# Hyperbolic CR singularities

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August 26, 2020

Obergugl, August 2020

# Surfaces with CR singularity

*Surfaces with CR singularity*: real analytic surface $M \subset (\mathbb{C}^2, 0)$:

$$M : z_2 = z_1\bar{z}_1 + \gamma(z_1^2 + \bar{z}_1^2) + O^3(z_1, \bar{z}_1), \quad \gamma \geq 0.$$

perturbation of the *Bishop quadric* $Q_\gamma : z_2 = z_1\bar{z}_1 + \gamma(z_1^2 + \bar{z}_1^2)$
$\gamma \in \mathbb{R}^+$ — *Bishop invariant*

If $\gamma \neq \frac{1}{2}$, the origin is an isolated *Cauchy-Riemann singularity* :
- $\forall\ p \neq 0$, "$\mathbb{C} \not\subset T_p M$" (ie. totally real at $p \neq 0$)
- $T_0 M = \{z_2 = 0\}$

$M$ is said to be :
- *elliptic* si $0 \leq \gamma < \frac{1}{2}$
- *hyperbolic* if $\gamma > \frac{1}{2}$
- *parabolic* if $\gamma = \frac{1}{2}$

# Normalization near an elliptic CR singularity

**Goal.**

- **Normal form** of $M$ with respect to holomorphic change of coordinates near the origin.

## Theorem (Moser-Webster 1983)

*If $0 < \gamma < \frac{1}{2}$, there exists a holomorphic change of variables near the origin such that $M$ reads*

$$x_2 = z_1 \bar{z}_1 + (\gamma + \delta x_2^s)(z_1^2 + \bar{z}_1^2), \quad y_2 = 0, \quad z_2 = x_2 + \mathrm{i} y_2$$

*avec $\delta = \pm 1$ si $s \in \mathbb{N}^*$ ou $\delta = 0$ si $s = \infty$.*

**Consequences**

$\rightsquigarrow$ *Flattening* : $M \subset \mathrm{Im}(z_2) = 0$.

$\rightsquigarrow$ *Local hull of holomorphy* : it is filled by an analytic family of holomorphic discs.

Complexification of $M$: $(z_1, z_2, \bar{z}_1, \bar{z}_2) \leftarrow (z_1, z_2, w_1, w_2) =: (z, w) \in \mathbb{C}^4$

$$\mathcal{M} \subset \mathbb{C}^4 : \begin{cases} z_2 = z_1 w_1 + \gamma(z_1^2 + w_1^2) + H(z_1, w_1) \\ w_2 = z_1 w_1 + \gamma(z_1^2 + w_1^2) + \bar{H}(w_1, z_1) \end{cases}$$

Canonical projections : $\pi_1(z, w) = z$ et $\pi_2(z, w) = w$ for $(z, w) \in \mathcal{M}$.

According to Moser-Webster, $\pi_1$ et $\pi_2$ are 2-1 branched coverings:

- $\pi_2(z, w) = \pi_2(z', w')$, $(z, w), (z', w') \in \mathcal{M}$
  $\implies$ unique solution $(z', w') =: \tau_1(z, w)$ with $z' \neq z$
- $\pi_1(z, w) = \pi_1(z', w')$, $(z, w), (z', w') \in \mathcal{M}$
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# Moser-Webster involutions

⤳ pair of holomorphic involutions: pour $\gamma > 0$

$$\tau_1 : \begin{cases} z_1' = -z_1 - \frac{1}{\gamma} w_1 + \underbrace{h_1(z_1, w_1)}_{\text{ord}_0 \geq 2} \\ w_1' = w_1 \end{cases} \quad -\;-\;-\;-\;-\;-\; \tau_1 \circ \tau_1 = Id$$

$$\tau_2 : \begin{cases} z_1' = z_1 \\ w_1' = -\frac{1}{\gamma} z_1 - w_1 + h_2(z_1, w_1) \end{cases} \quad -\;-\;-\;-\;-\;-\; \tau_2 \circ \tau_2 = Id$$

$$\tau_2 = \rho \tau_1 \rho, \quad \rho(z, w) := (\bar{w}, \bar{z})$$

## Proposition (Moser-Webster 1983)

*Holomorphic classification of surface $\mathcal{M} \in \mathbb{C}^4$ ⟷ Holomorphic classification of $(\tau_1, \tau_2)$*

**Remark.** Normal form of $M \subset \mathbb{C}^2$ ⟷ Normal form of $(\tau_1, \tau_2)$.

# Appropriate coordinates

$$\tau_1 : \begin{cases} \xi' = \lambda\eta + \text{h.o.t.} \\ \eta' = \lambda^{-1}\xi + \text{h.o.t.} \end{cases} , \quad \tau_2 : \begin{cases} \xi' = \lambda^{-1}\eta + \text{h.o.t.} \\ \eta' = \lambda\xi + \text{h.o.t.} \end{cases} ,$$

$$\sigma := \tau_1 \circ \tau_2 : \begin{cases} \xi' = \lambda^2\xi + \text{h.o.t.} \\ \eta' = \lambda^{-2}\eta + \text{h.o.t.} \end{cases} ,$$

$\lambda$ is a root of $\gamma\lambda^2 - \lambda + \gamma = 0$

## Remark

- elliptic surface $M$, $0 < \gamma < \frac{1}{2} \implies \lambda = \bar{\lambda}$ and $|\lambda| \neq 1$
  — origin is an hyperbolic fixed point of $\tau_1$, $\tau_2$ et $\tau_1 \circ \tau_2$
- hyperbolic surface $M$, $\gamma > \frac{1}{2} \implies |\lambda| = 1$
  — origin is an elliptic fixed point of $\tau_1$, $\tau_2$ et $\sigma = \tau_1 \circ \tau_2$

# Normal forms of involutions

## Theorem (Moser-Webster 1983, formal normal form)

*Assume: $\lambda$ not a root of unity*
*Conclusion : exists a unique formal normalized transformation $\psi$ s.t.*

$$\psi^{-1} \circ \tau_1 \circ \psi : \left\{ \begin{array}{l} \xi' = \Lambda(\xi\eta)\eta \\ \eta' = \Lambda^{-1}(\xi\eta)\xi \end{array} \right. , \quad \psi^{-1} \circ \tau_2 \circ \psi : \left\{ \begin{array}{l} \xi' = \Lambda^{-1}(\xi\eta)\eta \\ \eta' = \Lambda(\xi\eta)\xi \end{array} \right. ,$$

*where $\Lambda(t) \in \mathbb{C}[[t]]$. s.t. $\Lambda(t) = \bar{\Lambda}(t)$ (elliptic case) ou $\Lambda(t) \cdot \bar{\Lambda}(t) = 1$ (hyperbolic case).*

## Theorem (Moser-Webster 1983, Convergence in elliptic case)

*If $\lambda = \bar{\lambda}$ and $|\lambda| \neq 1$, then $\Lambda$ and $\psi$ are holomorphic on a neighborhood of the origin.*

# Main CR singularity results

Moser-Webster: $M_n \hookrightarrow \mathbb{C}^n$ with smallest dim. of complex tangent at 0 : $p = 1$.

- Smaller dimension : $M_m \hookrightarrow \mathbb{C}^n$, $m < n \rightsquigarrow$ Coffman [Houston '04, Pacific '06, Memoirs AMS'10]
- Higher degeneracy $p \geq 1 \rightsquigarrow$ Gong-S. [Invent. '16+ JDG '19]
- $\gamma = 0$ : No involution $\rightsquigarrow$ Holomorphic classification Huang-Yin [Invent. '09]
- Flattening in higher dimension $\rightsquigarrow$ Huang-Yin [Math. Ann.'16+Adv. Math.'17], Huang-Fan [GAFA '18]
- Hyperbolic exceptional case ($\lambda$ root of unity )$\rightsquigarrow$ on-going work with Martin Klimes.

# Non exceptional hyperbolic CR singularity

$|\lambda| = 1$ not a root of unity. Moser-Webster $\rightsquigarrow$ normalizing transformation $\psi$ might not converge at the origin.

## Definition

A CR singularity surface $M$ is said to be

- *non exceptionnal* if $\lambda$ is not a root of the unity.
- *non degenerate* if $\Lambda = \Lambda(t)$ is non constant.

## Theorem (Gong 1994: non exceptional degenerate case)

*Assumptions:*

1. $|\lambda| = 1$ and $\lambda$ satisfies *diophantine condition:*

$$|\lambda^n - 1| > \frac{c}{n^\delta}$$

2. $\tau_1$ et $\tau_2$ are *formally linearizable (i.e $\Lambda(\xi\eta) = \lambda$),*

*Then, $\psi$ is holomorphic in a neighborhood of the origin : $M$ is holomorphically equivalent to the quadric.*

# Non degenerate hyperbolic CR singularity surface

$$\tau_1 : \begin{cases} \xi' = \lambda\eta + \text{h.o.t.} \\ \eta' = \lambda^{-1}\xi + \text{h.o.t.} \end{cases} , \quad \tau_2 : \begin{cases} \xi' = \lambda^{-1}\eta + \text{h.o.t.} \\ \eta' = \lambda\xi + \text{h.o.t.} \end{cases}$$

$$\lambda := e^{\frac{i}{2}\alpha}, \frac{\alpha}{\pi} \in \mathbb{R} \setminus \mathbb{Q}, \quad \Lambda(\xi\eta) = \lambda + \sum_{n \geq 1} \tilde{c}_n(\xi\eta)^n.$$

## Theorem (S.-Zhao 2020)

*Assume $\Lambda(\xi\eta) \neq \lambda$. If $r > 0$ is small enough, there exists a "asymptotic full measure" parameters set $\mathcal{O}_r \subset\ ]-r^2, r^2[$ s.t. $\forall\ \omega \in \mathcal{O}_r$, $\exists\ \mu_\omega \in \mathbb{R}$ and an holomorphic transformation $\Psi_\omega$ on $\mathcal{C}_\omega^r := \{\xi\eta = \omega,\ |\xi|, |\eta| < r\}$ with $\Psi_\omega \circ \rho = \rho \circ \Psi_\omega$ and s.t. , on $\mathcal{C}_\omega^r$,*
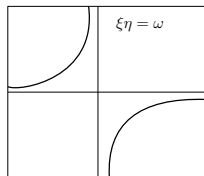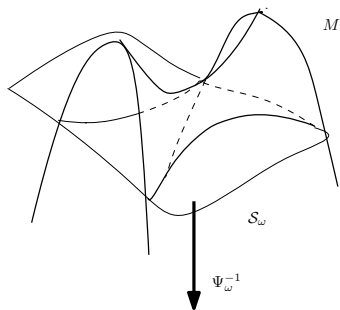
$$\Psi_\omega^{-1} \circ \tau_1 \circ \Psi_\omega : \begin{cases} \xi' = e^{\frac{i}{2}\mu_\omega}\eta \\ \eta' = e^{-\frac{i}{2}\mu_\omega}\xi \end{cases} , \quad \Psi_\omega^{-1} \circ \tau_2 \circ \Psi_\omega : \begin{cases} \xi' = e^{-\frac{i}{2}\mu_\omega}\eta \\ \eta' = e^{\frac{i}{2}\mu_\omega}\xi \end{cases} ,$$

Remark $\Psi_\omega(\mathcal{C}_\omega^r)$ is an holomorphic invariant set of $\tau_i$'s and their restriction is conjugated to a linear map . "Asymptotic full measure"$= \frac{|\mathcal{O}_r|}{2r^2} \xrightarrow{r \to 0} 1$.

# Geometric consequences

## Theorem (S.-Zhao 2020)

*Let $M$ be a surface with an hyberbolic CR singularity at the origin which is non exceptionnal andt non degenerate. Then: there exist a neighborhood of the origin and a family of holomorphic curves $\{S_\omega\}_{\omega \in \mathcal{O}}$ which intersects $M$ along holomorphic hyperbolas : 2 real curves which are simultaneously holomorphically mapped to the two branches of the hyperbolas $\xi\eta = \omega$, $\omega \neq 0$.*

# Intersection at the origin of $M$ by an holomorphic curve

### Theorem (Klingenberg 1985)

*Let $M$ be a surface with an hyberbolic CR singularity at the origin with $\lambda = e^{\frac{i}{2}\alpha}$ satisfiying the diophantine condition above. Then , there exists a unique holomorphic curve intersecting $M$ along 2 totally real curves interesecting transversally at the origin.*

Remarque. These are the "traces" of 2 lines $\xi\eta = 0$.

# KAM (Kolmogorv-Arnold-Moser) scheme — formulation

Pair of holomorphic involutions

$$\tau_1 : \begin{cases} \xi' = e^{\frac{i}{2}\alpha(\xi\eta)}\eta + p(\xi,\eta) \\ \eta' = e^{-\frac{i}{2}\alpha(\xi\eta)}\xi + q(\xi,\eta) \end{cases} , \quad \tau_2 = \rho \circ \tau_1 \circ \rho, \ \rho : (\xi,\eta) \mapsto (\bar{\xi},\bar{\eta})$$

restricted to a "crown" $\mathcal{C}_{\omega,\beta}^r := \{|\xi\eta - \omega| < \beta, \ |\xi|,|\eta| < r\}, \ \omega \in \mathcal{O} \subset\, ]-r^2,r^2[.$

- Non degeneracy: $\exists\, s \in \mathbb{N}^*, \forall\, \omega \in \mathcal{O}, \ |\alpha^{(s)}(\omega)| > \frac{1}{2}$,

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- Non degeneracy: $\exists s \in \mathbb{N}^*, \forall \omega \in \mathcal{O}, |\alpha^{(s)}(\omega)| > \frac{1}{2}$,
- Smallness: unique decomposition :

$$p(\xi,\eta) = p^{0,0}(\xi\eta) + \sum_{l\geq 1} p^{l,0}(\xi\eta)\xi^l + \sum_{j\geq 1} p^{0,j}(\xi\eta)\eta^j,$$

$$\|p\|_{\omega,\beta,r} := \sum_{l\cdot j=0} \sup_{|\xi\eta-\omega|<\beta} |p^{l,j}(\xi\eta)|r^{l+j} < \varepsilon, \quad \|q\|_{\omega,\beta,r} < \varepsilon$$

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- Skew condition: $\|e^{\frac{i}{2}\alpha(\xi\eta)}\eta q + e^{-\frac{i}{2}\alpha(\xi\eta)}\xi p\|_{\omega,\beta,r} < \varepsilon^{\frac{3}{2}}$

# KAM scheme — transformation

$$r \rightsquigarrow r_+ < r, \quad \mathcal{O} \rightsquigarrow \mathcal{O}_+ \subset \mathcal{O}, \quad \tau_i \rightsquigarrow \tau_{i,+} = \psi^{-1} \circ \tau_i \circ \psi$$

defined on a smaller "crown".

For $0 < r_+ < r$, $\beta_+ = \beta^{\frac{5}{4}}$, $\varepsilon_+ = \varepsilon^{\frac{5}{4}}$ ($\beta \sim \varepsilon^{\frac{1}{40s}}$)

$$\mathcal{O}_+ := \left\{ \omega \in \mathcal{O} \cap \, ] - r_+^2, r_+^2[ : |e^{ik\alpha(\omega)} - 1| > \varepsilon^{\frac{1}{64s}}, \quad 1 \le |k| \lesssim |\ln \varepsilon| \right\}$$

Using "approximate solutions of cohomological equations", one builds a transformation

$$\psi(\xi, \eta) = (\mathrm{Id} + \mathcal{U})(\xi, \eta) = \begin{pmatrix} \xi + u(\xi, \eta) \\ \eta + v(\xi, \eta) \end{pmatrix}$$

s.t. $\psi \circ \rho = \rho \circ \psi$, $\|u\|_{\omega, \beta_+, r_+}, \|v\|_{\omega, \beta_+, r_+} < \varepsilon^{\frac{49}{50}}$ and

$$\|\eta u + \xi v\|_{\omega, \beta_+, r_+} < \varepsilon^{\frac{61}{32}} + \varepsilon^{-\frac{1}{16}} \|e^{\frac{i}{2}\alpha(\xi\eta)}\eta q + e^{-\frac{i}{2}\alpha(\xi\eta)}\xi p\|_{\omega, \beta, r} < \varepsilon^{\frac{5}{4}}.$$

# KAM scheme — new perturbation

$$\leadsto \psi^{-1} \circ \tau_1 \circ \psi : \begin{cases} \xi' = e^{\frac{i}{2}\alpha_+(\xi\eta)}\eta + p_+(\xi,\eta) \\ \eta' = e^{-\frac{i}{2}\alpha_+(\xi\eta)}\xi + q_+(\xi,\eta) \end{cases} \quad \text{on } \mathcal{C}^{r_+}_{\omega,\beta_+}$$

$$\sup_{|\xi\eta-\omega|<\beta} |\alpha_+(\xi\eta) - \alpha(\xi\eta)| < \varepsilon.$$

New size: for $\varepsilon_+ = \varepsilon^{\frac{5}{4}}$,

$$\|p_+\|_{\omega,\beta_+,r_+}, \|q_+\|_{\omega,\beta_+,r_+} < \varepsilon^{\frac{61}{32}} + \varepsilon^{-\frac{1}{16}}\|e^{\frac{i}{2}\alpha(\xi\eta)}\eta q + e^{-\frac{i}{2}\alpha(\xi\eta)}\xi p\|_{\omega,\beta,r} < \varepsilon_+$$

New skew condition:

$$\|e^{\frac{i}{2}\alpha_+(\xi\eta)}\eta q_+ + e^{-\frac{i}{2}\alpha_+(\xi\eta)}\xi p_+\|_{\omega,\beta_+,r_+} < \varepsilon^{\frac{61}{32}} < \varepsilon^{\frac{15}{8}} = \varepsilon_+^{\frac{3}{2}}.$$

**Remarque.** In the computation of $e^{\frac{i}{2}\alpha_+(\xi\eta)}\eta q_+ + e^{-\frac{i}{2}\alpha_+(\xi\eta)}\xi p_+$, one needs to consider the part

$$\begin{pmatrix} (e^{\frac{i}{2}\alpha(\xi\eta+\eta u+\xi v+uv)} - e^{\frac{i}{2}\alpha(\xi\eta)})\eta \\ (e^{-\frac{i}{2}\alpha(\xi\eta+\eta u+\xi v+uv)} - e^{-\frac{i}{2}\alpha(\xi\eta)})\xi \end{pmatrix} \text{ of } \begin{pmatrix} p_+ \\ q_+ \end{pmatrix} :$$

$$e^{\frac{i}{2}\alpha(\xi\eta)}\eta \cdot (e^{-\frac{i}{2}\alpha(\xi\eta+\eta u+\xi v+uv)} - e^{-\frac{i}{2}\alpha(\xi\eta)})\xi$$
$$+ e^{-\frac{i}{2}\alpha(\xi\eta)}\xi \cdot (e^{\frac{i}{2}\alpha(\xi\eta+\eta u+\xi v+uv)} - e^{\frac{i}{2}\alpha(\xi\eta)})\eta$$
$$= (\xi\eta) \cdot \left(-\frac{i}{2}\alpha'(\xi\eta)(\eta u+\xi v+uv)\right) + (\xi\eta) \cdot \left(\frac{i}{2}\alpha'(\xi\eta)(\eta u+\xi v+uv)\right)$$
$$+ O^2(\eta u+\xi v+uv)$$
$$= O^2(\eta u+\xi v+uv)$$

# Proof of the Theorem — preparation

Start with involution

$$\tau_1 : \begin{cases} \xi' = e^{\frac{i}{2}\alpha}\eta + p(\xi,\eta) \\ \eta' = e^{-\frac{i}{2}\alpha}\xi + q(\xi,\eta) \end{cases} , \quad \frac{\alpha}{\pi} \in \mathbb{R} \setminus \mathbb{Q}$$

By an holomorphic transformation near the origin, it can be conjugate to

$$\begin{cases} \xi' = \left(e^{\frac{i}{2}\alpha} + \sum_{n=s}^{100s^2} \tilde{c}_n \cdot (\xi\eta)^n\right)\eta + \tilde{p}^{\geq 200s^2+2}(\xi,\eta) \\ \eta' = \left(e^{\frac{i}{2}\alpha} + \sum_{n=s}^{100s^2} \tilde{c}_n \cdot (\xi\eta)^n\right)^{-1}\xi + \tilde{q}^{\geq 200s^2+2}(\xi,\eta) \end{cases}$$

Remark Small divisors involved:

$$e^{\frac{i}{2}k\alpha} - 1, \quad 1 \leq |k| \leq 200s^2+2$$

# Proof of the Theorem — initial step of the KAM scheme

By an extra holomorphic change of coordinates, one can assume that

$$\tau : \begin{cases} \xi' = e^{\frac{i}{2}\alpha_0(\xi\eta)}\eta + p_0(\xi,\eta) \\ \eta' = e^{-\frac{i}{2}\alpha_0(\xi\eta)}\xi + q_0(\xi,\eta) \end{cases},$$

with $\alpha_0(\xi\eta) = \alpha \pm (\xi\eta)^s + \displaystyle\sum_{n=s+1}^{100s^2} c_n \cdot (\xi\eta)^n, \ c_n \in \mathbb{R}$

$$\operatorname{ord}_0 p_0(\xi,\eta), \operatorname{ord}_0 q_0(\xi,\eta) \geq 200s^2 + 2$$

Set :

$$\begin{aligned}
\varepsilon_0 &:= \max\{\|p_0\|_{\omega,\beta,r}, \ \|q_0\|_{\omega,\beta,r}\} \\
&\leq \max\{|p_0|_r, \ |q_0|_r\} \\
&\ll r^{200s^2}, \quad \forall \, \omega \in ]-r^2, r^2[
\end{aligned}$$

- If $\|e^{\frac{i}{2}\alpha_0(\xi\eta)}\eta q_0 + e^{-\frac{i}{2}\alpha_0(\xi\eta)}\xi p_0\|_{\omega,\beta,r} < \varepsilon_0^{\frac{3}{2}}$, $\forall \; \omega \in ]-r^2, r^2[$, then KAM scheme can be applied readily.

# Proof of the Theorem — perturbation the KAM sheme

- If $\|e^{\frac{i}{2}\alpha_0(\xi\eta)}\eta q_0 + e^{-\frac{i}{2}\alpha_0(\xi\eta)}\xi p_0\|_{\omega,\beta,r} < \varepsilon_0^{\frac{3}{2}}$, $\forall\ \omega \in ]-r^2, r^2[$, then KAM scheme can be applied readily.

- Otherwise, one can conjugate $\tau$ to a new involution, by excluding a small part of parameters of $\omega$,
  $\rightsquigarrow$ skew condition holds,
  $\Rightarrow$ KAM scheme can be applied.