The ∂-operator and real holomorphic vector fields

Friedrich Haslinger and Duong Ngoc Son

University of Vienna

Obergurgl, August 2020

Supported by the FWF grants P 28154-N35 and M 2472-N35

Introduction

Let (M, h) denote a manifold of complex dimension n with a Hermitian metric h, and let ψ be a smooth real-valued function on M. Consider the Segal-Bargmann spaces

$$A^2_{(p,0)}(M, h, e^{-\psi})$$

of holomorphic $(p, 0)$-forms $u = \sum_{|J|=p}{}' u_J \, dz^J$ such that

$$\int_M |u|_h^2 e^{-\psi} d\mathrm{vol}_h < \infty.$$

Here $J = (j_1, \ldots, j_p)$ are multiindices of length p and the summation is taken over increasing indices; in holomorphic coordinates, the metric h has the form $h_{j\bar{k}} dz^j \otimes dz^{\bar{k}}$, where $[h_{j\bar{k}}]$ is a positive definite Hermitian matrix with smooth coefficients; the volume element induced by the metric is denoted by $d\mathrm{vol}_h := \det(h_{j\bar{l}})\, d\lambda$; the metric h induces a metric on tensors of each degree, so for $(1,0)$-forms $u = u_j dz^j$ and $v = v_j dz^j$ one has $\langle u, v\rangle_h = h^{j\bar{k}} u_j v_{\bar{k}}$ and $|u|_h^2 = \langle u, u\rangle_h$, where $[h^{j\bar{k}}]$ is the transpose of the inverse matrix of $[h_{j\bar{k}}]$.

Under suitable conditions the complex derivative

$$\partial u := \sum_{|J|=p}{}' \sum_{j=1}^{n} \frac{\partial u_J}{\partial z_j}\, dz^j \wedge dz^J$$

is a densely defined, in general unbounded operator

$$\partial : A^2_{(p,0)}(M, h, e^{-\psi}) \longrightarrow A^2_{(p+1,0)}(M, h, e^{-\psi}), \quad 0 \leq p \leq n-1.$$

In order to determine the adjoint operator

$$\partial^* : A^2_{(p+1,0)}(M, h, e^{-\psi}) \longrightarrow A^2_{(p,0)}(M, h, e^{-\psi})$$

it is necessary to consider the nonvanishing Christoffel symbols for the Chern connection in local coordinates $z^1, \dots, z^n$:

$$\Gamma^i_{jk} = h^{i\bar{l}} \partial_j h_{k\bar{l}}, \quad \Gamma^{\bar{i}}_{\bar{j}\bar{k}} = \overline{\Gamma^i_{jk}}.$$

For a general Hermitian metric, the torsion tensor T^i_{jk} may be nontrivial; it is defined by

$$T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}, \quad T^{\bar{i}}_{\bar{j}\bar{k}} = \overline{T^i_{jk}},$$

the torsion $(1, 0)$-form is then obtained by taking the trace:

$$\tau = T^i_{ji} dz^j.$$

We use $h_{j\bar{k}}$ and its inverse $h^{\bar{k}l}$ to lower and raise indices. For example, raising and lowering indices of the torsion, we have

$$T_q{}^{pr} := T^{\bar{i}}_{\bar{j}\bar{k}} h_{q\bar{i}} h^{p\bar{j}} h^{r\bar{k}}.$$

In particular, for a $(0,1)$ form $w = w_{\bar{k}}\, d\bar{z}^k$, raising indices gives the "musical" operator $\sharp$ acting on w and to produce an $(1,0)$ vector field $w^\sharp := h^{k\bar{j}} w_{\bar{j}} \partial_k$. Now, if $(\bar{\partial}\psi - \bar{\tau})^\sharp$ is a holomorphic vector field the adjoint operator ∂^* on $\mathrm{dom}(\partial^*) \subset A^2_{(1,0)}(M, h, e^{-\psi})$ can be expressed in the form

$$\partial^* u = \langle u, \partial\psi - \tau \rangle_h.$$

If, in addition, the metric h is Kählerian one has $\tau = 0$ and thus

$$\partial^* u = h^{j\bar{k}} u_j \frac{\partial \psi}{\partial \bar{z}^k},$$

that means that the complex vector field

$$X := h^{j\bar{k}} \frac{\partial \psi}{\partial \bar{z}^k} \frac{\partial}{\partial z^j}$$

is holomorphic. In this case, the gradient field $\mathrm{grad}_h \psi$ is a *real holomorphic* vector field

We consider the ∂-complex

$$A^2(M,h,e^{-\psi}) \underset{\partial^*}{\overset{\partial}{\rightleftarrows}} A^2_{(1,0)}(M,h,e^{-\psi}) \underset{\partial^*}{\overset{\partial}{\rightleftarrows}} A^2_{(2,0)}(M,h,e^{-\psi}),$$

and the corresponding complex Laplacian

$$\widetilde{\Box}_p = \partial\partial^* + \partial^*\partial : A^2_{(1,0)}(M,h,e^{-\psi}) \longrightarrow A^2_{(1,0)}(M,h,e^{-\psi}),$$

which, under suitable assumptions, will be a densely defined self-adjoint operator.

In order to describe the formula for ∂^* on $(2,0)$-forms we write

$$v = \frac{1}{2} \sum_{j,k} v_{jk} dz^j \wedge dz^k = \sum_{j<k} v_{jk} dz^j \wedge dz^k,$$

where $v_{jk} = -v_{kj}$. Define an operator $T^\sharp \colon \Lambda^{2,0}(M) \to \Lambda^{1,0}(M)$ by

$$T^\sharp(v) = \frac{1}{2} T_p{}^{rs} v_{rs} dz^p.$$

Then

$$\partial^* v = P_{h,\psi}\left(-(\psi_{\bar{j}} - \tau_{\bar{j}}) v_{pq} h^{q\bar{j}} dz^p + T^\sharp(v)\right).$$

Here, $P_{h,\psi}$ is the orthogonal projection from $L^2_{(2,0)}(M, h, e^{-\psi})$ onto $A^2_{(2,0)}(M, h, e^{-\psi})$.
If h is Kähler and $(\bar{\partial}\psi)^\sharp$ is holomorphic then

$$\partial^* v = -\psi_{\bar{j}} v_{pq} h^{q\bar{j}} dz^p.$$

Holomorphic torsion

Definition

Let h be a Hermitian metric on a complex manifold. We say that h has *holomorphic torsion* if

$$\nabla_{\bar{l}} T_p{}^{rs} = 0,$$

where ∇ is the Chern connection.

h has holomorphic torsion if and only if the components of the torsion $T_p{}^{rs}$ (in any holomorphic coordinate frame) are holomorphic. Moreover, it implies that $\bar{\tau}^\sharp$ is a holomorphic $(1,0)$ vector field.

Let D_p^* and ∂_p^* be the Hilbert space adjoints of ∂ in the Lebesgue space $L^2_{(p+1,0)}(M, h, e^{-\psi})$ and $A^2_{(p+1,0)}(M, h, e^{-\psi})$, respectively.

In summary, we have the following

Theorem

Let (M,h) be a complete Hermitian manifold with weight $e^{-\psi}$. Assume that the torsion ${T_p}^{rs}$ of the Chern connection is holomorphic. If $(\bar\partial\psi)^\sharp$ is holomorphic, then for $\eta \in dom(D_p^*)$, $p \geqslant 0$, that is holomorphic in an open set $U \subset M$, $D_p^*\eta$ is also holomorphic in U. In particular, if ∂_p is densely defined in the Bergman space $A^2_{(p,0)}(M,h,e^{-\psi})$, then

$$D_p^*\eta = \partial_p^*\eta$$

for $\eta \in dom(\partial_p^*)$.

Example

Let $M = \mathbb{B}^n$ be the unit ball in $\mathbb{C}^n$ and let $h_{j\bar{k}} = (1 - |z|^2)^{-1}\delta_{jk}$ be a conformally flat metric. By direct computations, we find that the torsion

$$T_q{}^{pr} = z^p \delta^r_q - z^r \delta^p_q$$

is nontrivial (unless $n = 1$) and holomorphic. Let $\psi = \alpha \log(1 - |z|^2)$. Then

$$(\bar{\partial}\psi)^\sharp = -\alpha \sum_{j=1}^{n} z_j \frac{\partial}{\partial z^j}$$

is a holomorphic vector field.

Multiradial potential functions

We consider Kähler metrics on $\mathbb{C}^n$ with *multi-radial* potential functions

$$\chi(z_1, z_2, \ldots, z_n) = \tilde{\chi}(r_1, r_2, \ldots, r_n)$$

where $r_j = |z_j|^2$, $j = 1, \ldots, n$. For these metrics, we can determine explicitly the multi-radial weight functions ψ such that $(\bar{\partial}\psi)^\sharp$ is holomorphic.

Theorem

Let $\chi(z) = \tilde{\chi}(|z_1|^2, \dots, |z_n|^2)$ be a multi-radial potential function for a Kähler metric in $\mathbb{C}^n$. If $\psi(z) = \widetilde{\psi}(|z_1|^2, \dots, |z_n|^2)$ is a multi-radial weight function such that $(\bar{\partial}\psi)^\sharp$ is holomorphic, then

$$(\bar{\partial}\psi)^\sharp = \sum_{j=1}^{n} C_j z_j \frac{\partial}{\partial z_j},$$

where C_k's are real constant and

$$\tilde{\psi} = C_0 + \sum_{j=1}^{n} C_j r_j \frac{\partial \tilde{\chi}}{\partial r_j}.$$

For example, let $\tilde{\chi}$ have the following form

$$\tilde{\chi}(r_1, r_2, \dots, r_n) = F_1(r_1) + F_2(r_2) + \cdots + F_n(r_n),$$

where $r_j = |z_j|^2$, $j = 1, \dots, n$, with smooth real valued functions F_j, $j = 1, \dots, n$. Then we have a diagonal matrix

$$h_{j\bar{k}} = \delta_{jk}(F_j' + r_j F_j'').$$

We have to suppose that all entries satisfy $F_j' + r_j F_j'' > 0$.

For this metric, we can always find a weight function ψ such that $(\bar{\partial}\psi)^\sharp$ is holomorphic. In fact, we can determine all such multi-radial weight functions ψ.

Theorem

Let h be a Kähler metric on $\mathbb{C}^n$ *with a potential function*

$$\chi(z_1, z_2, \dots, z_n) = \sum_{j=1}^{n} F_j(|z_j|^2).$$

If $\psi(z_1, \dots, z_n) = \tilde{\psi}(|z_1|^2, \dots, |z_n|^2)$ *is a multi-radial weight, then* $(\bar{\partial}\psi)^\sharp$ *is holomorphic if and only if*

$$\psi(z_1, \dots, z_n) = C_0 + \sum_{j=1}^{n} C_j |z_j|^2 F_j'(|z_j|^2).$$

If this is the case, then we obtain the real holomorphic vector field

$$h^{j\bar{k}} \frac{\partial \psi}{\partial \bar{z}^k} \frac{\partial}{\partial z^j} = \sum_{j=1}^{n} C_j z_j \frac{\partial}{\partial z^j}.$$

Next we consider a non-decoupled example on $\mathbb{C}^2$ with potential function

$$\chi(z_1, z_2) = \frac{1}{4}|z_1|^4 + |z_1|^2|z_2|^2 + |z_1|^2 + |z_2|^2.$$

In the standard coordinates of $\mathbb{C}^2$, the metric is given by the matrix

$$\left[h_{j\bar{k}}\right] = \begin{pmatrix} |z_1|^2 + |z_2|^2 + 1 & \bar{z}_1 z_2 \\ z_1 \bar{z}_2 & |z_1|^2 + 1 \end{pmatrix},$$

with the determinant

$$\delta = \det\left[h_{j\bar{k}}\right] = |z_1|^4 + 2|z_1|^2 + |z_2|^2 + 1.$$

Let $\psi(z_1, z_2) = \frac{|z_1|^4}{2} + |z_1|^2|z_2|^2 + |z_1|^2$.

The corresponding Bergman space

$$A^2_{(0,0)}(\mathbb{C}^2, h, e^{-\psi}) = \left\{ f : \mathbb{C}^2 \longrightarrow \mathbb{C} \text{ entire} : \int_{\mathbb{C}^2} |f|^2 e^{-\psi} \delta \, d\lambda < \infty \right\}$$

has an orthogonal basis consisting of the functions

$$\{ z_1^k z_2^\ell \ : k \in \mathbb{N}, k \geq 2, \ \ell \in \mathbb{Z}, 0 \leq \ell \leq k-2 \}.$$

It follows that the operator ∂ is densely defined and we have for $u = u_1 \, dz_1 + u_2 \, dz_2 \in \operatorname{dom}(\partial^*)$

$$\partial^* u = z_1 u_1.$$

Thus, the adjoint ∂^* “forgets” the z_2-variable, although the weight and the metric both depend on z_2. Let

$$v = v_{12} \, dz_1 \wedge dz_2 \in A^2_{(2,0)}(\mathbb{C}^2, h, e^{-\psi}).$$

Then, by the same computation as above, we get

$$\partial^* v = P_{h,\psi}(-\psi_{\bar{j}} v_{12} h^{2\bar{j}}) dz_1 + P_{h,\psi}(-\psi_{\bar{j}} v_{21} h^{1\bar{j}}) dz_2 = z_1 v_{12} \, dz_2.$$

So we obtain for $\tilde{\Box} = \partial^* \partial + \partial \partial^*$ and $u \in A^2_{(1,0)}(\mathbb{C}^2, h, e^{-\psi}) \cap \mathrm{dom}(\tilde{\Box})$ that

$$\tilde{\Box} u = \left(u_1 + z_1 \frac{\partial u_1}{\partial z_1} \right) dz_1 + z_1 \frac{\partial u_2}{\partial z_1} dz_2 .$$

Theorem

The operator

$$\tilde{\Box} : A^2_{(1,0)}(\mathbb{C}^2, h, e^{-\psi}) \longrightarrow A^2_{(1,0)}(\mathbb{C}^2, h, e^{-\psi})$$

is densely defined and its spectrum consists of point eigenvalues with finite multiplicities. Precisely, for $k = 1, 2, \ldots,$ the eigenvalues are $\lambda_k = k + 1$, with multiplicity $2k - 1$.

Conformally Kähler metrics

Theorem

Let (M, h) be a Kähler manifold of dimension $n \geqslant 2$ and let $g = \phi^{-1} h$ be a conformally Kähler metric. Then the following are equivalent:

1. *g has holomorphic torsion,*
2. *$(\bar{\partial}\phi)^\sharp$ is holomorphic.*

Theorem

Let $M = \mathbb{B}^n$ and let

$$g_{j\bar{k}} = \frac{\delta_{jk}}{1 - |z|^2}$$

be a conformally flat metric on $\mathbb{B}^n$. If ψ is a real-valued function on $\mathbb{B}^n$ such that $(\bar{\partial}\psi)^\sharp$ is holomorphic, then

$$\psi(z) = A + B \log(1 - |z|^2)$$

for some real constants A and B.

In the sequel, we consider $U(n)$-invariant Kähler metrics and radial weights. Suppose that $h_{j\bar{k}}$ is a Kählerian metric induced by a radial potential $h(z) = \tilde{h}(|z|^2)$, where $\tilde{h}(r)$ is a real-valued function of a real variable. Precisely, we have

$$h_{j\bar{k}} = \partial_j \partial_{\bar{k}} \tilde{h}(|z|^2) = \tilde{h}'(|z|^2)\, \delta_{jk} + \tilde{h}''(|z|^2)\, \bar{z}_j z_k. \tag{1}$$

Thus, $h_{j\bar{k}}$ is a rank-one perturbation of a multiple of the identity matrix. For $h_{j\bar{k}}$ to be positive definite, we assume that $\tilde{h}'(r) > 0$ and $r\tilde{h}''(r) + \tilde{h}'(r) > 0$.

Theorem

Let g be the conformally $U(n)$-invariant Kähler metric

$$g_{j\bar{k}} = e^{\tilde{\sigma}(|z|^2)}\partial_j\partial_{\bar{k}}\tilde{h}(|z|^2)$$

and $\psi(z) = \tilde{\psi}(|z|^2)$ be a real-valued radial weight function. The vector field $(\bar{\partial}\psi - \bar{\tau})^\sharp$ and the torsion operator $T^\sharp$ are holmorphic if and only if

$$\tilde{\sigma}(r) = -\log(C_2 r\tilde{h}'(r) + C_3)$$

and

$$\tilde{\psi}(r) = -C_4\log(C_2 r\tilde{h}'(r) + C_3) + C_5,$$

where $C_4 = n - 1 - (C_1/C_2)$ and the constant C_3 has to be chosen such that $C_2 r\tilde{h}'(r) + C_3 > 0$. In this case we have for the vector field $(\bar{\partial}\psi - \bar{\tau})^\sharp = C_1\sum_{j=1}^n z^j\partial_j$ and for the torsion operator

$$T^\sharp(v) = -C_2\sum_{q=1}^{n} z^q v_{pq} dz^p.$$