}) &= \Phi_{1,1} + \tilde{F}_{1,1} \\
iQ(z, D_u\bar{f}_1(\bar{z},u)\cdot Q) - iQ(D_u f_1(z,u)\cdot Q,\bar{z}) &= \Phi_{2,2} + \tilde{F}_{2,2} \\
-\frac{1}{6}D_u^3 \operatorname{Re}(g_0)\cdot Q^3 + Q(z, D_u^2\bar{f}_1(\bar{z},u)\cdot Q^2) + Q(D_u^2 f_1(z,u)\cdot Q^2,\bar{z}) &= \Phi_{3,3} + \tilde{F}_{3,3}
\end{aligned} \tag{34}$$

At this point, the *existence of some formal normal form* follows by studying the injectivity of the linear operators appearing on the left hand side of (34) (as already explained in Beloshapka [Bel90]). We now explain how we can reach the normalization conditions from Section 2.4.

For the terms $\Phi_{p,0}$ (for $p \geq 0$) this is simply done by applying the conditions (9) to (34) and substituting the resulting expressions for $\operatorname{Im} g_0$ and g_p into the remaining equations.

In order to obtain the normalization conditions for the terms $\Phi_{p,1}$, we apply the operator $\mathcal{K}^*$ to lines 3 and 4 of (34), yielding after application of the normalization conditions (11) a system of implicit equations for f_p for $p \geq 2$. If we substitute the solution of this problem back into the remaining equations, we obtain (now already using the operator notation)

$$\begin{aligned}
-\frac{1}{2}\bar{\mathcal{K}}\Delta^2 f_0 &= \Phi_{3,2} - i\Delta\Phi_{2,1} + \hat{F}_{3,2} \\
\Delta \operatorname{Re}(g_0) - \bar{\mathcal{K}}\bar{f}_1 - \mathcal{K}f_1 &= \Phi_{1,1} + \hat{F}_{1,1} \\
i\bar{\mathcal{K}}\Delta f_1 - i\mathcal{K}\Delta f_1 &= \Phi_{2,2} + \hat{F}_{2,2} \\
-\frac{1}{6}\Delta^3 \operatorname{Re}(g_0) + \bar{\mathcal{K}}\Delta^2\bar{f}_1 + \mathcal{K}\Delta^2 f_1 &= \Phi_{3,3} + \hat{F}_{3,3},
\end{aligned} \tag{35}$$

We can then define the space of normal forms to be the kernel of the adjoint of the operator $\mathcal{L}\colon \mathbb{C}[\![u]\!]^n \times R[\![u]\!]^d \times \mathbb{C}[\![u]\!]^{n^2} \to \mathcal{R}^d_{3,2} \oplus \mathcal{R}^d_{1,1} \oplus \mathcal{R}^d_{2,2} \oplus \mathcal{R}^d_{3,3}$

$$\mathcal{L}(f_0, \operatorname{Re} g_0, f_1) = \begin{pmatrix} -\frac{1}{2}\bar{\mathcal{K}}\Delta^2 f_0 \\ \Delta \operatorname{Re}(g_0) - \bar{\mathcal{K}}\bar{f}_1 - \mathcal{K}f_1 \\ i\bar{\mathcal{K}}\Delta\bar{f}_1 - i\mathcal{K}\Delta f_1 \\ -\frac{1}{6}\Delta^3 \operatorname{Re}(g_0) + \bar{\mathcal{K}}\Delta^2\bar{f}_1 + \mathcal{K}\Delta^2 f_1 \end{pmatrix}$$

with respect to the Hermitian products on these spaces. The solution can be found by constructing the homogeneous terms in u (!) of f_0, ψ, f_1 inductively, since the right hand sides only contains terms of lower order homogeneity (and thus, found in a preceding step).

However, the f_1 enters the nonlinear terms in such a way as to render the system (35) *singular* when one attempts to interpret it as (a system of complete partial) differential equations, because the equation for the $(3,2)$-term contains in the $\tilde{F}_{3,2}$ an f_1'', thereby linking $\bar{f}_0'$ with f_1''; therefore, the appearance of f_0''' in the term $\tilde{F}_{3,3}$ acts as if it contained an f_1''', which exceeds the order of derivative f_1'' appearing in the linear part.

However, in the formal sense, a solution to this equation exists and is unique modulo $\ker \mathcal{L}$, which we know to be a finite dimensional space, and in particular unique if we require $(f_0, \operatorname{Re} g_0, f_1) \in \operatorname{Im} \mathcal{L}^*$. This gives us exactly exactly our normal form space, and thus gives Theorem 1.

6 Analytic solution to the weak conjugacy problem: Proof of Theorem 2

6.1 Step 1: Preparation

In this section, we shall first find a change of coordinates of the form $z' = f_0(w) + z$ and $w' = w + iG(z,w)$, where $G(0,w) = \bar{G}(0,w)$, in order to ensure the normalization conditions $\Phi_{p,0} = \Phi_{0,p} = 0$ for all non negative integers p. This condition is equivalent to the fact that the coordinates (z,w) are *normal* in the sense of Section 2.2. In particular, if we consider a complex defining equation $\tilde{\theta}$ for our perturbed quadric $\operatorname{Im} w' = Q(z', \bar{z}') + \tilde{\Phi}(z', \bar{z}', \operatorname{Re} w')$, then we see by Lemma 4 that (z,w) are normal coordinates if and only if

$$w + iG(z,w) = \tilde{\theta}(z + f_0(w), \bar{f}_0(w), w - iG(0,w)), \tag{36}$$

or eqivalently if and only if

$$\frac{1}{2}\left(G(z,w) + \bar{G}(0,w)\right) = \tilde{\varphi}\left(z + f_0(w), \bar{f}_0(w), w + \frac{i}{2}(G(z,w) - \bar{G}(0,w))\right) \tag{37}$$

We can thus first obtain $G(0,w)$ from the equation derived from (37) by putting $z = 0$:

$$G(0,w) = \tilde{\varphi}(f_0(w), \bar{f}_0(w), w)$$

and then define $G(z,w)$ by (36), obtaining

$$G(z,w) = \frac{1}{i}\left(\tilde{\theta}(z + f_0(w), \bar{f}_0(w), w - i\tilde{\varphi}(f_0(w), \bar{f}_0(w), w)) - w\right).$$

Summing up: we can therefore replace the given defining funtion by this new one, and assume from now on that $f_0 = 0$ and that the coordinates are already normal. This change of coordinates is rather standard and can be found in e.g. [BER99].

6.2 Step 2: Normalization of $(1,1)$, $(2,2)$, $(3,3)$, and $(2,1)$-terms

In this section we shall normalize further the equations of the manifold. Namely, we shall proceed a change of coordinates such that, not only, the manifold is prepared as in the previous section, but also its $(1,1)$, $(2,1)$, $(2,2)$, and $(3,3)$ terms belong to a subspace of normal forms. We will now (after having prepared with the given map f_0) only consider a change of coordinates of the form $z' = z+f(z,w) = z+f_1+f_2$ and $w' = w+g(z,w) = w+g_0$ which satisfies $f(0,w) = 0$, $g(0) = 0$ and $Df(0) = 0$, $Dg(0) = 0$. We assume that $\Phi_{p,0} = \Phi_{0,p} = 0$, $0 \le p$, i.e. that g has been chosen according to the solution of the implicit function theorem in the preceding subsection; with the preparation above, i.e. $\tilde\Phi_{p,0} = \tilde\Phi_{0,p} = 0$, and the restriction on f this amounts to $\operatorname{Im} g_0 = 0$. Using the left hand side of equations (30), (32),(33),(29) and (31) together with $f_0 = 0$, let us set

$$\begin{aligned}
L_{1,1}(f_1,g_0) &:= D_u \operatorname{Re}(g_0(u)) \cdot Q - Q(z,\bar f_1(\bar z,u)) - Q(f_1(z,u),\bar z) && (38)\\
L_{2,2}(f_1,g_0) &:= \frac{-1}{2} D_u^2 \operatorname{Im}(g_0) \cdot Q^2 + iQ(z, D_u\bar f_1(\bar z,u)\cdot Q) - iQ(D_u f_1(z,u)\cdot Q,\bar z) && (39)\\
L_{3,3}(f_1,g_0) &:= \frac{-1}{6} D_u^3 \operatorname{Re}(g_0)\cdot Q^3 + Q(z, D_u^2\bar f_1(\bar z,u)\cdot Q^2) + Q(D_u^2 f_1(z,u)\cdot Q^2,\bar z) && (40)\\
L_{2,1}(f_2) &= -Q(f_2,\bar z) && (41)\\
L_{3,1}(f_3) &= -Q(f_3,\bar z) && (42)\\
& && (43)
\end{aligned}$$

Therefore, equations (30),(32) and (33) read :

$$\begin{aligned}
L_{1,1}(f_1,g_0) &= \operatorname{Re}(D_u g_0(u))\Phi_{1,1} + Q(f_1,\bar f_1)\\
&\quad +\tilde\Phi_{1,1}(z,\bar z,u) - \Phi_{1,1}(z,\bar z,u)\\
&\quad +D_z\tilde\Phi_{1,1}(z,\bar z,u) f_1(z,u) + D_{\bar z}\tilde\Phi_{1,1}(z,\bar z,u)\overline{f_1(z,u)} && (44)\\
L_{2,2}(f_1,g_0) &= iQ(D_u f_1(Q+\Phi_{1,1}),\bar f_1) - iQ(f_1, D_u\bar f_1(Q+\Phi_{1,1}))\\
&\quad +2\operatorname{Re}(Q(iD_u f_1(u)\Phi_{1,1},\bar z)) + (110)_{2,2}\\
&\quad +\tilde\Phi_{2,2}(z,\bar z,u) - \Phi_{2,2}(z,\bar z,u) + Q(f_2,\bar f_2)\\
&\quad +\operatorname{Im}\left(iD_u g_0(u)\Phi_{2,2} + \frac{1}{2}D_u^2 g_0(u)(2\Phi_{1,1}Q + \Phi_{1,1}^2)\right) && (45)\\
L_{3,3}(f_1,g_0) &= Q(iD_u^2 f_1(Q+\Phi_{1,1})^2)),\bar f_1) + Q(f_1, -iD_u^2\bar f_1(Q+\Phi_{1,1})^2)\\
&\quad +2\operatorname{Re}\left(Q(iD_u f_1(u)\Phi_{2,2},\bar z) + \frac{1}{2}Q\left(\frac{1}{2}D_u^2 f_1(u)(2\Phi_{1,1}Q + \{\Phi^2\}_{2,2}),\bar z\right)\right)\\
&\quad +\operatorname{Im}\left(iD_u g_0(u)\Phi_{3,3} + \frac{1}{2}D_u^2 g_0(u)(2\Phi_{2,2}Q + \{\Phi^2\}_{3,3})\right.\\
&\quad \left. -\frac{i}{6}D_u^3 g_0(u)(3\Phi_{1,1}^2 Q + \Phi_{1,1}^3 + 3\Phi_{1,1}Q^2)\right)\\
&\quad +\tilde\Phi_{3,3}(z,\bar z,u) - \Phi_{3,3}(z,\bar z,u)\\
&\quad +(110)_{3,3} && (46)
\end{aligned}$$

Furthermore, equation (29) for $p = 2,3$ reads :

$$\begin{aligned}
L_{2,1}(f_2) &= \operatorname{Re}(D_u g_0(u))\,\Phi_{2,1} + Q(f_2,\bar f_1) + \tilde\Phi_{2,1}(z,\bar z,u) - \Phi_{2,1}(z,\bar z,u) + T_{2,1}\\
L_{3,1}(f_3) &= \operatorname{Re}(D_u g_0(u))\,\Phi_{3,1} + Q(f_3,\bar f_1) + \tilde\Phi_{3,1}(z,\bar z,u) - \Phi_{3,1}(z,\bar z,u) + T_{3,1}
\end{aligned} \qquad (47)$$

Let us recall that the operator Δ is given by $\Delta\colon \mathcal{R}_{p,q}[\![u]\!] \to \mathcal{R}_{p+1,q+1}[\![u]\!]$, $\Delta R(u) =$

$D_uR(u).Q(z,\bar z)$. Then we have

$$L_1(f_1,\mathrm{Re}(g_0)) = \begin{pmatrix} \Delta\,\mathrm{Re}(g_0) - 2\,\mathrm{Re}\,Q(f_1,\bar z)\\ -2\,\mathrm{Im}\,Q(\Delta f_1,\bar z)\\ -\frac{1}{6}\Delta^3\,\mathrm{Re}(g_0) + \mathrm{Re}\,Q(\Delta^2 f_1,\bar z)\end{pmatrix}. \tag{48}$$

Let us write

$$L_2(f_2,f_3) = \begin{pmatrix} -Q(f_2,\bar z)\\ -Q(f_3,\bar z)\end{pmatrix} \tag{49}$$

The system (44)–(47) now reads

$$L(f_1,f_2,f_3,\mathrm{Re}(g_0)) = \mathcal{G}(u, D_u^i f_1, D_u^j\,\mathrm{Re}(g_0), D_u^l f_2, \Phi_{123}) \tag{50}$$

where the indices ranges are: $0\le i\le 2$, $0\le j\le 3$, and $0\le l\le 1$. Also, Φ_{123} stands for $(\Phi_{1,1},\Phi_{2,2},\Phi_{3,3},\Phi_{2,1},\Phi_{3,1})$. Let us emphasize the dependence of $\mathcal{G}$ on Φ_{123} below. We have

$$\mathcal{G} = -(I - D_u\,\mathrm{Re}(g_0))\Phi_{123} + \tilde{\mathcal{G}}(u, D_u^i f_1, D_u^j\,\mathrm{Re}(g_0), D_u^k g_1, D_u^l f_2, \Phi_{123}) \tag{51}$$

where $D_u\,\mathrm{Re}(g_0)\Phi_{123}$ stands for

$$(D_u\,\mathrm{Re}(g_0)\Phi_{1,1}, D_u\,\mathrm{Re}(g_0)\Phi_{2,2}, D_u\,\mathrm{Re}(g_0)\Phi_{3,3}, D_u\,\mathrm{Re}(g_0)\Phi_{2,1}, D_u\,\mathrm{Re}(g_0)\Phi_{3,1}).$$

Furthermore, among Φ_{123}, the (i,j)-component of $\tilde{\mathcal{G}}$ depends only on $\Phi_{\le i-1,\le j-1}$.

Here, $\mathcal{G}$ is analytic in u in a neighborhood of the origin, polynomial in its other arguments and

$$L(f_1,f_2,f_3,\mathrm{Re}(g_0)) = \begin{pmatrix} L_1(f_1,\mathrm{Re}(g_0))\\ L_2(f_2,f_3)\end{pmatrix}. \tag{52}$$

The linear operator L_1 is defined from $(\mathrm{Re}(g_0), f_1)\in\mathbb{R}\{u\}^d\times\mathbb{C}\{u\}^{n^2}\cong\mathbb{R}\{u\}^{k_3+k_1}$ to $\mathcal{R}_{1,1}\{u\}\oplus\mathcal{R}_{2,2}\{u\}\oplus\mathcal{R}_{3,3}\{u\}\cong\mathbb{R}\{u\}^N$ for some N. The linear operator L_2 is defined from $(f_2,f_3)\in\mathbb{C}\{u\}^{n\binom{n+1}{2}}\times\mathbb{C}\{u\}^{n\binom{n+2}{3}}\cong\mathbb{R}\{u\}^{k_2+k_4}$ to $\mathcal{R}_{2,1}\{u\}\times\mathcal{R}_{3,1}\cong\mathbb{R}\{u\}^M$ for some M. Each of these spaces is endowed with the (modified) Fisher scalar product of $\mathbb{R}\{u\}$. Here we have set :

$$k_1 := 2n^2,\quad k_2 := 2n\binom{n+1}{2},\quad k_3 := d,\quad k_4 := 2n\binom{n+2}{3}. \tag{53}$$

Let $\mathcal{N}_1$ (resp. $\mathcal{N}_2$) be the orthogonal subspace to the image of L_1 (resp. L_2) with respect to that scalar product :

$$\begin{aligned}\mathcal{R}_{1,1}\{u\}\oplus\mathcal{R}_{2,2}\{u\}\oplus\mathcal{R}_{3,3}\{u\} &= \mathrm{Im}(L_1)\oplus^{\perp}\mathcal{N}_1\\ \mathcal{R}_{2,1}\{u\}\oplus\mathcal{R}_{3,1}\{u\} &= \mathrm{Im}(L_2)\oplus^{\perp}\mathcal{N}_2.\end{aligned} \tag{54}$$

These are the spaces of *normal forms* and they are defined to be the kernels of the adjoint operator with respect to the modified Fischer scalar product : $\mathcal{N}_1 = \ker L_1^*$, $\mathcal{N}_2 = \ker L_2^*$; in terms of the normal form spaces introduced in Section 2.4, we have in a natural way $\mathcal{N}_1\cong\mathcal{N}^1$ and $\mathcal{N}_2\cong\mathcal{N}_3^2$. Let π_i be the orthogonal projection onto the range of L_i and $\pi := \pi_1\oplus\pi_2$.

The set of the seven previous equations encoded in (50) has the seven real unknowns $\mathrm{Re}(f_1), \mathrm{Im}(f_1), \mathrm{Re}(f_2), \mathrm{Im}(f_2), \mathrm{Re}\,(f_3), \mathrm{Im}\,(f_3), \mathrm{Re}(g_0)$.

Let us project (50) onto the kernel of L^*, which is orthogonal to the image of L with respect to the Fischer inner product, i.e. we impose the normal form conditions (16).

Since Φ_{123} belongs to that space, we have

$$0 = -(I - (I-\pi)D_u \mathrm{Re}(g_0))\Phi_{123} + (I-\pi)\tilde{\mathcal{G}}(u, D_u^i f_1, D_u^j \mathrm{Re}(g_0), D_u^l f_2, \Phi_{123}).$$

In other words, we have obtained

$$\Phi_{123} = ((I - (I-\pi)D_u \mathrm{Re}(g_0))^{-1} (I-\pi)\tilde{\mathcal{G}}(u, D_u^i f_1, D_u^j \mathrm{Re}(g_0), D_u^l f_2, \Phi_{123}). \tag{55}$$

According to the triangular property mentioned above, we can express successively $\Phi_{1,1}, \cdots, \Phi_{3,3}$ as an analytic function of only $u, D_u^i f_1, D_u^j \mathrm{Re}(g_0), D_u^l f_2$. Substituting in (50) and projecting down onto the image of L, we obtain

$$L(f_1, f_2, f_3, \mathrm{Re}(g_0)) = \pi\mathcal{F}(u, D_u^i f_1, D_u^j \mathrm{Re}(g_0), D_u^l f_2, f_3) \tag{56}$$

The equations corresponding to L_2 then turn into a set of implicit equations for f_2 and f_3, which we can solve uniquely in terms of f_1 and $\mathrm{Re}\, g_0$. After substituting those solutions back into $\mathcal{F}$, we satisfy the normalization conditions in $\mathcal{N}_2$, and we turn up with a set of equations for f_1 and $\mathrm{Re}\, g_0$:

$$L_1(f_1, \mathrm{Re}(g_0)) = \pi_1\mathcal{F}_1(u, D_u^i f_1, D_u^j \mathrm{Re}(g_0)) \tag{57}$$

where the indices ranges are: $0 \le i \le 2$, $0 \le j \le 3$, and $0 \le l \le 1$. Here, $\mathcal{F}_1$ denotes an analytic function of its arguments at the origin.

From now on, $\mathrm{ord}_0 f$ will denote the order of $f(z, \bar{z}, u)$ w.r.t u at $u = 0$. Let us recall that we always have

$$\mathrm{ord}_0\, \tilde{\Phi}_{1,1} \quad \ge \quad 1 \tag{58}$$

We now claim that there is an analytic change of coordinates $z = z^* + f_1(z^*, w^*) + f_2(z^*, w^*) + f_3(z^*, w^*)$, $w = w^* + g_0(w^*)$ such that also the diagonal terms of the new equation of the manifold are in normal form, that is $(\Phi_{1,1}, \Phi_{2,2}, \Phi_{3,3}, \Phi_{2,1}, \Phi_{3,1}) \in \mathcal{N}_1 \times \mathcal{N}_2$. In fact, we shall prove that there is exists a unique $(f_1, \mathrm{Re}(g_0)) \in \mathrm{Im}(L_1^*)$ with this property; if we would like to have *all* solutions to that problem, we will see that we can construct a unique solution for any given "initial data" in $\ker L_1$. Instead of working directly on equation (57), we shall first "homogenize" the derivatives of that system. By this we mean, that we apply operator Δ^2 to the first coordinate of (57) and Δ to the second coordinate of (57). The resulting system reads

$$\tilde{L}_1(\tilde{f}_1, \mathrm{Re}(\tilde{g}_0)) = \tilde{\mathcal{F}}_1(u, D_u^i \tilde{f}_1, D_u^j \mathrm{Re}(\tilde{g}_0)) \tag{59}$$

where

$$\tilde{L}_1(\tilde{f}_1, \mathrm{Re}(\tilde{g}_0)) = \begin{pmatrix} \Delta^3 \mathrm{Re}(\tilde{g}_0) - 2\,\mathrm{Re}\, Q(\Delta^2 \tilde{f}_1, \bar{z}) \\ -2\,\mathrm{Im}\, Q(\Delta^2 \tilde{f}_1, \bar{z}) \\ -\frac{1}{6}\Delta^3 \mathrm{Re}(\tilde{g}_0) + \mathrm{Re}\, Q(\Delta^2 \tilde{f}_1, \bar{z}). \end{pmatrix} =: \mathcal{L}_1(D_u^2 \tilde{f}_1, D_u^3 \mathrm{Re}(\tilde{g}_0)) \tag{60}$$

Here, $\mathcal{L}_1$ denotes a linear operator on the finite dimensional vector spaces $\mathrm{Sym}^2(\mathbb{C}^d, \mathbb{C}^n) \times \mathrm{Sym}^3(\mathbb{C}^d, \mathbb{R}^d)$, and we have set $f_1 = j^1 f_1 + \tilde{f}_1$, $g_0 = j^2 g_0 + \tilde{g}_0$, and

$$\tilde{L}_1 := \tilde{\mathcal{D}} \circ L_1, \quad \tilde{\mathcal{F}}_1(u, D_u^i \tilde{f}_1, D_u^j \operatorname{Re}(\tilde{g}_0)) := \tilde{\mathcal{D}} \circ \pi_1 \circ \mathcal{F}_1(u, D_u^i f_1, D_u^j g_0),$$

where

$$\tilde{\mathcal{D}} := \begin{pmatrix} \Delta^2 & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & I \end{pmatrix}.$$

Using the right hand side of (44), (45), (46), and differentiating accordingly, we see that $\mathrm{ord}_0(\tilde{\mathcal{F}}(u, 0)) \geq 1$.

Let us set $\mathbf{m} = (m_1, m_3) = (2, 3)$ and $\mathcal{F}^{\geq 0}_{2,\mathbf{m}} := \left(\mathbb{A}_d^{k_1}\right)_{\geq m_1} \times \left(\mathbb{A}_d^{k_3}\right)_{\geq m_3}$ where the k_i's are defined in (53). Then a tuple of analytic functions

$$H := (H_1, H_3) = (\tilde{f}_1, \operatorname{Re}(\tilde{g}_0))$$

with $\mathrm{ord}_0\, f_1 \geq 2$, $\mathrm{ord}_0\, g_0 \geq 3$ is an element of $\mathcal{F}^{\geq 0}_{2,\mathbf{m}}$. Then, equation (59) reads :

$$\begin{aligned} \mathcal{S}(H) &= \tilde{\mathcal{F}}(u, j_u^{\mathbf{m}} H) &\quad (61)\\ \mathcal{S}(H) &:= \mathcal{L}_1(D_u^2 H_1, D_u^3 H_3). &\quad (62) \end{aligned}$$

Let us show that the assumptions of the Big denominators theorem 14 are satisfied. First of all, for any integer i, let us set $H^{(i)} := (H_1^{(m_1+i)}, H_3^{(m_3+i)})$. Their linear span will be denotes by $\mathcal{H}^{(i)}$. Then, for any i, $\mathcal{S}(H^{(i)})$ is homogeneous of degree of degree i. Let us consider the linear operator $d : (\tilde{f}_1, \operatorname{Re}(\tilde{g}_0)) \mapsto (D_u^2 \tilde{f}_1, D_u^3 \operatorname{Re}(\tilde{g}_0))$. It is one-to-one from $\mathcal{F}^{\geq 0}_{2,\mathbf{m}}$ and onto the space of $\mathrm{Sym}^2(\mathbb{C}^d, \mathbb{C}^n) \times \mathrm{Sym}^3(\mathbb{C}^d, \mathbb{R}^d)$-valued analytic functions in $(\mathbb{R}^d, 0)$. Let $V \in \mathrm{image}(\mathcal{S})$. We recall that $\mathcal{S} = \mathcal{L}_1 \circ d$. Let us set $K := (\mathcal{L}_1 \mathcal{L}_1^*)^{-1}(V)$. It is well defined since V is valued in the range of $\mathcal{L}_1$. Therefore, $\|K\| \leq \alpha \|V\|$ for some positive number α. On the other hand, we have $\mathcal{L}_1^* K \in \mathrm{image}\, d$, so we can (uniquely) solve the equation

$$d(\tilde{f}_1, \operatorname{Re}(\tilde{g}_0)) = \mathcal{L}_1^* K.$$

This solution now satisfies clearly :

$$\begin{aligned} \|\tilde{f}_1^{(i)}\| &\leq \frac{|||\mathcal{L}_1^*|||\alpha}{i^2} \|V^{(i)}\| \\ \|\operatorname{Re}(\tilde{g}_0^{(i)})\| &\leq \frac{|||\mathcal{L}_1^*|||\alpha}{i^3} \|V^{(i)}\| \\ \mathcal{S}(\tilde{f}_1, \operatorname{Re}(\tilde{g}_0)) &= \mathcal{L}_1 d(\tilde{f}_1, \operatorname{Re}(\tilde{g}_0)) = \mathcal{L}_1 \mathcal{L}_1^* K = V. \end{aligned}$$

Hence, $\mathcal{S}$ satisfies the Big Denominators property with respect to $\mathbf{m} = (m_1, m_3) = (2, 3)$.

On the other hand, let us show that $\tilde{\mathcal{F}}(u, j_m^{\mathbf{m}} H)$ strictly increases the degree by $q = 0$. This means that

$$\mathrm{ord}_0 \left(\tilde{\mathcal{F}}(u, j_m^{\mathbf{m}} H) - \tilde{\mathcal{F}}(u, j_m^{\mathbf{m}} \tilde{H}) \right) > \mathrm{ord}_0(H - \tilde{H}).$$

According to Corollary 16 of Appendix B, we just need to check that the system is regular.

So let us now prove that the analytic differential map $\tilde{\mathcal{F}}(u, j_u^{\mathbf{m}})$ is *regular* in the sense of definition 10. To do so, we have to differentiate each term of $\tilde{\mathcal{F}}(u, j_u^{\mathbf{m}})$ with respect to the unknowns and their derivatives and show that the vanishing order of the functions their multiplied by are greater or equal than number $p_{j,|\alpha|}$ as defined in (112) in definition 10. We recall that $q = 0$. Therefore, these number are either 0 (no condition) or 1 (vanishing condition). The later correspond to the vanishing at $u = 0$ of the coeffcient in front the highest derivative order of the unknown :

$$\frac{\partial \tilde{\mathcal{F}}_i}{\partial u_{j,\alpha}}(u, \partial H), \quad |\alpha| = m_j.$$

where $H = (H_1, \ldots, H_r) \in \widehat{\mathcal{F}}_{r,\mathbf{m}}^{\geq 0}$.

But this condition in turn is automatically fulfilled by the construction of the system, since we have put exactly the highest order derivatives appearing in each of the conjugacy equations appearing with a coefficient which is nonzero when evaluated at 0 into the linear part of the operator, and no of the operations which we applied to the system changes this appearance. Let us recall that $f_1(0) = \operatorname{Re} g(0) = 0$. As a conclusion, we see that the map $\tilde{\mathcal{F}}(u, j_u^{\mathbf{m}})$ is *regular*. Furthermore, according to (62), the linear operator $\mathcal{S}$ has the Big Denominator property of order $\mathbf{m} = (2, 1, 3)$. Then according the Big Denominator theorem 14 with $q = 0$, equation (61) has a unique solution $H^{\geq 0} \in \mathcal{F}_{2,\mathbf{m}}^{\geq 0} := \left(\mathbb{A}_d^{k_1}\right)_{\geq m_1} \times \left(\mathbb{A}_d^{k_3}\right)_{\geq m_3}$. This provides the terms of higher order in the expansions of f_1 and $\operatorname{Re} g_0$, and therefore, we proved the

Proposition 6. *There is exists a unique analytic map* $(f_1, \operatorname{Re}(g_0), f_2, f_3) \in \operatorname{Im}(L_1^*) \times \operatorname{Im}(L_2^*)$ *such that under the change of coordinates* $z = z^* + f_1(z^*, w^*) + f_2(z^*, w^*) + f_3(z^*, w^*)$, $w = w^* + g_0(w^*)$, *the* $(1,1)$, $(2,1)$, $(2,2)$ *and* $(3,3)$ *terms of the new equation of the manifold are in normal form, that is,* $\Phi \in \mathcal{N}^0 \cap \mathcal{N}^d \cap \mathcal{N}_{\leq 3}^1$ *as defined in Section 2.4.*

6.3 Normalization of terms $(m, 1)$, $m \geq 4$

Let us perform another change of coordinates of the form $z = z^* + \sum_{p \geq 4} f_p(z^*, w^*)$, $w = w^*$. According to (21)we obtain by extracting the $(p,1)$-terms, $p \geq 4$

$$-Q(f(z,u), \bar{z}) = \tilde{\Phi}_{*,1}(z + f(z,u), \bar{z}, u) - \Phi_{*,1}(z, u), \tag{63}$$

where $\tilde{\Phi}_{*,1}(z, \bar{z}, u) := \sum_{p \geq 4} \tilde{\Phi}_{p,1}(z, \bar{z}, u)$ is analytic at 0. We recall that $\tilde{\Phi}(z, 0, u) = \tilde{\Phi}(0, \bar{z}, u) = 0$. Therefore, by Taylor expanding, we obtain

$$\begin{aligned} \{\tilde{\Phi}_{\geq 3}(f, \bar{f}, u)\}_{*,1} \;=\; & \Big\{ \tilde{\Phi}_{\geq 3}(z + f_{\geq 2}(z,u), \bar{z}, u) \\ & + \frac{\partial \tilde{\Phi}_{\geq 3}}{\partial z}(f_{\geq 2}(z, u + iQ + i\Phi) - f_{\geq 2}(z,u)) \\ & + \frac{\partial \tilde{\Phi}_{\geq 3}}{\partial \bar{z}} \bar{f}_{\geq 2}(\bar{z}, u - iQ - i\Phi) + \cdots \Big\}_{*,1} \end{aligned}$$

Since $\tilde{\Phi}_{p,0} = 0$ for all integer p, the previous equality reads

$$\{\tilde{\Phi}_{\geq 3}(f, \bar{f}, u)\}_{*,1} = \tilde{\Phi}_{*,1}(z + f_{\geq 2}(z,u), \bar{z}, u).$$

6.3.1 A linear map

In this section we consider the linear map $\mathcal{K}$, which maps a germ of holomorphic function $f(z)$ at the origin to

$$\mathcal{K}(f) = Q(f(z), \bar{z}). \tag{64}$$

This complex linear operator $\mathcal{K}$ is valued in the space of power series in $z, \bar{z}$, valued in $\mathbb{C}^d$ which are linear in $\bar{z}$. We will first restrict $\mathcal{K}$ to a map $\mathcal{K}_m$ on the space of homogeneous polynomials of degree m in z, with values in $\mathbb{C}^n$, For any $C, \delta > 0$, let us define the Banach space

$$\mathcal{B}_{n,C,\delta} := \{f = \sum_m f_m, f_m \in \mathcal{H}_{n,m}, \|f_m\| \leq C\delta^m\}. \tag{65}$$

Then, the map $\mathcal{K}_m$ is valued in the space $\mathcal{R}_{m,1}$ of polynomials in z and $\bar{z}$, valued in $\mathbb{C}^d$, which are linear in $\bar{z}$ and homogeneous of degree m in z. Let us consider the space $\mathcal{R}_{*,1} := \bigoplus_m \mathcal{R}_{m,1}$ as well as

$$\{f = \sum_m f_m \in \mathcal{R}_{*,1}, \|f_m\| \leq C\delta^m\}$$

where $\|.\|$ denotes the modified Fischer norm and C, δ a positive numbers. The latter is a Banach space denoted $\mathcal{R}_{*,1}(C, \delta)$.

In particular, let us note that if we write $P_k = \sum_j P_k^j(z)\bar{z}_j$ with $P_k^j \in \mathcal{H}_m$, then

$$\left\|P_k\right\|^2 = (m+1)\sum_{j=1}^{n}\left\|P_k^j\right\|^2. \tag{66}$$

Let us write $P_k = \bar{z}^t\mathbf{P}_k$ where $\mathbf{P}_k = (P_k^1, \ldots, P_k^n)^t$. We can now formulate

Lemma 7. *There exists a constant $C > 0$ such that for all $m \geq 0$, we have that*

$$\|f\| \leq \frac{C}{\sqrt{(m+1)}}\|\mathcal{K}_m f\|.$$

In particular, $\mathcal{K}$ has a bounded inverse on its image : if $g \in \mathcal{R}_{*,1}(M, \delta) \cap Im\mathcal{K}$, then $\mathcal{K}^{-1}(g) \in \mathcal{B}_{M,\delta}$ and

$$\|\mathcal{K}^{-1}(g)\| \leq C\|g\|.$$

Proof. We consider the $n \times (nd)$-matrix J defined by

$$J = \begin{pmatrix} J_1 \\ \vdots \\ J_d \end{pmatrix}. \tag{67}$$

Since $\langle\cdot,\cdot\rangle$ is nondegenerate, we can choose an invertible $n\times n$-submatrix $\tilde{J}$ from J, composed of the rows in the spots $(j_1, \ldots, j_n)$; let $k(j_\ell)$ denote which J_k the row j_ℓ belongs to. Then, if $\mathcal{K}_m f = P$, we have for every $k = 1, \ldots, d$ that $\bar{z}^t J_k f = \bar{z}^t\mathbf{P}_k$. Hence, by complexification we see that $J_k f = \mathbf{P}_k$.

Let $\tilde{P} = (P^{j_1}_{k(j_1)}, \dots P^{j_n}_{k(j_n)})^t$. Then $\tilde{J}f = \tilde{P}$, and we can write $f = (\tilde{J})^{-1}\tilde{P}$. Hence,

$$\|f\|^2 \le C \sum_{\ell=1}^{n} \left\| P^{j_\ell}_{k(j_\ell)} \right\|^2 \le \frac{C}{m+1} \|P\|^2 ,$$

by the observation in (66). □

In order to find an explicit complementary space to image $\mathcal{K}_m$, we will use the Fischer inner product to compute its adjoint $\mathcal{K}^*_m$. We first note, that since the components of $\mathcal{R}_{m,1}$ are orthogonal to one another, if we write $\mathcal{K}_m = (\mathcal{K}^1_m, \dots, \mathcal{K}^d_m)$, then $\mathcal{K}^*_m = (\mathcal{K}^1_m)^* + \cdots + (\mathcal{K}^d_m)^*$. The adjoints of the maps $\mathcal{K}^k_m$, $k = 1, \dots, d$ are computed via

$$\begin{aligned} \left\langle \mathcal{K}^k_m f, P_k \right\rangle &= \left\langle \bar{z}^t J_k f, \sum_j P^j_k \bar{z}_j \right\rangle \\ &= \left\langle \sum_{p,q=1}^{n} (J_k)^p_q \bar{z}_p f^q, \sum_j P^j_k \bar{z}_j \right\rangle \\ &= \frac{1}{m+1} \sum_{p,q=1}^{n} (J_k)^p_q \left\langle f^q, P^p_k \right\rangle \\ &= \frac{1}{m+1} \sum_{p,q=1}^{n} \left\langle f^q, \overline{(J_k)^p_q} P^p_k \right\rangle \end{aligned} \tag{68}$$

to be given by

$$(m+1)((\mathcal{K}^k_m)^* P_k)^q = \sum_{p=1}^{n} (J_k)^q_p P^p_k = \sum_{p=1}^{n} (J_k)^q_p \frac{\partial}{\partial \bar{z}_p} P_k, \tag{69}$$

or in more compact notation,

$$(m+1)(\mathcal{K}^k_m)^* P_k = \left(J_k \frac{\partial}{\partial \bar{z}} \right) P_k. \tag{70}$$

We now define the subspace $\mathcal{N}^1_{m,1}$ to consist of the elements of the kernel of $\mathcal{K}^*_m$, i.e.

$$\mathcal{N}^1_{m,1} := \left\{ P = (P_1, \dots P_d)^t \in \mathcal{R}_{m,1} \colon \sum_{k=1}^{d} \left(J_k \frac{\partial}{\partial \bar{z}} \right) P_k = \sum_{k=1}^{d} J_k \mathbf{P}_k = 0 \right\}. \tag{71}$$

Proposition 8. *There exists a holomorphic transformation $z = z^* + f_{\ge 4}(z, w)$, $w = w^*$ such that, the new equation of the manifold satisfies*

$$\Phi_{p,1} \in \mathcal{N}_{p,1}, \quad p \ge 4.$$

Proof. Let $\pi_{*,1}$ be the orthogonal projection onto the range of $\mathcal{K}$. Then since we want $\Phi_{*,1}$ to belong the normal forms space $\mathcal{N}^1_{*,1}$, we have to solve

$$-\mathcal{K}(f) := -Q(f(z,u), \bar{z}) = \pi_{*,1} \tilde{\Phi}_{*,1}(z + f(z,u), \bar{z}, u).$$

According to Lemma 7, the latter has an analytic solution by the implicit function theorem and we are done. □

7 Convergence of the formal normal form

We are now going to prove convergence of the formal normal form in Section 5 under the additional condition of Theorem 3 on the formal normal form. The goal of this section is to show that one can, under this additional condition, replace the nonlinear terms in the conjugacy equations for the terms of order up to $(3,3)$, by another system which allows for the application of the big denominator theorem.

We are again going to consider two real-analytic Levi-nondegenerate submanifolds of $\mathbb{C}^N$, but we now need to use their *complex defining equations* $w = \theta(z,\bar z,\bar w)$ and $w = \tilde\theta(z,\bar z,\bar w)$, respectively, where θ and $\tilde\theta$ are germs of analytic maps at the origin in $\mathbb{C}^n\times\mathbb{C}^n\times\mathbb{C}^d$ valued in $\mathbb{C}^d$; analogously to the real defining functions, we think about $\tilde\theta$ as the "old" and about θ as the "new" defining equation.

When dealing with the complex defining function, we will usually write $\chi = \bar z$ and $\tau = \bar w$. Recall that a map $\theta\colon \mathbb{C}^{2n+d}\to\mathbb{C}^d$ determines a real submanifold if and only if the reality relation

$$\tau = \theta(z,\chi,\bar\theta(\chi,z,\tau)) \tag{72}$$

holds. θ is obtained from a real defining equation $\operatorname{Im} w = \varphi(z,\bar z,\operatorname{Re} w)$ by solving the equation

$$\frac{w-\bar w}{2i} = \varphi\left(z,\bar z,\frac{w+\bar w}{2}\right)$$

for w.

We will already at the outset prepare our conjugacy equation so that (z,w) are *normal coordinates* for these submanifolds, i.e. that $\theta(z,0,\tau) = \theta(0,\chi,\tau) = \tau$ and we assume that $\tilde\theta(z',0,\tau') = \tilde\theta(0,\chi',\tau') = \tau'$. In terms of the original "real" defining function this means $\varphi(z,0,s) = \tilde\varphi(z,0,s) = 0$ (and analogously for $\tilde\varphi$).

If our real real defining function, as assumed before, satisfies $\varphi(z,\bar z,s) = Q(z,\bar z) + \Phi(z,\bar z,s)$, we can write

$$\theta(z,\chi,\tau) = \tau + 2iQ(z,\chi) + S(z,\chi,\tau).$$

S can be further decomposed as

$$S(z,\chi,\tau) = \sum_{j,k=1}^{\infty} S_{j,k}(\tau) z^j\chi^k.$$

Here we think of $S_{j,k}$ as a power series in τ taking values in the space of multilinear maps on $(\mathbb{C}^n)^{j+k}$ which are symmetric in their first j and in their last k variables separately, taking values in $\mathbb{C}^d$ (i.e. polynomials in z and χ homogeneous of degree j in z and of degree k in χ), and for any such map L, write $Lz^j\chi^k$ for $L(\underbrace{z,\dots,z}_{j\text{times}},\underbrace{\chi,\dots,\chi}_{k\text{times}})$.

We note for future reference the following simple observations:

$$S_{1,\ell} = 2i\Phi_{1,\ell},\quad S_{\ell,1} = 2i\Phi_{\ell,1},\qquad \ell\geq 1,\quad S_{2,2} = 2i\left(\Phi_{2,2} + i\Phi'_{1,1}(Q+\Phi_{1,1})\right),$$
$$S_{2,3} = 2i\left(\Phi_{2,3} + i\Phi'_{1,2}(Q+\Phi_{1,1}) + i\Phi'_{1,1}\Phi_{1,2}\right),\qquad S_{3,2} = 2i\left(\Phi_{3,2} + i\Phi'_{2,1}(Q+\Phi_{1,1}) + i\Phi'_{1,1}\Phi_{2,1}\right). \tag{73}$$

and

$$
\begin{aligned}
\Phi_{2,2} &= \frac{1}{2i}S_{2,2} - \frac{1}{4i}S'_{1,1}(2iQ + S_{1,1}) \\
\Phi_{2,3} &= \frac{1}{2i}S_{2,3} - \frac{1}{4i}S'_{1,1}S_{1,2} - \frac{1}{4i}S'_{1,2}(2iQ + S_{1,1}), \\
\Phi_{3,2} &= \frac{1}{2i}S_{3,2} - \frac{1}{4i}S'_{1,1}S_{2,1} - \frac{1}{4i}S'_{2,1}(2iQ + S_{1,1}) \\
\Phi_{3,3} &= \frac{1}{2i}S_{3,3} - \frac{1}{4i}S'_{2,2}(2iQ + S_{1,1}) - \frac{1}{8i}S'_{1,1}(2S_{2,2} + S'_{1,1}(2iQ + S_{1,1}))+ \\
&\quad - \frac{1}{4i}S'_{1,2}S_{2,1} - \frac{1}{4i}S'_{2,1}S_{1,2} + \frac{1}{16i}S''_{1,1}(2iQ + S_{1,1})^2.
\end{aligned}
\tag{74}
$$

Furthermore, from the fact that $\theta(z, \chi, \bar{\theta}(\chi, z, w)) = w$, we obtain the following equations relating $S_{j,k}$ and their conjugates:

$$
S_{1,\ell}(w)+\bar{S}_{\ell,1}(w) = 0, \quad S_{2,2}-S'_{1,1}(2iQ-\bar{S}_{1,1})+\bar{S}_{2,2} = 0, \quad S_{2,3}-S'_{1,2}(2iQ-\bar{S}_{1,1})+S'_{1,1}\bar{S}_{2,1}+\bar{S}_{3,2} = 0 \tag{75}
$$

A map $H = (f, g)$ maps the manifold defined by $w = \theta(z, \bar{z}, \bar{w})$ into the one defined by $w' = \tilde{\theta}(z', \bar{z}', \bar{w}')$ if and only if the following equation is satisfied:

$$
g(z, \theta(z, \chi, \tau)) = \tilde{\theta}(f(z, \theta(z, \chi, \tau)), \bar{f}(\chi, \tau), \bar{g}(\chi, \tau)). \tag{76}
$$

An equivalent equation is (after application of (72))

$$
g(z, w) = \tilde{\theta}(f(z, w), \bar{f}(\chi, \bar{\theta}(\chi, z, w)), \bar{g}(\chi, \bar{\theta}(\chi, z, w))). \tag{77}
$$

If we set $\chi = 0$ in (77), the assumed normality of the coordinates, i.e. the equation $\theta(z, 0, w) = 0$, is equivalent $g(z, w) = \tilde{\theta}(f(z, w), \bar{f}(0, w), \bar{g}(0, w))$; in particular, for $w = \theta(z, \chi, \tau)$, we have the (also equivalent) condition

$$
g(z, \theta(z, \chi, \tau)) = \tilde{\theta}(f(z, \theta(z, \chi, \tau)), \bar{f}(0, \theta(z, \chi, \tau)), \bar{g}(0, \theta(z, \chi, \tau))). \tag{78}
$$

On the other hand setting $z = 0$, observing $\theta(0, \chi, \tau) = \tau$, and using (the conjugate of) (76) we also have

$$
\bar{g}(\chi, \tau) = \bar{\tilde{\theta}}(\bar{f}(\chi, \tau), f(0, \tau), g(0, \tau)) \tag{79}
$$

Combining this with (76), we obtain the following equivalent equation, which now guarantees the normality of (z, w):

$$
\begin{aligned}
&\tilde{\theta}(f(z, \theta(z, \chi, \tau)), \bar{f}(0, \theta(z, \chi, \tau)), \bar{g}(0, \theta(z, \chi, \tau))) \\
&\quad = \tilde{\theta}\left(f(z, \theta(z, \chi, \tau)), \bar{f}(\chi, \tau), \bar{\tilde{\theta}}(\bar{f}(\chi, \tau), f(0, \tau), g(0, \tau))\right).
\end{aligned}
\tag{80}
$$

Lastly, we can use one of the equations implicit in (80) to eliminate $\operatorname{Im} g$ from it. This is easiest done using (37), which (after extending to complex w) becomes

$$
(\operatorname{Im} g)(0, w) = \tilde{\varphi}(f(0, w), \bar{f}(0, w), (\operatorname{Re} g)(0, w)). \tag{81}
$$

Substituting this relation into (80) eliminates the dependence on $\operatorname{Im} g$ completely from the equation, only $\operatorname{Re} g$ appears now.

We now substitute $f = z + f_{\geq 2}(z,w)$, where f only contains terms of quasihomogeneity greater than 1, and write

$$f_{\geq 2}(z,w) = \sum_{k\geq 0} f_k(w) z^k, \quad g(0,w) = w + g_0(w);$$

we also write $\psi = \operatorname{Re} g_0$ for brevity. Let us first disentangle the equation (81). In our current notation, this reads

$$(\operatorname{Im} g_0)(w) = \tilde{\varphi}(f_0(w), \bar{f}_0(w), w + \psi(w)). \tag{82}$$

By virtue of the fact that $\tilde{\varphi}(z,0,s) = 0$, this exposes $\operatorname{Im} g_0$ as an nonlinear expression in f_0, $\bar{f}_0$, and ψ.

We can thus rewrite (80) as

$$\begin{aligned} &\tilde{\theta}\left(z + f_{\geq 2}, \bar{f}_0 \circ \theta, \theta + \psi \circ \theta + i\tilde{\varphi}(f_0 \circ \theta, \bar{f}_0 \circ \theta, \theta + \psi \circ \theta)\right) \\ &\quad = \tilde{\theta}\left(z + f_{\geq 2}, \chi + \bar{f}_{\geq 2}, \bar{\tilde{\theta}}(\chi + \bar{f}_{\geq 2}, f_0, \tau + \psi + i\tilde{\varphi}(f_0(w), \bar{f}_0(w), w + \psi(w)))\right), \end{aligned} \tag{83}$$

where we abbreviate $f_{\geq 2} = f_{\geq 2}(z, \theta(z,\chi,\tau))$ and $\bar{f}_{\geq 2} = \bar{f}_{\geq 2}(\chi,\tau)$.

We will now extract terms which are linear in the variables $f_{\geq 2}$, $\bar{f}_{\geq 2}$, and ψ from this equation. We rewrite:

$$\begin{aligned} &\tilde{\theta}\left(z + f_{\geq 2}, \bar{f}_0 \circ \theta, \theta + \psi \circ \theta + i\tilde{\varphi}(f_0 \circ \theta, \bar{f}_0 \circ \theta, \theta + \psi \circ \theta)\right) \\ &\quad = \tau + 2iQ(z,\chi) + S + \psi \circ \theta + 2iQ(z, \bar{f}_0 \circ \theta) + \dots \end{aligned}$$

$$\begin{aligned} &\tilde{\theta}\left(z + f_{\geq 2}, , \chi + \bar{f}_{\geq 2}, \bar{\tilde{\theta}}(\chi + \bar{f}_{\geq 2}, f_0, \tau + \psi + i\tilde{\varphi}(f_0(w), \bar{f}_0(w), w + \psi(w)))\right) \\ &\quad = \bar{\tilde{\theta}}(\chi + \bar{f}_{\geq 2}, f_0, \tau + \psi + i\tilde{\varphi}(f_0(w), \bar{f}_0(w), w + \psi(w))) + 2iQ(z + f_{\geq 2}, , \chi + \bar{f}_{\geq 2}) + \dots \\ &\quad = \tau + 2iQ(z,\chi)\psi - 2iQ(f_0,\chi) + 2iQ(z, \bar{f}_{\geq 2}) + 2iQ(f_{\geq 2}, \chi) + \dots, \end{aligned}$$

where we will elaborate on the terms which appear in the dots a bit below.

We can thus further express the conjugacy equation (83) in the following form:

$$\begin{aligned} &\psi \circ \theta - \psi + 2iQ(z, \bar{f}_0 \circ \theta) + 2iQ(f_0, \chi) - 2iQ(z, \bar{f}_{\geq 2}) - 2iQ(f_{\geq 2}, \chi) \\ &\quad = \tilde{\mathcal{T}}\left(z, \chi, \tau, f_0, \bar{f}_0, \psi, f_0 \circ \theta, \bar{f}_0 \circ \theta, \psi \circ \theta, f_{\geq 2}, \bar{f}_{\geq 2}\right) - S, \end{aligned} \tag{84}$$

where $\tilde{\mathcal{T}}$ has the property that in the further expansion to follow, it will only create "non-linear terms".

We now restrict (84) to the space of space of power series which are homogeneous of degree up to at most 3 in z and χ. By replacing the compositions $\psi \circ \theta$, $\bar{f}_0 \circ \theta$, and $f_j \circ \theta$, for $j \leq 3$, by their Taylor expansions, we get

$$\psi(\tau + 2iQ(z,\chi) + S(z,\chi,\tau)) = \sum_{k=0}^{3} \psi^{(k)}(\tau) \left(2iQ(z,\chi) + S(z,\chi,\tau)\right)^k, \qquad \operatorname{mod}\ (z)^4 + (\chi)^4$$

$$\bar{f}_0(\tau + 2iQ(z,\chi) + S(z,\chi,\tau)) = \sum_{k=0}^{3} \bar{f}_0^{(k)}(\tau) \left(2iQ(z,\chi) + S(z,\chi,\tau)\right)^k, \qquad \operatorname{mod}\ (z)^4 + (\chi)^4$$

$$f_j(\tau + 2iQ(z,\chi) + S(z,\chi,\tau)) = \sum_{k=0}^{3-j} \bar{f}_j^{(k)}(\tau) \left(2iQ(z,\chi) + S(z,\chi,\tau)\right)^k, \qquad \operatorname{mod}\ (z)^4 + (\chi)^4$$

the resulting equations, ordered by powers of (z,χ), writing $h=(f_0,\bar f_0,\psi)$, and saving space by setting $\varphi^{\leq j}=(\varphi,\varphi',\dots,\varphi^{(j)})$ and

$$S^{<p,<q}=(S_{k,\ell}\colon k<p,\ell\leq q \text{ or } k\leq p,\ell<q)\,,$$

become

$$\begin{array}{lr}
z\chi & -\psi'Q+Q(z,\bar f_1)+Q(f_1,\chi)=\dfrac{S_{1,1}}{2i}+\tilde{\mathcal{T}}_{1,1}\left(h^{\leq 1},f_1,\bar f_1\right)\\
z^2\chi & -2iQ(z,\bar f_0'Q)+Q(f_2,\chi)=\dfrac{S_{2,1}}{2i}+\tilde{\mathcal{T}}_{2,1}\left(h^{\leq 1},f_1^{\leq 1},\bar f_1,S_{1,1}\right)\\
z^3\chi & Q(f_3,\chi)=\dfrac{S_{3,1}}{2i}+\tilde{\mathcal{T}}_{3,1}\left(h^{\leq 1},f_1^{\leq 1},f_2,\bar f_1,S^{<3,<1}\right)\\
z\chi^2 & 2iQ(f_0'Q,\chi)+Q(z,\bar f_2)=\dfrac{S_{1,2}}{2i}+\tilde{\mathcal{T}}_{1,2}\left(h^{\leq 1},f_1,\bar f_1,S_{1,1}\right)\\
z\chi^3 & Q(z,\bar f_3)=\dfrac{S_{1,3}}{2i}+\tilde{\mathcal{T}}_{1,3}\left(h^{\leq 1},f_1,f_2,\bar f_1,S^{<1,<3}\right)\\
z^2\chi^2 & -i\psi''Q^2+2iQ(f_1'Q,\chi)=\dfrac{S_{2,2}}{2i}+\tilde{\mathcal{T}}_{2,2}\left(h^{\leq 2},f_1^{\leq 1},f_2,\bar f_1,\bar f_2,S^{<2,<2}\right)\\
z^2\chi^3 & -2Q(f_0''Q^2,\chi)=\dfrac{S_{2,3}}{2i}+\tilde{\mathcal{T}}_{2,3}\left(h^{\leq 2},f_1^{\leq 1},f_2,\bar f_1,\bar f_2,S^{<2,<3}\right)\\
z^3\chi^2 & 2iQ(f_2'Q,\chi)+2Q(z,\bar f_0''Q^2)=\dfrac{S_{3,2}}{2i}+\tilde{\mathcal{T}}_{3,2}\left(h^{\leq 2},f_1^{\leq 2},f_2,\bar f_1,\bar f_2,S^{<3,<2}\right)\\
z^3\chi^3 & \dfrac{2}{3}\psi'''Q^3-2Q(f_1''Q^2,\chi)=\dfrac{S_{3,3}}{2i}+\tilde{\mathcal{T}}_{3,3}\left(h^{\leq 3},f_1^{\leq 2},f_2^{\leq 1},\bar f_1,\bar f_2,S^{<3,<3}\right)
\end{array}$$

The "nonlinear terms" $\tilde{\mathcal{T}}_{(p,q)}$ have the property that the derivatives of highest order appearing in each line, if they appear in the nonlinear part, then their coefficient vanishes when evaluated at $\tau=0$. (One can go through very similar arguments as in Section 3 to convince oneself of that fact).

This system has the problem that the equations for the $z^2\chi$ and $z^3\chi$ involve f_1' and that the equation for $z^3\chi^2$ inolves f_1'', which effectively turns the full system of equations *singular*: In order to see that, consider the last two lines of the preceding system, brought to the same order of differentiation:

$$\begin{array}{lr}
z^3\chi^2 & 2iQ(f_2''Q^2,\chi)+2Q(z,\bar f_0'''Q^3)=\dfrac{S_{3,2}'Q}{2i}+\hat{\mathcal{T}}_{3,2}\left(h^{\leq 3},f_1^{\leq 3},f_2^{\leq 1},\bar f_1^{\leq 1},\bar f_2^{\leq}1,\hat S^{<3,<2}\right)\\
z^3\chi^3 & \dfrac{2}{3}\psi'''Q^3-2Q(f_1''Q^2,\chi)=\dfrac{S_{3,3}}{2i}+\tilde{\mathcal{T}}_{3,3}\left(h^{\leq 3},f_1^{\leq 2},f_2^{\leq 1},\bar f_1,\bar f_2,S^{<3,<3}\right)
\end{array}$$

and note that in the nonlinear terms, the order of differentiation of f_1 in the first line is 3 in the nonlinear part while it is 2 in the linear part on the second line. This behaviour has to be excluded.

However, we have improved the system from (34), since the equations for $z\chi^2$ and for $z^2\chi^3$ do not have this problem. We can thus use our crucial assumptions, namely that

$$\Phi_{1,2}'(Q+\Phi_{1,1})+\Phi_{1,1}'\Phi_{1,2}=0. \tag{85}$$

Under this assumption, (75) implies that $S_{1,2}=-\bar S_{2,1}$, $S_{1,3}=-\bar S_{3,1}$, $S_{3,2}=-\bar S_{2,3}$, and we can replace the equations for these terms with their conjugate equations, therefore eliminating the derivatives of too high order. Indeed, among the previous equations, consider each pair of equations of the form $L_{p,q} = \frac{S_{p,q}}{2i} + \tilde{\mathcal{T}}_{pq}$ and $(*) L_{q,p} = \frac{S_{q,p}}{2i} + \tilde{\mathcal{T}}_{qp}$. Assume that $\tilde{\mathcal{T}}_{qp}$ involves higher derivatives than $\tilde{\mathcal{T}}_{pq}$. Since $\bar{S}_{pq} = -S_{qp}$, we have

$$\tilde{\mathcal{T}}_{qp} = L_{q,p} - \frac{S_{q,p}}{2i} = L_{q,p} + \frac{\bar{S}_{p,q}}{2i} = L_{q,p} - \bar{L}_{p,q} + \overline{\tilde{\mathcal{T}}}_{pq}.$$

Hence, we can replace equation (*) by $\bar{L}_{p,q} = \frac{S_{q,p}}{2i} + \overline{\tilde{\mathcal{T}}}_{pq}$, lowering thereby the order of the differentials invloved. Therefore, we obtain a system of the form

$$\begin{array}{lrl}
z\chi & -\psi' Q + Q(z, \bar{f}_1) + Q(f_1, \chi) = & \frac{S_{1,1}}{2i} + \tilde{\mathcal{T}}_{1,1}\left(h^{\leq 1}, f_1, \bar{f}_1\right) \\
z^2\chi & -2iQ(z, \bar{f}_0' Q) + Q(f_2, \chi) = & \frac{S_{2,1}}{2i} + \overline{\tilde{\mathcal{T}}}_{1,2}\left(\bar{h}^{\leq 1}, \bar{f}_1, f_1, \bar{S}_{1,1}\right) \\
z^3\chi & Q(f_3, \chi) = & \frac{S_{3,1}}{2i} + \overline{\tilde{\mathcal{T}}}_{1,3}\left(\bar{h}^{\leq 1}, \bar{f}_1, \bar{f}_2, f_1, \bar{S}^{<1,<3}\right) \\
z\chi^2 & 2iQ(f_0' Q, \chi) + Q(z, \bar{f}_2) = & \frac{S_{1,2}}{2i} + \tilde{\mathcal{T}}_{1,2}\left(h^{\leq 1}, f_1, \bar{f}_1, S_{1,1}\right) \\
z\chi^3 & Q(z, \bar{f}_3) = & \frac{S_{1,3}}{2i} + \tilde{\mathcal{T}}_{1,3}\left(h^{\leq 1}, f_1, f_2, \bar{f}_1, S^{<1,<3}\right) \\
z^2\chi^2 & -i\psi'' Q^2 + 2iQ(f_1' Q, \chi) = & \frac{S_{2,2}}{2i} + \tilde{\mathcal{T}}_{2,2}\left(h^{\leq 2}, f_1^{\leq 1}, f_2, \bar{f}_1, \bar{f}_2, S^{<2,<2}\right) \\
z^2\chi^3 & -2Q(f_0'' Q^2, \chi) = & \frac{S_{2,3}}{2i} + \tilde{\mathcal{T}}_{2,3}\left(h^{\leq 2}, f_1^{\leq 1}, f_2, \bar{f}_1, \bar{f}_2, S^{<2,<3}\right) \\
z^3\chi^2 & -2Q(z, \bar{f}_0'' Q^2) = & \frac{S_{3,2}}{2i} + \overline{\tilde{\mathcal{T}}}_{3,2}\left(\bar{h}^{\leq 2}, \bar{f}_1^{\leq 1}, \bar{f}_2, f_1, f_2, \bar{S}^{<2,<3}\right) \\
z^3\chi^3 & \frac{2}{3}\psi''' Q^3 - 2Q(f_1'' Q^2, \chi) = & \frac{S_{3,3}}{2i} + \tilde{\mathcal{T}}_{3,3}\left(h^{\leq 3}, f_1^{\leq 2}, f_2^{\leq 1}, \bar{f}_1, \bar{f}_2, S^{<3,<3}\right)
\end{array}$$

The equations for the $(2,1)$, the $(3,1)$ and the $(3,2)$ term now depend nonlinearly on the conjugate $\bar{S}_{p,q}$, which we replace by their conjugates (i.e. the unbarred terms) using the rules (75). After that, we can use the implicit function theorem in order to eliminate the dependence of the $\tilde{\mathcal{T}}_{p,q}$ on the $S_{p,q}$, obtaining the equivalent system of equations

$$\begin{array}{lr}
z\chi & -\psi' Q + Q(z,\bar{f}_1) + Q(f_1,\chi) = \dfrac{S_{1,1}}{2i} + \mathcal{T}_{1,1}\left(h^{\leq 1}, f_1, \bar{f}_1\right) \\
z^2\chi & -2iQ(z,\bar{f}_0'Q) + Q(f_2,\chi) = \dfrac{S_{2,1}}{2i} + \mathcal{T}_{1,3}\left(h^{\leq 1}, \bar{f}_1, f_1\right) \\
z^3\chi & Q(f_3,\chi) = \dfrac{S_{3,1}}{2i} + \mathcal{T}_{3,1}\left(h^{\leq 1}, \bar{f}_1, \bar{f}_2, f_1\right) \\
z\chi^2 & 2iQ(f_0'Q,\chi) + Q(z,\bar{f}_2) = \dfrac{S_{1,2}}{2i} + \mathcal{T}_{1,2}\left(h^{\leq 1}, f_1, \bar{f}_1,\right) \\
z\chi^3 & Q(z,\bar{f}_3) = \dfrac{S_{1,3}}{2i} + \mathcal{T}_{1,3}\left(h^{\leq 1}, f_1, f_2, \bar{f}_1\right) \\
z^2\chi^2 & -i\psi'' Q^2 + 2iQ(f_1'Q,\chi) = \dfrac{S_{2,2}}{2i} + \mathcal{T}_{2,2}\left(h^{\leq 2}, f_1^{\leq 1}, f_2, \bar{f}_1, \bar{f}_2\right) \\
z^2\chi^3 & -2Q(f_0''Q^2,\chi) = \dfrac{S_{2,3}}{2i} + \mathcal{T}_{2,3}\left(h^{\leq 1}, f_1^{\leq 1}, f_2, \bar{f}_1, \bar{f}_2\right) \\
z^3\chi^2 & -2Q(z,\bar{f}_0''Q^2) = \dfrac{S_{3,2}}{2i} + \mathcal{T}_{2,3}\left(h^{\leq 2}, f_1^{\leq 1}, f_2, \bar{f}_1, \bar{f}_2\right) \\
z^3\chi^3 & \dfrac{2}{3}\psi''' Q^3 - 2Q(f_1''Q^2,\chi) = \dfrac{S_{3,3}}{2i} + \mathcal{T}_{3,3}\left(h^{\leq 3}, f_1^{\leq 2}, f_2^{\leq 1}, \bar{f}_1, \bar{f}_2\right)
\end{array}$$

We use this system and substitute it (and its appropriate derivatives) into (74) in order to obtain equations for the $\Phi_{p,q}$, leading to

$$\begin{array}{lr}
z\chi & -\psi' Q + Q(z,\bar{f}_1) + Q(f_1,\chi) = \Phi_{1,1} + \mathcal{T}_{1,1}\left(h^{\leq 1}, f_1, \bar{f}_1\right) \\
z^2\chi & -2iQ(z,\bar{f}_0'Q) + Q(f_2,\chi) = \Phi_{2,1} + \overline{\mathcal{T}}_{1,2}\left(\bar{h}^{\leq 1}, \bar{f}_1, f_1\right) \\
z^3\chi & Q(f_3,\chi) = \Phi_{3,1} + \overline{\mathcal{T}}_{1,3}\left(\bar{h}^{\leq 1}, \bar{f}_1, \bar{f}_2, f_1\right) \\
z\chi^2 & 2iQ(f_0'Q,\chi) + Q(z,\bar{f}_2) = \Phi_{1,2} + \mathcal{T}_{1,2}\left(h^{\leq 1}, f_1, \bar{f}_1,\right) \\
z\chi^3 & Q(z,\bar{f}_3) = \Phi_{1,3} + \mathcal{T}_{1,3}\left(h^{\leq 1}, f_1, f_2, \bar{f}_1\right) \\
z^2\chi^2 & i\left(Q(f_1'Q,\chi) - Q(z,\bar{f}_1'Q)\right) = \Phi_{2,2} + \tilde{\mathcal{S}}_{2,2}\left(h^{\leq 2}, f_1^{\leq 1}, \bar{f}_1^{\leq 1}, f_2, \bar{f}_2\right) \\
z^2\chi^3 & -iQ(z,\bar{f}_2'Q) = \Phi_{2,3} + \tilde{\mathcal{S}}_{2,3}\left(h^{\leq 2}, f_1^{\leq 1}, f_2, \bar{f}_1, \bar{f}_2\right) \\
z^3\chi^2 & iQ(f_2'Q,\chi) = \Phi_{3,2} + \overline{\tilde{\mathcal{S}}}_{3,2}\left(\bar{h}^{\leq 2}, \bar{f}_1^{\leq 1}, \bar{f}_2, f_1, f_2\right) \\
z^3\chi^3 & \dfrac{1}{6}\psi''' Q^3 - \dfrac{1}{2}\left(Q(f_1''Q^2,\chi) + Q(z,\bar{f}_1''Q^2)\right) = \Phi_{3,3} + \tilde{\mathcal{S}}_{3,3}\left(h^{\leq 3}, f_1^{\leq 2}, f_2^{\leq 1}, \bar{f}_1^{\leq 2}, \bar{f}_2^{\leq 1}\right)
\end{array} \tag{86}$$

This system is now "well graded" so that we can expose it as a system of PDEs which allows for the application of the big denominator theorem. However, we first single out the equations for $z^2\chi$, $z^3\chi$, $z\chi^2$, $z\chi^3$:

$$\begin{array}{lr}
z^2\chi & -2iQ(z,\bar{f}_0'Q) + Q(f_2,\chi) = \Phi_{2,1} + \overline{\mathcal{T}}_{1,2}\left(\bar{h}^{\leq 1}, \bar{f}_1, f_1\right) \\
z^3\chi & Q(f_3,\chi) = \Phi_{3,1} + \overline{\mathcal{T}}_{1,3}\left(\bar{h}^{\leq 1}, \bar{f}_1, \bar{f}_2, f_1\right) \\
z\chi^2 & 2iQ(f_0'Q,\chi) + Q(z,\bar{f}_2) = \Phi_{1,2} + \mathcal{T}_{1,2}\left(h^{\leq 1}, f_1, \bar{f}_1,\right) \\
z\chi^3 & Q(z,\bar{f}_3) = \Phi_{1,3} + \mathcal{T}_{1,3}\left(h^{\leq 1}, f_1, f_2, \bar{f}_1\right)
\end{array} \tag{87}$$

Applying the adjoint operator $\mathcal{K}^*$ to the system (87) and using the normalization conditions (11) for the $(1,p)$ and $(p,1)$-terms for $p = 2,3$ transforms them into a system of implicit equations for f_2 and f_3 in term of $h^{\leq 1}$, f_1 and their conjugates:

$$\begin{aligned} &z^2\chi \qquad && \mathcal{K}^*\mathcal{K} f_2 = \mathcal{K}^*(2iQ(z,\bar{f}_0'Q) + \overline{\mathcal{T}}_{1,2}) \\ &z^3\chi && \mathcal{K}^*\mathcal{K} f_3 = \mathcal{K}^*\overline{\mathcal{T}}_{1,3} \end{aligned} \tag{88}$$

By the fact that $\mathcal{K}^*\mathcal{K}$ is invertible (on the image of $\mathcal{K}^*$, where the right hand side lies), we can solve this equation for f_2 and f_3 and substitute the result into the "remaining" equations to obtain the following system:

$$\begin{aligned} &z\chi && -\psi' Q + Q(z,\bar{f}_1) + Q(f_1,\chi) = \Phi_{1,1} + \mathcal{T}_{1,1}\left(h^{\leq 1}, f_1, \bar{f}_1\right) \\ &z^2\chi^2 && i\left(Q(f_1'Q,\chi) - Q(z,\bar{f}_1'Q)\right) = \Phi_{2,2} + \mathcal{S}_{2,2}\left(h^{\leq 2}, f_1^{\leq 1}, \bar{f}_1^{\leq 1}, f_2, \bar{f}_2\right) \\ &z^3\chi^3 && \frac{1}{6}\psi''' Q^3 - \frac{1}{2}\left(Q(f_1''Q^2,\chi) + Q(z,\bar{f}_1''Q^2)\right) = \Phi_{3,3} + \mathcal{S}_{3,3}\left(h^{\leq 3}, f_1^{\leq 2}, f_2^{\leq 1}, \bar{f}_1^{\leq 2}, \bar{f}_2^{\leq 1}\right) \\ &z^3\chi^2 && -2Q(z,\bar{f}_0''Q^2) = \Phi_{3,2} - i\Phi_{2,1}'Q + \overline{\mathcal{S}}_{3,2}\left(\bar{h}^{\leq 2}, \bar{f}_1^{\leq 1}, \bar{f}_2, f_1, f_2\right) \end{aligned} \tag{89}$$

While coupled in the nonlinear parts, the linear parts of the equations corresponding to the diagonal terms of type $(1,1)$, $(2,2)$, and $(3,3)$ on the one hand and of the off-diagonal terms of type $(3,2)$ (we drop from now on the conjugate term $(2,3)$) on the other hand are *decoupled*, the diagonal terms only depending on f_1 and ψ, the off-diagonal terms on f_0 and their derivatives.

We thus obtain the linear operator $\mathcal{L}$ already introduced in Section 5, if we rewrite everything in terms of our operators Δ, $\mathcal{K}$ and $\bar{\mathcal{K}}$ (see section 2.4),

$$\begin{aligned} &z\chi && -\Delta\psi + \bar{\mathcal{K}}\bar{f}_1 + \mathcal{K} f_1 = \Phi_{1,1} + \mathcal{T}_{1,1} \\ &z^2\chi^2 && i\left(\mathcal{K}\Delta f_1 - \bar{\mathcal{K}}\Delta\bar{f}_1\right) = \Phi_{2,2} + \mathcal{S}_{2,2} \\ &z^3\chi^3 && \frac{1}{6}\Delta^3\psi - \frac{1}{2}\left(\mathcal{K}\Delta^2 f_1 + \bar{\mathcal{K}}\Delta^2\bar{f}_1\right) = \Phi_{3,3} + \mathcal{S}_{3,3} \end{aligned} \tag{90}$$

The equation determining f_0 can be rewritten as

$$-2\bar{\mathcal{K}}\Delta^2\bar{f}_0 = \Phi_{3,2} - i\Delta\Phi_{2,1} + \overline{\mathcal{S}}_{3,2} \tag{91}$$

Let us stress that even though the linear terms here are the same as in Section 5, the nonlinear terms are not the same as we had in that section, and an elimination of the derivatives of "bad order" like we did here is only possible under some restriction.

However, with this in mind, we can completely proceed as in the proof of Theorem 2: we first project the equations on the normal form space $\mathcal{N}^{\mathrm{off}} \times \mathcal{N}^d$, and obtain an equation of the form

$$\begin{aligned} -2\bar{\mathcal{K}}\Delta^2\bar{f}_0 &= \pi_0\overline{\mathcal{S}}_{3,2} \\ -\Delta\psi + \bar{\mathcal{K}}\bar{f}_1 + \mathcal{K} f_1 &= \pi_1\mathcal{T}_{1,1} \\ i\left(\mathcal{K} f_1 - \bar{\mathcal{K}}\bar{f}_1\right) &= \pi_2\mathcal{S}_{2,2} \\ \frac{1}{6}\Delta^3\psi - \frac{1}{2}\left(\mathcal{K}\Delta^2 f_1 + \bar{\mathcal{K}}\Delta^2\bar{f}_1\right) &= \pi_3\mathcal{S}_{3,3} \end{aligned} \tag{92}$$

We now "homogenize" the degree of differentials of these equations again, obtaining a system of the form

$$\begin{aligned}
-2\bar{\mathcal{K}}\Delta^3 \bar{f}_0 &= \mathcal{F}_{3,2}\left(h^{\le 3}, f_1^{\le 2}, \bar{f}_1^{\le 2}\right)\\
-\Delta^3\psi + \bar{\mathcal{K}}\Delta^2 \bar{f}_1 + \mathcal{K}\Delta^2 f_1 &= \mathcal{F}_{1,1}\left(h^{\le 3}, f_1^{\le 2}, \bar{f}_1^{\le 2}\right)\\
i\left(\mathcal{K}\Delta^2 f_1 - \bar{\mathcal{K}}\Delta^2 \bar{f}_1\right) &= \mathcal{F}_{2,2}\left(h^{\le 3}, f_1^{\le 2}, \bar{f}_1^{\le 2}\right)\\
\frac{1}{6}\Delta^3\psi - \frac{1}{2}\left(\mathcal{K}\Delta^2 f_1 + \bar{\mathcal{K}}\Delta^2 \bar{f}_1\right) &= \pi_3\mathcal{S}_{3,3}\left(h^{\le 3}, f_1^{\le 2}, \bar{f}_1^{\le 2}\right)
\end{aligned} \tag{93}$$

Next, we substitute f_0, $\operatorname{Re} g_0$, and f_1 with $\tilde{f}_0 = f_0 - j^3 f_0$, $\operatorname{Re}\tilde{\psi} = \psi - j^3\psi$, and $\tilde{f}_1 = f_1 - j^2 f_1$ and obtain

$$\begin{aligned}
-2\bar{\mathcal{K}}\Delta^3 \bar{\tilde{f}}_0 &= \tilde{\mathcal{F}}_{3,2}\left(\tilde{h}^{\le 3}, \tilde{f}_1^{\le 2}, \bar{\tilde{f}}_1^{\le 2}\right)\\
-\Delta^3\tilde{\psi} + \bar{\mathcal{K}}\Delta^2 \bar{\tilde{f}}_1 + \mathcal{K}\Delta^2 \tilde{f}_1 &= \tilde{\mathcal{F}}_{1,1}\left(\tilde{h}^{\le 3}, \tilde{f}_1^{\le 2}, \bar{\tilde{f}}_1^{\le 2}\right)\\
i\left(\mathcal{K}\Delta^2 \tilde{f}_1 - \bar{\mathcal{K}}\Delta^2 \bar{\tilde{f}}_1\right) &= \tilde{\mathcal{F}}_{2,2}\left(\tilde{h}^{\le 3}, \tilde{f}_1^{\le 2}, \bar{\tilde{f}}_1^{\le 2}\right)\\
\frac{1}{6}\Delta^3\psi - \frac{1}{2}\left(\mathcal{K}\Delta^2 \tilde{f}_1 + \bar{\mathcal{K}}\Delta^2 \bar{\tilde{f}}_1\right) &= \tilde{\mathcal{F}}_{3,3}\left(\tilde{h}^{\le 3}, \tilde{f}_1^{\le 2}, \bar{\tilde{f}}_1^{\le 2}\right).
\end{aligned} \tag{94}$$

We can now apply the Big Denominator theorem 14 to this system, just as we did in the proof of Theorem 2. The setup is the same, with $\operatorname{Re}(\tilde{g}_0)$ now replaced by (ψ, f_0), and the details are completely analogous to the details carried out in the proof of Theorem 2 and therefore left to the reader.

8 On the Chern-Moser normal form

As we have already pointed out above, our normal form necessarily cannot agree with the normal form of Chern-Moser in the case $d = 1$ (which we assume from now on). The reason is that we do not have a choice of which normal form space to use for the diagonal terms-the operator associated to all diagonal terms is injective, and we need to use its full adjoint. In the Chern-Moser case, the equation for the $(1,1)$-term, (with our notations from above)

$$\Phi_{1,1} = \Delta\psi - \mathcal{K} f_1 - \bar{\mathcal{K}}\bar{f}_1 + \dots,$$

is rather special, because *the operator* $f_1 \mapsto \operatorname{Re}\mathcal{K} f_1$ *is surjective.* (One can check that the weaker condition $\operatorname{image}\Delta \subset \operatorname{image}\operatorname{Re}\mathcal{K}$ happens if and only if $d = 1$).

This means that if we look at the normal form condition for the $(1,p)$-terms, which just becomes $\Phi_{1,p} = 0$ (because $\mathcal{K}$ is surjective, $\mathcal{K}^*$ is injective, and hence $\Phi_{1,p} = 0$ if and only if $\mathcal{K}^*\Phi_{1,p} = 0$), we can naturally also use it for the $(1,1)$-term and just request that $\Phi_{1,1} = 0$. A tricky point is that even though $\operatorname{Re}\mathcal{K}$ is surjective (as a map on $\mathcal{H}_1[\![u]\!]$)), it is not injective. By considering the polar decomposition $z + f_1(z,u) = U(u)(I + R(u))z$ with U unitary with respect to Q, i.e. $Q(U(u)z, \overline{U}(u)\bar{z}) = Q(z,\bar{z})$, the equation for the $(1,1)$-term becomes *an implicit equation for* R in terms of all the other variables, because

$$\begin{aligned}
Q(z + f_1(z,u), \bar{z} + \bar{f}_1(z,u)) &= Q(U(u)(I + R(u))z, \overline{U}(u)(I + R(u))\bar{z})\\
&= Q(z,\bar{z}) + 2\operatorname{Re} Q(R(u)z,\bar{z}) + Q(R(u)z, R(u)\bar{z}).
\end{aligned}$$

. We can then use the implicit function theorem to solve the $(1,1)$, $(2,1)$, and $(3,1)$-equations under the requirement $\Phi_{1,1} = \Phi_{2,1} = \Phi_{3,1} = 0$ jointly for R, f_2, and f_3 in terms of U and $\operatorname{Re} g_0$ and substitute the result back in all the other equations as we did before. If we follow this procedure and go through with the rest of the arguments following (87) with the appropriate changes, we obtain the Chern-Moser normal form; one just has to note that $u\mathrm{tr}\varphi = \Delta^*\varphi$.

A Computations

We recall that $\Phi_{p,0} = \Phi_{0,q} = 0$. Therefore, $(Q+\Phi)^l$ contains no terms (p,q) with $p < l$ or $q < l$. As a consequence, we have

$$
\begin{aligned}
(23)_{p,0} &= 0 && (95)\\
(23)_{p,1} &= iD_u g_{p-2}(u)\Phi_{2,1} + iD_u g_{p-1}(u)\Phi_{1,1} && (96)\\
(23)_{2,2} &= iD_u g_0(u)\Phi_{2,2} + iD_u g_1(u)\Phi_{1,2}\\
&\quad + \frac{1}{2}D_u^2 g_0(u)(2\Phi_{1,1}Q + \Phi_{1,1}^2) && (97)\\
(23)_{3,3} &= iD_u g_0(u)\Phi_{3,3} + iD_u g_1(u)\Phi_{2,3} + iD_u g_2(u)\Phi_{1,3}\\
&\quad + \frac{1}{2}D_u^2 g_0(u)(2\Phi_{2,2}Q + \{\Phi^2\}_{3,3}) + \frac{1}{2}D_u^2 g_1(u)(2\Phi_{1,2}Q + \{\Phi^2\}_{2,3})\\
&\quad - \frac{i}{6}D_u^3 g_0(u)(3\Phi_{1,1}^2 Q + \Phi_{1,1}^3 + 3\Phi_{1,1}Q^2) && (98)\\
(23)_{3,2} &= iD_u g_0(u)\Phi_{3,2} + iD_u g_1(u)\Phi_{2,2} + iD_u g_2(u)\Phi_{1,2}\\
&\quad + \frac{1}{2}D_u^2 g_0(u)(2\Phi_{2,1}Q + \{\Phi^2\}_{3,2}) + \frac{1}{2}D_u^2 g_1(u)(2\Phi_{1,1}Q + \{\Phi^2\}_{2,2}) && (99)\\
(23)_{3,1} &= iD_u g_0(u)\Phi_{3,1} + iD_u g_1(u)\Phi_{2,1} + iD_u g_2(u)\Phi_{1,1} && (100)
\end{aligned}
$$

To obtain $\bar{g}_{\geq 3}(z, u - iQ) - \bar{g}_{\geq 3}(z, u - iQ - i\Phi)$, we just use the previous result and substitute g_k in $\bar{g}_k$ and i by $-i$. We have, using essentially the same computations :

$$
\begin{aligned}
(24)_{p,1} &= (24)_{p,0} = 0 && (101)\\
(24)_{2,2} &= Q(iD_u f_0(u)\Phi_{2,1} + iD_u f_1(u)\Phi_{1,1}, \bar{C}\bar{z}) && (102)\\
(24)_{3,3} &= Q(iD_u f_0(u)\Phi_{3,2} + iD_u f_1(u)\Phi_{2,2} + iD_u f_2(u)\Phi_{1,2}, \bar{C}\bar{z})\\
&\quad + \frac{1}{2}Q(D_u^2 f_0(u)(2\Phi_{2,1}Q + \{\Phi^2\}_{3,2}) + \frac{1}{2}D_u^2 f_1(u)(2\Phi_{1,1}Q + \{\Phi^2\}_{2,2}), \bar{C}\bar{z}) && (103)\\
(24)_{3,2} &= Q(iD_u f_0(u)\Phi_{3,1} + iD_u f_1(u)\Phi_{2,1} + iD_u f_2(u)\Phi_{1,1}, \bar{C}\bar{z}) && (104)
\end{aligned}
$$

We have

$$
Q\left(f_{\geq 2}, \bar{f}_{\geq 2}\right) = \sum_{k,l\geq 0} \frac{i^{k+l}(-1)^l}{k!l!} Q\left(D_u^k f_{\geq 2}(z,u)(Q+\Phi)^k, D_u^l \bar{f}_{\geq 2}(\bar{z},u)(Q+\Phi)^l\right).
$$

The function $D_u^k f_{j'}(z,u)(Q+\Phi)^k$ (resp. $D_u^l \bar{f}_j(\bar{z},u)(Q+\Phi)^l$) has only terms (p,q) with $p \geq j' + k$ and $q \geq k$ (rep. $p \geq l$ and $q \geq l + j$). Hence, the function $Q(D_u^k f_{j'}(z,u)(Q +$

$\Phi)^k, D_u^l \bar{f}_j(z,u)(Q+\Phi)^l)$ contains only terms (p,q) with $p \geq j' + k + l$ and $q \geq j + k + l$. we have

$$\begin{aligned}
Q\left(f_{\geq 2}, \bar{f}_{\geq 2}\right)_{p,0} &= Q(f_p, \bar{f}_0) && (105)\\
Q\left(f_{\geq 2}, \bar{f}_{\geq 2}\right)_{p,1} &= Q(f_p, \bar{f}_1) + iQ(Df_{p-1}(Q+\Phi_{1,1}), \bar{f}_0) - iQ(f_{p-1}, D_u\bar{f}_0(Q+\Phi_{1,1})) && (106)\\
Q\left(f_{\geq 2}, \bar{f}_{\geq 2}\right)_{2,2} &= Q(f_2, \bar{f}_2) + iQ(Df_1(Q+\Phi_{1,1}), \bar{f}_1) - iQ(f_1, D\bar{f}_1(Q+\Phi_{1,1}))\\
&\quad -\frac{1}{2}\left(Q(f_0, D_u^2\bar{f}_0(Q+\Phi_{1,1})^2) + Q(D_u^2 f_0(Q+\Phi_{1,1})^2, \bar{f}_0)\right)\\
&\quad -Q\left(D_u f_0(u)(Q+\Phi_{1,1}), D_u\bar{f}_0(u)(Q+\Phi_{1,1})\right) && (107)\\
Q\left(f_{\geq 2}, \bar{f}_{\geq 2}\right)_{3,3} &= Q(f_3, \bar{f}_3) + iQ(Df_0\Phi_{3,1} + Df_1\Phi_{2,1} + Df_2(Q+\Phi_{1,1}), \bar{f}_2)\\
&\quad -iQ(f_2, D_u\bar{f}_0\Phi_{1,3} + D_u\bar{f}_1\Phi_{1,2} + D_u\bar{f}_2(Q+\Phi_{1,1}))\\
&\quad +Q(i(D_u f_0\Phi_{3,2} + D_u f_2(Q+\Phi_{1,1}) - \frac{1}{2}(D_u^2 f_0(Q+\Phi_{1,1})\Phi_{2,1} + D_u^2 f_1(Q+\Phi_{1,1})^2)), \bar{f}_1)\\
&\quad +Q(f_1, -i(D_u\bar{f}_0\Phi_{2,3} + D_u\bar{f}_2(Q+\Phi_{1,1}) - \frac{1}{2}(D_u^2\bar{f}_0(Q+\Phi_{1,1})\Phi_{1,2} + D^2\bar{f}_1(Q+\Phi_{1,1})^2)))\\
&\quad Q(\frac{-i}{3}D_u^3 f_0(Q+\Phi_{1,1})^3 + \frac{-1}{2}(D_u^2 f_0(Q+\Phi_{1,1})\Phi_{2,2} + D_u^2 f_1(Q,\Phi_{1,1})\Phi_{1,2}), \bar{f}_0)\\
&\quad Q(f_0, \frac{i}{3}D_u^3\bar{f}_0(Q+\Phi_{1,1})^3 + \frac{-1}{2}(D_u^2\bar{f}_0(Q+\Phi_{1,1})\Phi_{2,2} + D_u^2\bar{f}_1(Q,\Phi_{1,1})\Phi_{2,1}))\\
&\quad +Q(-i(D_u f_0\Phi_{3,3} + D_u f_1\Phi_{2,3} + D_u f_2\Phi_{2,3}), \bar{f}_0)\\
&\quad +Q(f_0, i(D_u\bar{f}_0\Phi_{3,3} + D_u\bar{f}_1\Phi_{3,2} + D_u\bar{f}_2\Phi_{3,2}))\\
&\quad \frac{-i}{2}Q\left(D_u f_0(u)(Q+\Phi_{1,1}), D_u^2\bar{f}_0(u)(Q+\Phi_{1,1})^2\right)\\
&\quad +\frac{i}{2}Q\left(D_u^2 f_0(u)(Q+\Phi_{1,1})^2, D_u\bar{f}_0(u)(Q+\Phi_{1,1})\right)\\
&\quad +Q(iD_u f_1(z,u)(Q+\Phi_{1,1}), D_u\bar{f}_1(\bar{z},u)(Q+\Phi_{1,1})) && (108)\\
Q\left(f_{\geq 2}, \bar{f}_{\geq 2}\right)_{3,2} &= Q(f_3, \bar{f}_2) - iQ(f_2, D_u\bar{f}_1(Q+\Phi_{1,1}) + D_u\bar{f}_0\Phi_{1,2})\\
&\quad -iQ(f_1, D_u\bar{f}_0(Q+\Phi_{1,1})^2 + D_u\bar{f}_1\Phi_{2,1}) - iQ(f_0, D_u\bar{f}_1\Phi_{3,1} + D_u\bar{f}_0\Phi_{3,2})\\
&\quad +Q\left(D_u f_1(z,u)(Q+\Phi_{1,1}), D_u\bar{f}_0(u)(Q+\Phi_{1,1})\right) - \frac{1}{2}Q(D_u^2 f_1(Q+\Phi_{1,1})^2, \bar{f}_0) && (109)
\end{aligned}$$

We have

$$\tilde{\Phi}_{\geq 3}\left(f, \bar{f}, \frac{1}{2}(g+\bar{g})\right) - \tilde{\Phi}_{\geq 3}\left(Cz, \bar{C}\bar{z}, su\right) = \sum_{\substack{|\alpha|+|\beta|+|\gamma|=k\\ k\geq 1}} \frac{1}{\alpha!\beta!\gamma!}\frac{\partial^k\tilde{\Phi}_{\geq 3}}{\partial z^\alpha\bar{z}^\beta u^\gamma}(Cz, \bar{C}\bar{z}, su) f_{\geq 2}^\alpha \bar{f}_{\geq 2}^\beta \left(\frac{1}{2}(g_{\geq 3} + \bar{g}_{\geq 3})\right)^\gamma \tag{110}$$

where $\alpha, \beta \in \mathbb{N}^n$ and $\gamma \in \mathbb{N}^d$. Hence, the (p,q) term of $\tilde{\Phi}_{\geq 3}\left(f, \bar{f}, \frac{1}{2}(g+\bar{g})\right) - \tilde{\Phi}_{\geq 3}\left(Cz, \bar{C}\bar{z}, su\right)$ is a sum of terms of the form

$$\left\{\frac{\partial^k\tilde{\Phi}_{\geq 3}}{\partial z^\alpha\partial\bar{z}^\beta\partial u^\gamma}(Cz, \bar{C}\bar{z}, su)\right\}_{p_1,q_1} \{f_{\geq 2}^\alpha\}_{p_2,q_2}\{\bar{f}_{\geq 2}^\beta\}_{p_3,q_3}\left\{\left(\frac{1}{2}(g_{\geq 3} + \bar{g}_{\geq 3})\right)^\gamma\right\}_{p_4,q_4} \tag{111}$$

with $\sum_{i=1}^4 p_i = p, \sum_{i=1}^4 q_i = q$.

Let us first compute $\{f^{\alpha}_{\geq 2}\}_{p_2,q_2}$ with $p_2, q_2 \leq 3$. In the following computations, f, g are considered as vector valued functions except when computing $f^{\alpha}, (g+\bar{g})^{\gamma}$ where f, g are considered as scalar functions and α, γ as an integers.

In the sums below, the terms appear with some positive multiplicity that we do not write since we are only interested in a lower bound of vanishing order of the terms. Fron these computations, we easily obtain $\{\bar{f}^{\alpha}_{\geq 2}\}_{p_2,q_2}$ in the following way : replace f_k by $\bar{f}_k$ in formula defining $\{f^{\alpha}\}_{p,q}$ in order to obtain $\{\bar{f}^{\alpha}\}_{q,p}$. Furthermore, we have

$$\left\{\frac{\partial^k \tilde{\Phi}_{\geq 3}}{\partial z^{\alpha}\partial \bar{z}^{\beta}\partial u^{\gamma}}\right\}_{p_1,q_1} = \frac{\partial^k \tilde{\Phi}_{p_1+|\alpha|,q_1+|\beta|}}{\partial z^{\alpha}\bar{z}^{\beta}u^{\gamma}}$$

Let us set as notation

$$Re(g) := \frac{g+\bar{g}}{2} = \frac{g(z,u+i(Q(z,\bar{z})+\Phi(z,\bar{z},u))) + \bar{g}(\bar{z},u-i(Q(z,\bar{z})+\Phi(z,\bar{z},u))}{2}.$$

B Big denominators theorem for non-linear systems of PDEs

In this section we recall one of the main results of article [Sto16] about local analytic solvability of some non-linear systems of PDEs that have the “big denominators property”.

B.1 The problem

Let $r \in \mathbb{N}^*$ and let $\mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{N}^r$ be a fixed multiindex. Let us denote $\mathbb{A}^k_n$ (resp. $\left(\mathbb{A}^k_n\right)_{>d}$, $\widehat{\mathbb{A}^k_n}$, $\left(\mathbb{A}^k_n\right)^{(i)}$) the space of k-tuples of germs at $0 \in \mathbb{R}^n$ (or $\mathbb{C}^n$) of analytic functions (resp. vanishing at order d at the origin, formal power series maps, homogeneous polynomials of degree i) of n variables. Let us set

$$\mathcal{F}^{\geq 0}_{r,\mathbf{m}} := (\mathbb{A}_n)_{\geq m_1} \times (\mathbb{A}_n)_{\geq m_2} \times \cdots \times (\mathbb{A}_n)_{\geq m_r}$$

Given $F = (F_1, ..., F_r) \in \mathcal{F}^{\geq 0}_{r,\mathbf{m}}$ and $x \in (\mathbb{R}^n, 0)$, let us denote

$$j^{\mathbf{m}}_x F := (j^{m_1}_x F_1, \ \cdots \ , \ j^{m_r}_x F_r), \quad J^{\mathbf{m}}\mathcal{F}^{\geq 0}_{r,\mathbf{m}} := \left\{(x, \ j^{\mathbf{m}}_x F), \quad x \in (\mathbb{R}^n, 0), \ F \in \mathcal{F}^{\geq 0}_{r,\mathbf{m}}\right\}.$$

Definition 9. *A map $\mathcal{T} : \mathcal{F}^{\geq 0}_{r,\mathbf{m}} \to \mathbb{A}^s_n$ is a* **differential analytic map of order m** *at the point $0 \in \mathbb{A}^k_n$ if there exists an analytic map germ*

$$W : \left(J^{\mathbf{m}}\mathcal{F}^{\geq 0}_{r,\mathbf{m}}, 0\right) \to \mathbb{R}^s$$

such that $\mathcal{T}(F)(x) = W(x, j^{\mathbf{m}}_x F)$ for any $x \in \mathbb{R}^n$ close to 0 and any function germ $F \in \mathcal{F}^{\geq 0}_{r,\mathbf{m}}$ such that $j^m_0 F$ is close to 0.

Denote by

$$v = (x_1, ..., x_n, u_{j,\alpha}), \quad 1 \leq j \leq r, \ \alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n, \ |\alpha| \leq m_j$$

the local coordinates in $J^{\mathbf{m}}\mathbb{A}^r_n$, where $u_{j,\alpha}$ corresponds to the partial derivative $\partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$ of the j-th component of a vector function $F \in \mathbb{A}^r_n$. As usual, we have set $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

Definition 10. *Let q be a nonnegative integer. Let $\mathcal{T} : \mathcal{F}^{\geq 0}_{r,\mathbf{m}} \to \mathbb{A}^s_n$ be a map.*

- *We shall say that it* **increases the order at the origin** *(resp. strictly) by q if for all $(F, G) \in (\mathcal{F}^{\geq 0}_{r,\mathbf{m}})^2$ then*

$$ord_0 \left(\mathcal{T}(F) - \mathcal{T}(G)\right) \geq ord_0(F - G) + q,$$

(resp. $>$ instead of $\geq$).

- *Assume that $\mathcal{T}$ is an analytic differential map of order $\mathbf{m}$ defined by a map germ $W : \left(J^{\mathbf{m}}\mathcal{F}^{\geq 0}_{r,\mathbf{m}}, 0\right) \to \mathbb{R}^s$ as in Definition 9. We shall say that it is* **regular** *if, for any formal map $F = (F_1, \dots, F_r) \in \widehat{\mathcal{F}}^{\geq 0}_{r,\mathbf{m}}$, then*

$$ord_0 \left(\frac{\partial W_i}{\partial u_{j,\alpha}}(x, \partial F)\right) \geq p_{j,|\alpha|},$$

where

$$p_{j,|\alpha|} = \max(0, |\alpha| + q + 1 - m_j) \tag{112}$$

We have set $\partial F := \left(\frac{\partial^{|\alpha|} F_i}{\partial x^\alpha},\ 1 \leq i \leq r,\ 0 \leq |\alpha| \leq m_i\right)$.

Let us consider linear maps :

1. $$\mathcal{S} : \mathcal{F}^{\geq 0}_{r,\mathbf{m}} \ \to \ \mathbb{A}^s_n,$$

   that increases the order by q and is homogenous, i.e $\mathcal{S}\left(\mathcal{F}^{(i)}_{r,\mathbf{m}}\right) \subset (\mathbb{A}^s_n)^{(q+i)}$.

2. $$\pi : \mathbb{A}^s_n \to Image\,(\mathcal{S}) \subset \mathbb{A}^s_n$$

   is a projection onto $Image\,(\mathcal{S})$.

Let us consider a differential analytic map of order $\mathbf{m}$, $\mathcal{T} : \mathcal{F}^{\geq 0}_{r,\mathbf{m}} \ \to \ \mathbb{A}^s_n$.
We consider the equation

$$\mathcal{S}(F) = \pi\left(\mathcal{T}(F)\right) \tag{113}$$

In [Sto16], we gave a sufficient condition on the triple $(\mathcal{S}, \mathcal{T}, \pi)$ under which equation (113) has a solution $F \in \mathcal{F}^{\geq 0}_{r,\mathbf{m}}$; this condition is called the "Big Denominators property" of the triple $(\mathcal{S}, \mathcal{T}, \pi)$ defined below.

B.2 Big denominators. Main theorem

Now we can define the big denominators property of the triple $(\mathcal{S}, \mathcal{T}, \pi)$ in equation (113).

Definition 11. *The triple of maps $(\mathcal{S}, \mathcal{T}, \pi)$ of form (B.1) has* **big denominators property of order m** *if there exists an nonnegative integer q such that the following holds:*

1. *$\mathcal{T}$ is an regular analytic differential map of order $\mathbf{m}$ that strictly increases the order by q and $j_0^{q-1}\mathcal{T}(0) = 0$, i.e. $\mathcal{T}(F)(x) = W(x, j^{\mathbf{m}}_x F)$ for any $x \in \mathbb{R}^n$ close to 0 and any function germ $F \in \mathcal{F}^{\geq 0}_{r,\mathbf{m}}$ such that $j_0^m F$ is close to 0 and $ord_0(W(x, 0)) \geq q$.*

2. $\mathcal{S} : \mathcal{F}^{\geq 0}_{r,\mathbf{m}} \to \mathbb{A}^s_n$ *is linear , increases the order by* q *and is homogenous, i.e.* $\mathcal{S}\left(\mathcal{F}^{(i)}_{r,\mathbf{m}}\right) \subset (\mathbb{A}^s_n)^{(q+i)}$.

3. *the linear map* $\pi : \mathbb{A}^s_n \to Image(\mathcal{S}) \subset \mathbb{A}^s_n$ *is a projection.*

4. *the map* $\mathcal{S}$ *admits right-inverse* $\mathcal{S}^{-1} : Image(S) \to \mathbb{A}^r_n$ *such that the composition* $\mathcal{S}^{-1} \circ \pi$ *satisfies:*

   there exists $C > 0$ *such that for any* $G \in \mathbb{A}^s_n$ *of order* $> q$, *one has for all* $1 \leq j \leq r$, *and all integer* i,

$$\left\| \left(\mathcal{S}_j^{-1} \circ \pi(G)\right)^{(i+m_j)} \right\| \leq C \frac{\|G^{(i+q)}\|}{(i+m_j+q)\cdots(i+q+1)}. \tag{114}$$

   where $\mathcal{S}_i^{-1}$ *denotes the ith component of* $\mathcal{S}^{-1}$, $1 \leq i \leq r$.

Remark 12. Let $i \geq 0$ and let $F = (F_1, ..., F_k) \in \left(\mathbb{A}^k_n\right)^{(i)}$. Let $F_j = \sum F_{j,\alpha} x^\alpha$ where the sum is taken over all $j = 1, ..., k$ and all multiindexes $\alpha = (\alpha_1, ..., \alpha_n)$ such that $|\alpha| = \alpha_1 + \cdots + \alpha_n = i$. The norm $||F||$ used in (114) is either

- $$||F_j|| = \sum_{|\alpha|=i} |F_{j,\alpha}|, \quad ||F|| = max\left(||F_1||, \cdots, ||F_k||\right).$$

- or the modified Fisher-Belitskii norm

$$||F_j||^2 = \sum_{|\alpha|=i} \frac{\alpha!}{|\alpha|!} |F_{j,\alpha}|^2, \quad ||F||^2 = ||F_1||^2 + \cdots + ||F_k||^2.$$

Remark 13. In practice, for each i, there is a decomposition into direct sums $\mathcal{F}^{(i)}_{r,\mathbf{m}} = L_i \oplus K_i$ with $\mathcal{S}_{|L_i}$ is a bijection onto its range. The chosen right inverse is then the one with zero component along K_i. For instance, the case of the modified Fisher-Belitskii norm, $K_i := \ker \mathcal{S}_i^*$ is the natural one, where $\mathcal{S}_i^*$ denotes the adjoint of $\mathcal{S}_i$ w.r.t. the scalar product.

Theorem 14. *[Sto16][theorem 7] Let us consider a system of analytic non-linear pde's such as equation (113) :*

$$\mathcal{S}(F) = \pi\left(W(x, j^{\mathbf{m}}_x F)\right). \tag{115}$$

If the triple $(\mathcal{S}, \mathcal{T}, \pi)$ *has big denominators property of order* $\mathbf{m}$, *according to definition 11, then the equation has an analytic solution* $F \in \mathcal{F}^{\geq 0}_{r,\mathbf{m}}$.

Remark 15. The precise statement of [Sto16] holds for $F \in \mathcal{F}^{>0}_{r,\mathbf{m}}$ and where the order of $W(x, 0)$ at the origin is greater than q. The shift by 1 (i.e $F \in \mathcal{F}^{\geq 0}_{r,\mathbf{m}}$ and where the order of $W(x, 0)$ at the origin is greater or equal to q) of the above statement, doesn't affect its proof.

B.3 Application

In this section we shall devise the strictly increasing condition in more detail. We look for a formal solution $F^{\geq 0}=\sum_{i\geq 0}F^{(i)}$ to (115). As above, $F^{(i)}$ stands for $(F_1^{(m_1+i)},\dots,F_r^{(m_r+i)})$. We define

$$\mathcal{S}(F^{(i+1)}) := \left[\pi W\left(x, j_x^{\mathbf{m}}\sum_{j\geq 0}^{i}F^{(j)}\right)\right]^{(i+q+1)}.$$

Here $[G]^{(i)}$ denotes the homogenous part of degree i of G in the Taylor expansion at the origin. Therefore $F:=\sum_{i\geq}F^{(i)}$ is a solution of (115) if

$$\mathrm{ord}_0\left(W\left(x, j_x^{\mathbf{m}}\sum_{j\geq 0}F^{(j)}\right)-W\left(x, j_x^{\mathbf{m}}\sum_{j\geq 0}^{i}F^{(j)}\right)\right)>i+q+1. \tag{116}$$

Indeed, we would have

$$\begin{aligned}\mathcal{S}\left(\sum_{i\geq 0}F^{(i)}\right) &= \sum_{i\geq 0}\left[\pi W\left(x, j_x^{\mathbf{m}}\sum_{j\geq 0}^{i}F^{(j)}\right)\right]^{(i+q+1)}\\ &= \sum_{i\geq 0}\left[\pi W\left(x, j_x^{\mathbf{m}}\sum_{j\geq 0}F^{(j)}\right)\right]^{(i+q+1)} = \pi W\left(x, j_x^{\mathbf{m}}F\right)\end{aligned}$$

We emphasize that condition (116) just means that W strictly increases the order by q as defined in Definition 10. Let us look closer to that condition. Let us denote $F^{\leq i}:=\sum_{j\geq 0}^{i}F^{(j)}$ and $F^{>i}:=\sum_{j>i}F^{(j)}$. Let us Taylor expand $W(x,j_x^{\mathbf{m}}F)$ at $F^{\leq i}$. We thus have

$$\begin{aligned}W(x,j_x^{\mathbf{m}}F)-W(x,j_x^{\mathbf{m}}F^{\leq i}) &= \sum\frac{\partial W}{\partial u_{j,\alpha}}((x,j_x^{\mathbf{m}}F^{\leq i}))\frac{\partial^{|\alpha|}F_j^{>i}}{\partial x^{\alpha}}\\ &\quad+\frac{1}{2}\sum\frac{\partial W}{\partial u_{j,\alpha}\partial u_{j',\alpha'}}((x,j_x^{\mathbf{m}}F^{\leq i}))\frac{\partial^{|\alpha|}F_j^{>i}}{\partial x^{\alpha}}\frac{\partial^{|\alpha'|}F_{j'}^{>i}}{\partial x^{\alpha'}}+\cdots\end{aligned}$$

We recall that $\mathrm{ord}_0 F_j^{>i}>m_j+i$ and when considering a coordinate $u_{j,\alpha}$, we have $|\alpha|\leq m_j$. Hence, we have

$$\mathrm{ord}_0\frac{\partial^{|\alpha|}F_j^{>i}}{\partial x^{\alpha}}>m_j+i-|\alpha|.$$

In order that the first derivative part of this Taylor expansion satisfies (116), it is sufficient that

$$\mathrm{ord}_0\frac{\partial W}{\partial u_{j,\alpha}}((x,j_x^{\mathbf{m}}F^{\leq i}))\geq|\alpha|-m_j+q+1.$$

This is nothing but the *regularity condition* as defined in Definition (10). Let us consider the other terms in the Taylor expansion. We have, for instance,

$$\mathrm{ord}_0\frac{\partial^{|\alpha|}F_j^{>i}}{\partial x^{\alpha}}\frac{\partial^{|\alpha'|}F_{j'}^{>i}}{\partial x^{\alpha'}}\geq m_j+i+1-|\alpha|+m_{j'}+i+1-|\alpha'|$$

If $i + 1 > q$, then not only the second but also any higher order derivative part of this Taylor expansion satisfies (116).

Corollary 16. *If $q = 0$ and if the system is regular, then it strictly increases the order by 0.*

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