Symmetries of almost CR structures

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Almost CR structures

Almost CR structure (of hypersurface type) ... smooth manifold M of dim $2n+1$, $n>1$ together with

- distribution $\mathcal{H} \subset TM$ of dimension $2n$, and
- *almost complex structure* J on $\mathcal{H}$, i.e., $J : \mathcal{H} \to \mathcal{H}$ is an endomorphism with the property $J^2 = -\mathrm{id}$.

Almost CR structure is *non–degenerate* ... $\mathcal{H}$ is completely non–integrable; defines a contact structure on M.

Endomorphism J extends by complex linearity to an endomorphism of the complexification $\mathbb{C}\mathcal{H}$; it decomposes as

$$\mathbb{C}\mathcal{H} = \mathcal{H}^{1,0} \oplus \mathcal{H}^{0,1}$$

into *holomorphic* and *anti–holomorphic* parts ... eigenbundles for eigenvalues i and $-i$ of J.

Almost CR structures

Almost CR structure is *partially integrable* ...

- $[\mathcal{H}^{1,0}, \mathcal{H}^{1,0}] \subset \mathcal{H}^{1,0} \oplus \mathcal{H}^{0,1}$, or equivalently,
- $[\xi, \eta] - [J(\xi), J(\eta)] \in \Gamma(\mathcal{H})$ for all $\xi, \eta \in \Gamma(\mathcal{H})$.

The component of $[\mathcal{H}^{1,0}, \mathcal{H}^{1,0}]$ in $\mathcal{H}^{0,1}$ corresponds (up to multiple) to the complexification of the Nijenhuis tensor

$$N(\xi, \eta) = J([\xi, \eta] - [J(\xi), J(\eta)]) - [J(\xi), \eta] - [\xi, J(\eta)]$$

for all $\xi, \eta \in \Gamma(\mathcal{H})$.

Almost CR structure is *integrable* ... the Nijenhuis tensor vanishes.

Signature of almost CR structure ... the signature of Levi form.

Assumptions

... oriented non–degenerate partially integrable almost CR structure $(M, \mathcal{H}, J)$ *of hypersurface type; arbitrary signature* (p, q).

Almost CR structures

Consider

- $PSU(p+1, q+1)$ ... projectivization of the group of matrices preserving the pseudo–Hermitian form on $\mathbb{C}^{n+2}$

$$m(u,v) = u_0\overline{v_{n+1}} + u_{n+1}\overline{v_0} + \sum_{k=1}^{p} u_k\overline{v_k} - \sum_{k=p+1}^{n} u_k\overline{v_k}$$

- P ... stabilizer of the complex line generated by first basis vector in the standard basis of $\mathbb{C}^{n+2}$.

Maximally symmetric model $PSU(p+1, q+1)/P$ ... smooth real hypersurface in $\mathbb{C}P^{n+1}$; can be viewed as the projectivization of the null cone of m in $\mathbb{C}^{n+2}$.

Geometrically, CR structures in question can be viewed as curved versions of maximally symmetric model.

CR symmetries

Local CR transformation of $(M, \mathcal{H}, J)$ ... local diffeomorphism of M such that its tangent map preserves $\mathcal{H}$ and restriction to $\mathcal{H}$ is complex linear

Definition

Local symmetry s_x at x ... local CR transformation such that

- $s_x(x) = x$, and
- $T_x s_x = -\mathrm{id}$ on $\mathcal{H}$.

CR transformations of the model $PSU(p+1, q+1)/P$ ... left multiplications by elements of $PSU(p+1, q+1)$

CR symmetries

Theorem

There is infinite number of symmetries at each point kP of $PSU(p+1, q+1)/P$ given by matrices of the form $ks_{Z,z}k^{-1}$ for all $Z \in \mathbb{C}^{n*}$ and $z \in \mathbb{R}^*$, where E denotes identity matrix

$$s_{Z,z} = \begin{pmatrix} -1 & -Z & iz+\frac{1}{2}ZIZ^* \\ 0 & E & -IZ^* \\ 0 & 0 & -1 \end{pmatrix}.$$

1. There exists an infinite number of involutive symmetries at each point characterized by $z = 0$. For each such symmetry, there is a different metric preserved by this symmetry compatible with the CR geometry.
2. There exists an infinite number of non–involutive symmetries at each point characterized by $z \neq 0$. They do not preserve any metric compatible with the CR geometry.

CR symmetries

Weyl connections ... admissible connections satisfying certain normalization condition; CR geometries carry two CR-invariants

- Nijenhuis tensor N (structure torsion),
- Chern-Moser/Weyl tensor W (trace-free part of curvature of arbitrary Weyl connection).

Theorem

1. *If there is local symmetry at $x \in M$, then $N(x) = 0$.*
2. *If there is non–involutive local symmetry at x, then $W(x) = 0$.*
3. *There is at most one local symmetry at x with $W(x) \neq 0$.*
4. *Local symmetry at x is involutive if and only if there is invariant Weyl connection ∇ at x defined locally near x.*
5. *It holds $\nabla_\xi W(x) = 0$ for invariant Weyl connection ∇ at x for each $\xi \in \mathcal{H}$.*

Symmetric CR geometries

Symmetric geometry . . . symmetry at each point.
Symmetric CR structures are integrable; admit only systems of involutive symmetries for non-vanishing W.

Theorem

Suppose $(M, \mathcal{H}, J)$ is symmetric CR geometry. Then either

1. $W = 0$*; CR geometry is locally equivalent to the model* $PSU(p+1, q+1)/P$*, or*
2. $W \neq 0$*; group generated by symmetries is a Lie group that acts transitively on* M*, i.e., CR geometry is homogeneous;* (M, S) *is homogeneous reflexion space, where* S *is the smooth system of uniquely given symmetries.*

Reflexion space (M, S) . . . space M together with map $S : M \times M \to M$ such that for all $x, y, z \in M$

- $S(x, x) = x$,
- $S(x, S(x, y)) = y$,
- $S(x, S(y, z)) = S(S(x, y), S(x, z))$.

Symmetric CR geometries

We use the notation $S(x,-) = s_x$.

Then $T_x s_x : T_x M \to T_x M$ is involutive for each $x \in M$, and $T_x M$ decomposes into ± 1–eigenspaces, where

- $\mathcal{H}_x$ is the -1–eigenspace,
- there is the one–dimensional 1–eigenspace $T_x^+ M$ complementary to $\mathcal{H}$.

Since $T_x^+ M$ is involutive it determines a foliation $\mathcal{F}$ on M and

- $M = K/H$, where K is the Lie group generated by symmetries from S and H is the stabilizer of a point,
- the stabilizer L of the leaf F going through eH is a closed subgroup of K,
- $N = K/L$ is symmetric space,
- $F = L/H$.

Symmetric CR geometries

Theorem

Let K be the Lie group generated by all symmetries on a non–flat symmetric CR geometry $(M, \mathcal{H}, J)$. Assume

$$\mathrm{Ad}(H^0)|_{\mathfrak{n}/\mathfrak{h}} = \mathrm{Ad}(H)|_{\mathfrak{n}/\mathfrak{h}},$$

where H^0 denotes connected component of identity of the stabilizer $H \subset K$ of a point and $\mathfrak{n}$ is the 1–eigenspace of $s = diag(-1, E, -1)$ in $\mathfrak{k}$. There exist

- *a distinguished Weyl connection ∇ preserving the corresponding Reeb field,*
- *a K–invariant contact form θ,*
- *a K–invariant pseudo–Riemannian metric $\bar{g}$ on $\mathcal{H}$, and*
- *a K–invariant Webster metric g on TM,*

such that ...

Symmetric CR geometries

Theorem

1. $\nabla \bar{g} = 0, \nabla g = 0,$
2. $g|_{\mathcal{H}} = \bar{g}$ *and the Reeb field of* ∇ *is orthogonal to* $\mathcal{H}$ *and has length* 1*,*
3. *choosing the Reeb field of* ∇ *as a trivialization of* $TM/\mathcal{H} \otimes \mathbb{C}$*, the pseudo–Riemannian metric* $\bar{g}$ *on* $\mathcal{H}$ *coincides with the real part of the Levi form up to constant multiple,*
4. *the symmetry at* x *is linear in geodesic coordinates of* ∇ *at* x*, reverses the directions of* $\mathcal{H}_x$ *and preserves the direction of the Reeb field of* ∇ *at* x*.*

Literature

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