Sphere maps and the reflection map

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Spheres and their mappings

The *unit sphere* S^{2n-1} in $\mathbb{C}^n$:

$$S^{2n-1} = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : \|z\|^2 = |z_1|^2 + \cdots + |z_n|^2 = 1\}.$$

A *sphere map* $H : S^{2n-1} \to S^{2m-1}, m \geq n$ is a holomorphic map $H : U \subset \mathbb{C}^n \to \mathbb{C}^m$, where U is an open neighborhood of S^{2n-1}, satisfying $H(S^{2n-1}) \subset S^{2m-1}$, which means:

$$\|H(z)\|^2 = 1, \qquad \text{for all } z \in S^{2n-1}.$$

Forstnerič '89: Sphere maps are rational maps with poles outside of S^{2n-1}.

Let $H = \frac{P}{Q}$ be a rational map, where $P = (P_1, \dots, P_m)$, such that each P_j and Q are polynomials with no common factors. The *degree* $\deg(H)$ of H is defined as

$$\deg(H) = \max(\deg P_1, \dots, \deg P_m, \deg Q).$$

The homogeneous sphere maps

Example

The *homogeneous sphere map* $H_n^d : S^{2n-1} \to S^{2K(n,d)-1}$ of degree d in $\mathbb{C}^n$, for $K(n,d) = \binom{n+d-1}{d}$, is defined as

$$H_n^d(z) := \left(\sqrt{\binom{\alpha}{d}} z^\alpha \right)_{|\alpha|=d}.$$

Rudin '84: The homogeneous sphere map is up to unitary equivalence unique among all homogeneous polynomial sphere maps.

The group invariant sphere maps

D'Angelo '88 defined another class of sphere maps:

Definition

Define $G^\ell : S^3 \to S^{2\ell+3}$ for $\ell \geq 0$ by

$$G^\ell(z,w) = \left(z^{2\ell+1}, c_1^\ell z^{2\ell-1} w, \ldots, c_\ell^\ell z w^\ell, w^{2\ell+1}\right),$$

where $(c_k^\ell)^2 = \left(\frac{1}{4}\right)^{\ell-k} \sum_{j=k}^{\ell} \binom{2\ell+1}{2j}\binom{j}{k}$ for $1 \leq k \leq \ell$.

These maps are invariant under a fixed-point-free finite unitary group.
D'Angelo–Kos–Riehl '03: The degree of a monomial sphere map from S^3 into $S^{2N-1} \subset \mathbb{C}^N$ is bounded by $2N-3$.

The tensor operation for sphere maps

For $v = (v_k)_{1\le k\le n} \in \mathbb{C}^n$ and $w = (w_j)_{1\le j\le m} \in \mathbb{C}^m$ two vectors, the tensor product of v and w is given by:

$$v \otimes w = (v_1 w_1, \ldots, v_1 w_m, \ldots, v_n w_1, \ldots, v_n w_m) \in \mathbb{C}^{nm}\,.$$

Definition

Let $H : S^{2n-1} \to S^{2m-1}$ and $G : S^{2n-1} \to S^{2\ell-1}$ be CR maps and $A \subseteq \mathbb{C}^m$ be a linear subspace. Decompose $H = H_A \oplus H_{A^\perp} \in A \oplus A^\perp = \mathbb{C}^m$ and define

$$E_{(A,G)}H := (H_A \otimes G) \oplus H_{A^\perp},$$

which is called the *tensor product of H by G on A*.

$E_{(A,G)}H$ is again a sphere map.

D'Angelo '88: Every polynomial sphere map of degree d in $\mathbb{C}^n$ can be tensored to H_n^d (after possibly applying a unitary transformation and a projection onto $\mathbb{C}^{K(n,d)}$).

The reflection map

Definition (D'Angelo '03)

Let $H = \frac{P}{Q} : U \subset \mathbb{C}^n \to \mathbb{C}^m$ be a sphere map of degree d, where $P = (P_1, \ldots, P_m)$ and $Q \neq 0$ on U. The *reflection map* V_H of H is defined by the following equation, evaluated on S^{2n-1}:

$$V_H X \cdot \frac{\bar{H}_n^d}{\bar{Q}} = X \cdot \bar{H}, \quad X \in \mathbb{C}^m .$$

Here, $a \cdot b = \sum_{j=1}^n a_j b_j$ for $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathbb{C}^n$.

The reflection matrix is obtained by homogenization of H: Write $H = \frac{1}{Q} \sum_{k=0}^d P^k$, where P^k is homogeneous of order k. Then, on S^{2n-1},

$$X \cdot \bar{H} = X \cdot \frac{1}{\bar{Q}} \sum_{k=0}^{d} \bar{P}^k(\bar{z}) \|z\|^{2(d-k)} = V_H X \cdot \frac{\bar{H}_n^d}{\bar{Q}}.$$

V_H is a $K(n,d) \times m$-matrix with holomorphic entries.

Examples

The reflection map of $H_1(z,w) = (z, zw, w^2)$ is given by

$$V_{H_1} = \left(\begin{array}{ccc} z & 0 & 0 \\ \frac{w}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{array} \right).$$

The reflection map of $H_2(z,w) = (z, \cos(t)w, \sin(t)zw, \sin(t)w^2)$, $t \in [0, 2\pi)$ is given by

$$V_{H_2} = \left(\begin{array}{cccc} z & 0 & 0 & 0 \\ \frac{w}{\sqrt{2}} & \frac{\cos(t)z}{\sqrt{2}} & \frac{\sin(t)}{\sqrt{2}} & 0 \\ 0 & \cos(t)w & 0 & \sin(t) \end{array} \right).$$

Finite nondegeneracy for sphere maps

Denote by $\mathcal{V}$ the set of CR vector fields tangent to S^{2n-1}.

Definition (Lamel '01)

Let $p \in S^{2n-1}$. A sphere map $H : S^{2n-1} \to S^{2m-1}$ has *degeneracy* s at p, if

$$\max_{\ell \in \mathbb{N}} \left[\dim_{\mathbb{C}} \operatorname{span}_{\mathbb{C}} \left\{ L_1 \cdots L_k H(z)|_{z=p} : L_j \in \mathcal{V}, k \leq \ell \right\} \right] = m - s.$$

If $s = 0$, the map is called *finitely nondegenerate* at p.

Example

Example

H_n^d is finitely nondegenerate at each point of S^{2n-1}.

Idea for $n = 2$ and $H = H_2^d : S^3 \to S^{2d+1} \subset \mathbb{C}^{d+1}$: Apply $\bar{L}$ to $H \cdot \bar{H} = 1$, use the commutator relations of $L, \bar{L}$ and $T = [L, \bar{L}]$ and the homogeneity of H_2^d to get that the vectors $(L^k \bar{H})$ for $k = 0, 1 \ldots, d$ form an orthogonal system in $\mathbb{C}^{d+1}$.

Holomorphic nondegeneracy for sphere maps

Definition (Lamel–Mir '17)

A sphere map $H : S^{2n-1} \to S^{2m-1}$ is called *holomorphically degenerate* if there exists a nontrivial holomorphic map $Y : U \to \mathbb{C}^m$, where U is an open neighborhood of S^{2n-1}, such that, on S^{2n-1},

$$Y \cdot \bar{H} = 0.$$

If a map is not holomorphically degenerate it is called *holomorphically nondegenerate*.

Example

H_n^d is holomorphically nondegenerate in S^{2n-1}.

Nondegeneracy conditions via reflection map

Theorem

Let $H : S^{2n-1} \to S^{2m-1}$ be a sphere map and $p \in S^{2n-1}$.

(a) *H is of degeneracy s at p if and only if $\ker V_H$ is of dimension s at p.*
In particular, H is finitely nondegenerate at p if and only if V_H is of rank m at p.

(b) *H is holomorphically nondegenerate if and only if V_H is generically of rank m in S^{2n-1}.*

Idea: (a) Since $X \cdot \bar{H} = V_H X \cdot \bar{H}_n^d$, apply CR vector fields to get $X \cdot \bar{L}^\alpha \bar{H} = V_H X \cdot \bar{L}^\alpha \bar{H}_n^d$. This leads to $A_p^\gamma(\bar{H})X = A_p^\gamma(\bar{H}_n^d)V_H X$, where A_p^γ is a matrix-valued map, and use the finite nondegeneracy of H_n^d.
(b) Use the holomorphic nondegeneracy of H_n^d.

Examples

The reflection map of $H_1(z,w) = (z, zw, w^2)$ is given by

$$V_{H_1} = \begin{pmatrix} z & 0 & 0 \\ \frac{w}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The reflection map of $H_2(z,w) = (z, \cos(t)w, \sin(t)zw, \sin(t)w^2)$, $t \in [0, 2\pi)$ is given by

$$V_{H_2} = \begin{pmatrix} z & 0 & 0 & 0 \\ \frac{w}{\sqrt{2}} & \frac{\cos(t)z}{\sqrt{2}} & \frac{\sin(t)}{\sqrt{2}} & 0 \\ 0 & \cos(t)w & 0 & \sin(t) \end{pmatrix}.$$

Applications

Example

For each $\ell \geq 0$, the group invariant map G^ℓ is finitely nondegenerate.

Theorem

Let $H : S^{2n-1} \to S^{2m-1}$ be a monomial sphere map of degree d and define $s_H = \min_{p \in S^{2n-1}} s(p)$, where $s(p)$ is the degeneracy of H at p.
Then H is of degeneracy s_H in $U := \{z_1 \cdots z_n \neq 0\} \cap S^{2n-1}$, while for points in S^{2n-1}, which belong to the complement of U, the degeneracy is at least s_H.

The invariant s_H is called *generic degeneracy*. The points in S^{2n-1} at which H is of degeneracy $s(H)$ form an open dense subset of S^{2n-1} (Lamel '01).

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The invariant s_H is called *generic degeneracy*. The points in S^{2n-1} at which H is of degeneracy $s(H)$ form an open dense subset of S^{2n-1} (Lamel '01).

Infinitesimal deformations of sphere maps

Definition

Let $H : S^{2n-1} \to S^{2m-1}$ be a sphere map. A holomorphic map $X : U \to \mathbb{C}^m$, for an open neighborhood U of S^{2n-1}, is called an *infinitesimal deformation of H*, if on S^{2n-1},

$$\mathrm{Re}(X \cdot \bar{H}) = 0.$$

The space of infinitesimal deformations of H is denoted by $\mathfrak{hol}(H)$.

If $H = \mathrm{id}$, then X is an infinitesimal automorphism of S^{2n-1}. The space of *infinitesimal automorphisms* of S^{2n-1} is denoted by $\mathfrak{hol}(S^{2n-1})$.

Motivation - local rigidity

Smooth curves of maps give rise to infinitesimal deformations: Let H_t be smooth curve of sphere maps, i.e. $H_t \cdot \bar{H}_t = 1$. Writing $V = \frac{d}{dt}\big|_{t=0} H_t$ we have that $V \cdot \bar{H}_0 + H_0 \cdot \bar{V} = 0$, i.e. $V \in \mathfrak{hol}(H_0)$.

Infinitesimal deformations are useful to study *local rigidity*, the CR analogue of the notion of *stability* of smooth mappings of smooth manifolds due to Mather '68.

Definition

A map $H : M \to M'$ is called *locally rigid* if there is a neighborhood of H in the space of holomorphic mappings (equipped with its natural topology), which only consists of maps belonging to the orbit $G \cdot H$, where $G = \mathrm{Aut}(M) \times \mathrm{Aut}(M')$ and for $g = (g_1, g_2) \in G$ define the G-action by $g \cdot H = g_2 \circ H \circ g_1^{-1}$.

A sufficient condition for local rigidity

Definition

For a map H we denote the space of *trivial infinitesimal deformations* by $\mathfrak{aut}(H)$, given by $\mathfrak{aut}(H) = H_*(\mathfrak{hol}(M)) + \mathfrak{hol}(M')|_H$.

Theorem (della Sala–Lamel–R. '19)

Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be compact, generic real-analytic submanifolds. Assume that M is minimal. Consider the class $\mathcal{F}$ of CR maps sending M into M', which are finitely nondegenerate at all points of M. Let $H : M \to M'$ be a map in $\mathcal{F}$ satisfying $\mathfrak{hol}(H) = \mathfrak{aut}(H)$, then H is locally rigid.

This is only a sufficient condition: The homogeneous sphere map $H_2^2(z,w) = (z^2, \sqrt{2}zw, w^2)$ is locally rigid but satisfies $27 = \dim \mathfrak{hol}(H) > \dim \mathfrak{aut}(H) = 19$.

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Infinitesimal deformations of sphere maps

Question

Are there sphere maps which admit only trivial infinitesimal deformations?

Good candidates are the group invariant maps G^ℓ. Let N be the real dimension of the space of nontrivial infinitesimal deformations of G^ℓ:

ℓ	4	7	10	12	13	16	17	19	22	24	25	27	28	31
N	10	24	28	14	32	36	32	40	80	18	48	40	52	96

For values of ℓ which are less than 31 and not included in the above table, the corresponding map G^ℓ only admits trivial infinitesimal deformations.

Infinitesimal deformations of sphere maps

Theorem

(a) *Let $H : S^{2n-1} \to S^{2m-1}$ be a holomorphically nondegenerate sphere map of degree d, then*

$$\dim \mathfrak{hol}(H) \leq \dim \mathfrak{hol}(H_n^d) = \binom{2d+n}{d} K(n,d)^2.$$

(b) *If $H : S^{2n-1} \to S^{2m-1}$ is a polynomial sphere map of degree d, it holds that $\dim \mathfrak{hol}(H) = \dim \mathfrak{hol}(H_n^d)$ if and only if H is unitarily equivalent to H_n^d.*

Idea:
(a) $X \cdot \bar{H} = V_H X \cdot \bar{H}_n^d$ implies $X \in \mathfrak{hol}(H) \Leftrightarrow V_H X \in \mathfrak{hol}(H_n^d)$.
(b) The reflection map V_H is invertible and $X \cdot {}^t V_H^{-1} \bar{H} = X \cdot \bar{H}_n^d$.
Take $X = H_n^d$ such that $1 = V_H^{-1} H_n^d \cdot \bar{H}$.

References

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