Regularity of CR maps into uniformly pseudoconvex hypersurfaces

Josef E. Greilhuber

University of Vienna

Obergurgl, August 2020

The regularity problem for CR maps

Under which conditions is a CR map $h : M \to M'$ between CR manifolds C^∞-smooth?

The regularity problem for CR maps

Under which conditions is a CR map $h : M \to M'$ between CR manifolds C^∞-smooth?

Example 0 (Lewy 1956)

If the Levi form of a hypersurface $M \subset \mathbb{C}^N$ has two nonzero eigenvalues of different signs, any CR function extends to a holomorphic function on both sides of M.

The regularity problem for CR maps

Under which conditions is a CR map $h : M \to M'$ between CR manifolds C^∞-smooth?

Example 0 (Lewy 1956)

If the Levi form of a hypersurface $M \subset \mathbb{C}^N$ has two nonzero eigenvalues of different signs, any CR function extends to a holomorphic function on both sides of M.

Example 1 (Forsterič 1989)

A CR map $h : \mathbb{S}^{2N-1} \to \mathbb{S}^{2N'-1}$ of class $C^{N'-N+1}$ extends to a rational map on a neighborhood of $\mathbb{S}^{2N-1} \subset \mathbb{C}^N$.

Examples of irregularity

If the source is pseudoconvex and the target is not, things can go *horribly* wrong!

Examples of irregularity

If the source is pseudoconvex and the target is not, things can go *horribly* wrong!

Example 2 (Berhanu & Xiao 2017)

If $M \subset \mathbb{C}^N$ is strongly pseudoconvex, then for any $k \in \mathbb{N}$ there exists a CR function ϕ which is C^k, but nowhere C^∞-smooth.

Examples of irregularity

If the source is pseudoconvex and the target is not, things can go *horribly* wrong!

Example 2 (Berhanu & Xiao 2017)

If $M \subset \mathbb{C}^N$ is strongly pseudoconvex, then for any $k \in \mathbb{N}$ there exists a CR function ϕ which is C^k, but nowhere C^∞-smooth.

Sketch of proof: At any $p \in M$, take Z_p with $\Re(Z_p) > 0$ on $M \backslash \{p\}$ and $Z_p(p) = 0$. Then $Z_p^{\kappa+\frac{1}{2}}$ (standard branch cut) is C^κ but not $C^{\kappa+1}$ on M.

Examples of irregularity

If the source is pseudoconvex and the target is not, things can go *horribly* wrong!

Example 2 (Berhanu & Xiao 2017)

If $M \subset \mathbb{C}^N$ is strongly pseudoconvex, then for any $k \in \mathbb{N}$ there exists a CR function ϕ which is C^k, but nowhere C^∞-smooth.

Sketch of proof: At any $p \in M$, take Z_p with $\Re(Z_p) > 0$ on $M \backslash \{p\}$ and $Z_p(p) = 0$. Then $Z_p^{\kappa+\frac{1}{2}}$ (standard branch cut) is C^κ but not $C^{\kappa+1}$ on M. For a dense sequence $(p_\kappa)_{\kappa=k}^\infty$ and rapidly decaying $\lambda_\kappa \in \mathbb{C}$, the sum $\sum_\kappa \lambda_\kappa Z_{p_\kappa}^{\kappa+\frac{1}{2}}$ works, by a Cantor diagonal argument.

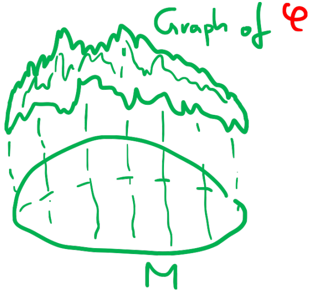
Examples of irregularity

If the source is pseudoconvex and the target is not, things can go *horribly* wrong!

Example 2 (Berhanu & Xiao 2017)

If $M \subset \mathbb{C}^N$ is strongly pseudoconvex, then for any $k \in \mathbb{N}$ there exists a CR function ϕ which is C^k, but nowhere C^∞-smooth.

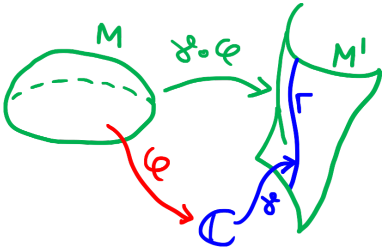
Sketch of proof: At any $p \in M$, take Z_p with $\Re(Z_p) > 0$ on $M \setminus \{p\}$ and $Z_p(p) = 0$. Then $Z_p^{\kappa+\frac{1}{2}}$ (standard branch cut) is C^κ but not $C^{\kappa+1}$ on M. For a dense sequence $(p_\kappa)_{\kappa=k}^\infty$ and rapidly decaying $\lambda_\kappa \in \mathbb{C}$, the sum $\sum_\kappa \lambda_\kappa Z_{p_\kappa}^{\kappa+\frac{1}{2}}$ works, by a Cantor diagonal argument.



Examples of irregularity

Example 3

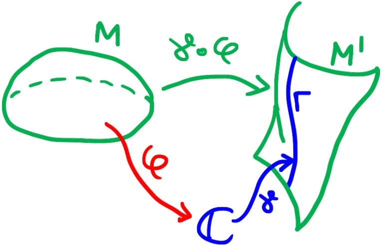
If $M' \in \mathbb{C}^{N'}$ contains a complex curve parametrized by γ, then $h := \gamma \circ \phi : M \to M'$ is a C^k, but nowhere C^∞-smooth CR map.



Examples of irregularity

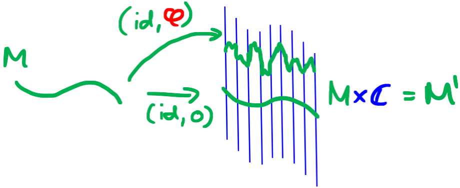
Example 3

If $M' \in \mathbb{C}^{N'}$ contains a complex curve parametrized by γ, then $h := \gamma \circ \phi : M \to M'$ is a C^k, but nowhere C^∞-smooth CR map.



Example 4

If $M' = M \times \mathbb{C}$, the CR map $h : M \to M'$ given by $h(q) = (q, \phi(q))$ is a C^k, but nowhere C^∞-smooth map.



Some necessary definitions

For $X \subseteq \mathbb{C}^{N'}$, let $\mathscr{I}_X(p') \subseteq C^\infty(\mathbb{C}^{N'}, p')$ be the vanishing ideal of X at p.

Some necessary definitions

For $X \subseteq \mathbb{C}^{N'}$, let $\mathscr{I}_X(p') \subseteq C^\infty(\mathbb{C}^{N'}, p')$ be the vanishing ideal of X at p.

Complex gradients and their derivatives

Consider a CR map $h : M \to M' \subseteq \mathbb{C}^{N'}_w$. We define

$$
\begin{aligned}
r_0(p) &:= \dim_{\mathbb{C}} \left\langle \left\{ \rho_w \circ h(p) : \rho \in \mathscr{I}_{h(M)}(h(p)) \right\} \right\rangle, \\
r_k(p) &:= \dim_{\mathbb{C}} \left\langle \left\{ \bar{L}_1 \dots \bar{L}_j (\rho_w \circ h)(p) : \right. \right. \\
&\qquad \left. \left. \rho \in \mathscr{I}_{h(M)}(h(p)), \bar{L}_1, \dots, \bar{L}_j \in \mathcal{V}_p(M), 0 \le j \le k \right\} \right\rangle,
\end{aligned}
$$

Some necessary definitions

For $X \subseteq \mathbb{C}^{N'}$, let $\mathscr{I}_X(p') \subseteq C^\infty(\mathbb{C}^{N'}, p')$ be the vanishing ideal of X at p.

Complex gradients and their derivatives

Consider a CR map $h : M \to M' \subseteq \mathbb{C}^{N'}_w$. We define

$$r_0(p) := \dim_{\mathbb{C}} \left\langle \left\{ \rho_w \circ h(p) : \rho \in \mathscr{I}_{h(M)}(h(p)) \right\} \right\rangle,$$
$$r_k(p) := \dim_{\mathbb{C}} \left\langle \left\{ \bar{L}_1 \dots \bar{L}_j (\rho_w \circ h)(p) : \right.\right.$$
$$\left.\left. \rho \in \mathscr{I}_{h(M)}(h(p)), \bar{L}_1, \dots, \bar{L}_j \in \mathcal{V}_p(M), 0 \le j \le k \right\} \right\rangle,$$

Formal complex manifolds

A *formal complex manifold tangential to infinite order to* $X \subseteq \mathbb{C}^{N'}$ *at* p' is a power series $\Gamma \in \mathbb{C}[[t_1, \dots, t_k]]^{N'}$ such that $\gamma_0 = p'$, $\mathrm{rk}(\gamma_t) = k$ and such that $\mathcal{T}\rho \circ \Gamma \equiv 0$ as power series for $\rho \in \mathscr{I}_X(p')$.

Theorem (Lamel & Mir 2018)

Let $M \subset \mathbb{C}^N$ be a minimal CR submanifold and $h : M \to \mathbb{C}^{N'}$ be a CR map of class $C^{N'-k+l}$ for some $k, l \in \mathbb{N}$. Assume that $r_k \geq l$.

Then there exists a dense open subset $O \subseteq \mathrm{SingSupp}(h)^\circ$ such that for every $p \in O$, there exists a neighborhood $V \subseteq O$ of p, an integer $r \geq 1$ and a C^1-smooth CR family of formal complex submanifolds $(\Gamma_\xi)_{\xi \in V}$ of dimension r through $h(V)$ for which Γ_ξ is tangential to infinite order to $h(M)$ at $h(\xi)$, for every $\xi \in V$.

In particular, $h(O) \subseteq \mathcal{E}_{h(M)}$.

The Levi null space

The *Levi null space* $\mathcal{N} \subseteq T^{0,1}M$ of a CR manifold M is given by $\mathcal{N}_p = \{\bar{L}|_p : [\bar{L}, \Gamma] \in \Gamma_p(T^{0,1}M \oplus T^{1,0}M) \text{ for all } \Gamma \in \Gamma_p(T^{1,0}M)\}$.

The Levi foliation (Sommer 1959, Freeman 1974)

Let M be a CR submanifold such that $\mathcal{N}$ has constant rank across M. Then there exists a foliation η of M by complex manifolds such that $\mathcal{N}_p = T_p^{0,1}\eta$ at all $p \in M$.

Uniformly pseudoconvex hypersurfaces

The Levi null space

The *Levi null space* $\mathcal{N} \subseteq T^{0,1}M$ of a CR manifold M is given by $\mathcal{N}_p = \{\bar{L}|_p : [\bar{L}, \Gamma] \in \Gamma_p(T^{0,1}M \oplus T^{1,0}M)$ for all $\Gamma \in \Gamma_p(T^{1,0}M)\}$.

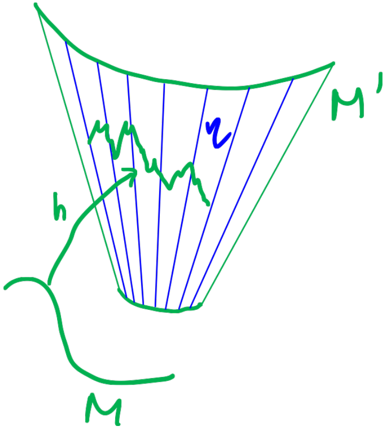
The Levi foliation (Sommer 1959, Freeman 1974)

Let M be a CR submanifold such that $\mathcal{N}$ has constant rank across M. Then there exists a foliation η of M by complex manifolds such that $\mathcal{N}_p = T_p^{0,1}\eta$ at all $p \in M$.

Sketch of sketch of proof. Let $N = \Re(\mathcal{N})$. Check that $\Gamma_q(N) = \Gamma_q(T^cM) \cap \mathfrak{n}(\Gamma_q(T^cM))$, a Lie subalgebra of $\Gamma_q(TM)$. Frobenius theorem $\Rightarrow$ Existence of η. As $N_q \subseteq T_q\mathbb{C}^n$ are complex subspaces, η consists of complex manifolds.

CR maps into uniformly pseudoconvex hypersurfaces

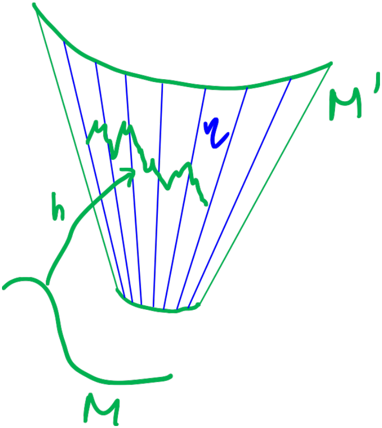
We'll consider M' uniformly pseudoconvex with Levi foliation η, and look at maps $h : M \to M'$.



CR maps into uniformly pseudoconvex hypersurfaces

We'll consider M' uniformly pseudoconvex with Levi foliation η, and look at maps $h : M \to M'$.

In view of the formal foliation theorem, the existence of the Levi foliation might spell doom!

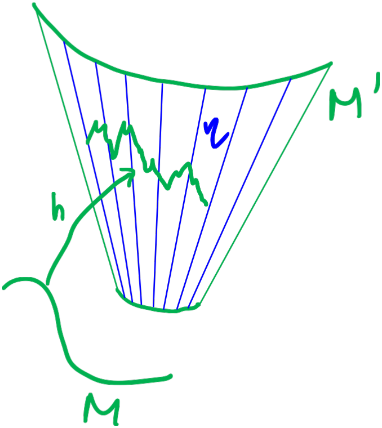


CR maps into uniformly pseudoconvex hypersurfaces

We'll consider M' uniformly pseudoconvex with Levi foliation η, and look at maps $h : M \to M'$.

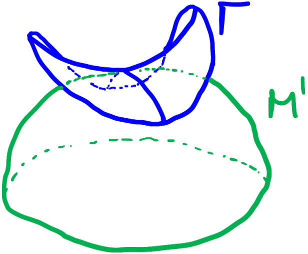
In view of the formal foliation theorem, the existence of the Levi foliation might spell doom!

But pseudoconvexity makes it easy to understand tangency of (formal) complex manifolds.



Lemma 1

If a formal complex curve $\Gamma = p' + t\gamma_t + \frac{t^2}{2}\gamma_{tt} + \dots$ is tangential to infinite order to M', then $\gamma_t \in T\eta$.

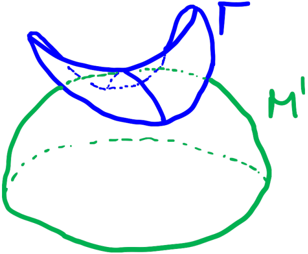


CR maps into uniformly pseudoconvex hypersurfaces

Lemma 1

If a formal complex curve $\Gamma = p' + t\gamma_t + \frac{t^2}{2}\gamma_{tt} + \dots$ is tangential to infinite order to M', then $\gamma_t \in T\eta$.

We get a CR map $\gamma_t : M \to T\eta$!

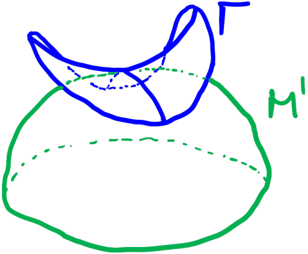


CR maps into uniformly pseudoconvex hypersurfaces

Lemma 1

If a formal complex curve $\Gamma = p' + t\gamma_t + \frac{t^2}{2}\gamma_{tt} + \dots$ is tangential to infinite order to M', then $\gamma_t \in T\eta$.

We get a CR map $\gamma_t : M \to T\eta$!



Lemma 2

The map R defined by $R(\bar{L}, \psi) = \mathbb{P}_{T^\perp\eta}(\bar{L}\psi)$ for $\bar{L} \in \Gamma(T^{0,1}M')$ and $\psi \in \Gamma(T\eta)$ is a tensor, and $T_p^{0,1}\eta \subseteq \ker R_p(\cdot, \psi)$.

CR maps into uniformly pseudoconvex hypersurfaces

Proposition 1

Let M' be a uniformly pseudoconvex hypersurface with Levi foliation η, M be a minimal CR submanifold and $h : M \to M'$ be a C^1 CR map. Suppose there exists $p \in M$ mapped to $p' := h(p)$, and a C^1 CR family of formal complex curves $(\Gamma_q)_{q \in O}$ defined on a neighborhood $O \subseteq M$ of p such that Γ_q is tangential to second order to M' at $h(q)$ for all $q \in O$. Define R as in Lemma 2.

Then $\gamma_t(p) \in T_{p'}\eta$ and $h_* T_p^{0,1} M \subseteq \ker R_{p'}(\cdot, \gamma_t(p))$.

CR maps into uniformly pseudoconvex hypersurfaces

Proposition 1

Let M' be a uniformly pseudoconvex hypersurface with Levi foliation η, M be a minimal CR submanifold and $h : M \to M'$ be a C^1 CR map. Suppose there exists $p \in M$ mapped to $p' := h(p)$, and a C^1 CR family of formal complex curves $(\Gamma_q)_{q \in O}$ defined on a neighborhood $O \subseteq M$ of p such that Γ_q is tangential to second order to M' at $h(q)$ for all $q \in O$. Define R as in Lemma 2.

Then $\gamma_t(p) \in T_{p'}\eta$ and $h_* T_p^{0,1} M \subseteq \ker R_{p'}(\cdot, \gamma_t(p))$.

Sketch of proof. If $h_* \bar{L}|_p = 0$, nothing to show. If not, there's a 2D slice $S \subset M$ through p with $T_p^{1,0} S = \left\langle \bar{L}|_p \right\rangle_{\mathbb{C}}$, which is immersed into M' $\to$ differentiate γ_t along that.

CR maps into uniformly pseudoconvex hypersurfaces

Measuring the size of the kernel

For $q \in M'$, define

$$\nu_q = \max_{0 \neq V \in T_q \eta} \dim_{\mathbb{C}} \ker R_q(\cdot, V) - \dim_{\mathbb{C}} \eta$$

CR maps into uniformly pseudoconvex hypersurfaces

Measuring the size of the kernel

For $q \in M'$, define

$$\nu_q = \max_{0 \neq V \in T_q \eta} \dim_{\mathbb{C}} \ker R_q(\cdot, V) - \dim_{\mathbb{C}} \eta$$

The function $q \to \nu_q$ is upper semicontinuous on M'.

Measuring the size of the kernel

For $q \in M'$, define

$$\nu_q = \max_{0 \neq V \in T_q \eta} \dim_{\mathbb{C}} \ker R_q(\cdot, V) - \dim_{\mathbb{C}} \eta$$

The function $q \to \nu_q$ is upper semicontinuous on M'.

Proof. Let $K := \dim_{\mathbb{C}} \eta$. Then $\nu_p < \ell \Leftrightarrow \dim_{\mathbb{C}} \ker R_p(\cdot, V) \leq \ell + K$

CR maps into uniformly pseudoconvex hypersurfaces

Measuring the size of the kernel

For $q \in M'$, define

$$\nu_q = \max_{0 \neq V \in T_q \eta} \dim_{\mathbb{C}} \ker R_q(\cdot, V) - \dim_{\mathbb{C}} \eta$$

The function $q \to \nu_q$ is upper semicontinuous on M'.

Proof. Let $K := \dim_{\mathbb{C}} \eta$. Then $\nu_p < \ell \Leftrightarrow \dim_{\mathbb{C}} \ker R_p(\cdot, V) \leq \ell + K$
$\Leftrightarrow$ For every $V \in T_p \eta$, *some* $(N' - 1 - \ell - K)$-minor of $R(\cdot, V)$ is nonzero.

CR maps into uniformly pseudoconvex hypersurfaces

Measuring the size of the kernel

For $q \in M'$, define

$$\nu_q = \max_{0 \neq V \in T_q \eta} \dim_{\mathbb{C}} \ker R_q(\cdot, V) - \dim_{\mathbb{C}} \eta$$

The function $q \to \nu_q$ is upper semicontinuous on M'.

Proof. Let $K := \dim_{\mathbb{C}} \eta$. Then $\nu_p < \ell \Leftrightarrow \dim_{\mathbb{C}} \ker R_p(\cdot, V) \leq \ell + K$
$\Leftrightarrow$ For every $V \in T_p \eta$, *some* $(N' - 1 - \ell - K)$-minor of $R(\cdot, V)$ is nonzero.
$\Leftrightarrow$ Square sum $\sum_j |m_j(q, V)|^2$ over all minors nonvanishing on $\mathbb{S}^{2K-1}$.

Measuring the size of the kernel

For $q \in M'$, define

$$\nu_q = \max_{0 \neq V \in T_q \eta} \dim_{\mathbb{C}} \ker R_q(\cdot, V) - \dim_{\mathbb{C}} \eta$$

The function $q \to \nu_q$ is upper semicontinuous on M'.

Proof. Let $K := \dim_{\mathbb{C}} \eta$. Then $\nu_p < \ell \Leftrightarrow \dim_{\mathbb{C}} \ker R_p(\cdot, V) \leq \ell + K$
$\Leftrightarrow$ For every $V \in T_p \eta$, *some* $(N' - 1 - \ell - K)$-minor of $R(\cdot, V)$ is nonzero.
$\Leftrightarrow$ Square sum $\sum_j |m_j(q, V)|^2$ over all minors nonvanishing on $\mathbb{S}^{2K-1}$.
$\Leftrightarrow$ By compactness of $\mathbb{S}^{2K-1}$, for a neighborhood O of q, $\sum_j |m_j(q, V)|^2$ is nonzero for $(q, V) \in O \times \mathbb{S}^{2K-1}$.

CR maps into uniformly pseudoconvex hypersurfaces

Measuring the size of the kernel

For $q \in M'$, define

$$\nu_q = \max_{0 \neq V \in T_q\eta} \dim_{\mathbb{C}} \ker R_q(\cdot, V) - \dim_{\mathbb{C}} \eta$$

The function $q \to \nu_q$ is upper semicontinuous on M'.

Proof. Let $K := \dim_{\mathbb{C}} \eta$. Then $\nu_p < \ell \Leftrightarrow \dim_{\mathbb{C}} \ker R_p(\cdot, V) \leq \ell + K$
$\Leftrightarrow$ For every $V \in T_p\eta$, *some* $(N' - 1 - \ell - K)$-minor of $R(\cdot, V)$ is nonzero.
$\Leftrightarrow$ Square sum $\sum_j |m_j(q, V)|^2$ over all minors nonvanishing on $\mathbb{S}^{2K-1}$.
$\Leftrightarrow$ By compactness of $\mathbb{S}^{2K-1}$, for a neighborhood O of q, $\sum_j |m_j(q, V)|^2$ is nonzero for $(q, V) \in O \times \mathbb{S}^{2K-1}$. $\Leftrightarrow \nu_q < \ell$ for all $q \in O$.

CR maps into uniformly pseudoconvex hypersurfaces

Proposition 2

Let $M' \subset \mathbb{C}^{N'}$ be a uniformly pseudoconvex hypersurface with Levi foliation η, M be a *minimal* CR manifold and $h : M \to M'$ be a $C^{N'-1}$-regular CR map $h : M \to M'$ mapping $p \in M$ to $p' \in M'$.

If $\nu_{p'} = 0$, then there exists an open neighborhood O of p such that each connected open set $\tilde{O} \subseteq \mathrm{SingSupp}(h)^\circ$ is mapped into a single leaf $\eta_{h(q)}$.

CR maps into uniformly pseudoconvex hypersurfaces

Proposition 2

Let $M' \subset \mathbb{C}^{N'}$ be a uniformly pseudoconvex hypersurface with Levi foliation η, M be a *minimal* CR manifold and $h : M \to M'$ be a $C^{N'-1}$-regular CR map $h : M \to M'$ mapping $p \in M$ to $p' \in M'$.

If $\nu_{p'} = 0$, then there exists an open neighborhood O of p such that each connected open set $\tilde{O} \subseteq \mathrm{SingSupp}(h)^\circ$ is mapped into a single leaf $\eta_{h(q)}$.

Sketch of proof. Choose $O' \subseteq M'$ s.t. $\nu_{O'} = 0$ and let $O = h^{-1}(O')$. Suppose $\tilde{O} \subseteq \mathrm{SingSupp}(h)^\circ \cap O$ is connected. By prop. 1 and the formal foliation theorem, $h_* T^{0,1}\tilde{O} \subseteq T\eta$ *on a dense open subset*, but that is a closed property, thus $h_* T^{0,1}\tilde{O} \subseteq T\eta$.

CR maps into uniformly pseudoconvex hypersurfaces

Proposition 2

Let $M' \subset \mathbb{C}^{N'}$ be a uniformly pseudoconvex hypersurface with Levi foliation η, M be a *minimal* CR manifold and $h : M \to M'$ be a $C^{N'-1}$-regular CR map $h : M \to M'$ mapping $p \in M$ to $p' \in M'$.

If $\nu_{p'} = 0$, then there exists an open neighborhood O of p such that each connected open set $\tilde{O} \subseteq \mathrm{SingSupp}(h)^\circ$ is mapped into a single leaf $\eta_{h(q)}$.

Sketch of proof. Choose $O' \subseteq M'$ s.t. $\nu_{O'} = 0$ and let $O = h^{-1}(O')$. Suppose $\tilde{O} \subseteq \mathrm{SingSupp}(h)^\circ \cap O$ is connected. By prop. 1 and the formal foliation theorem, $h_* T^{0,1}\tilde{O} \subseteq T\eta$ *on a dense open subset*, but that is a closed property, thus $h_* T^{0,1}\tilde{O} \subseteq T\eta$.
For generic $q \in \tilde{O}$, $h^{-1}(h(q))$ is a manifold integrating $T^{0,1}M$, thus minimality implies that $h^{-1}(h(q))$ is open $\Rightarrow h_* T_q\tilde{O} \in T_q\eta$. *Thats a closed property too.* Thus $h_* T\tilde{O} \subseteq T\eta$, finish by connectedness of $\tilde{O}$. □

CR maps into uniformly pseudoconvex hypersurfaces

Proposition 3

Let $M' \subset \mathbb{C}^{N'}$ be a uniformly pseudoconvex hypersurface with Levi foliation η, M be a *pseudoconvex* hypersurface with at least n_+ positive Levi eigenvalues and $h : M \to M'$ be a $C^{N'-n_+}$-regular *CR transversal* CR map mapping $p \in M$ to $p' \in M'$.

If $\nu_{p'} < n_+$, then h is C^∞-smooth on a dense open subset of some neighborhood of p.

CR maps into uniformly pseudoconvex hypersurfaces

Proposition 3

Let $M' \subset \mathbb{C}^{N'}$ be a uniformly pseudoconvex hypersurface with Levi foliation η, M be a *pseudoconvex* hypersurface with at least n_+ positive Levi eigenvalues and $h : M \to M'$ be a $C^{N'-n_+}$-regular *CR transversal* CR map mapping $p \in M$ to $p' \in M'$.

If $\nu_{p'} < n_+$, then h is C^∞-smooth on a dense open subset of some neighborhood of p.

Sketch of proof. Choose $O' \subseteq M'$ s.t. $\nu_{O'} < n_+$ and let $O = h^{-1}(O')$. CR transversality and strong pseudoconvexity of a $(1 + 2n_+)$-dimensional slice of M through q yield first $r_1 > n_+$ and then $\dim_{\mathbb{C}} h_* T_q^{0,1} M \geq n_+$ at any $q \in M$, contradicting prop. 1 and the formal foliation theorem on O. □

Maps into boundaries of classical symmetric domains

Bounded symmetric domain

A bounded domain $\Omega \subseteq \mathbb{C}^N$ s.t. at each $p \in \Omega$ there exists a biholomorphic involution with p as isolated fixed point.

Maps into boundaries of classical symmetric domains

Bounded symmetric domain

A bounded domain $\Omega \subseteq \mathbb{C}^N$ s.t. at each $p \in \Omega$ there exists a biholomorphic involution with p as isolated fixed point.

Irreducible bounded symmetric domains fall into four series of *classical symmetric domains* and two exceptional domains (E. Cartan 1935).

Maps into boundaries of classical symmetric domains

Bounded symmetric domain

A bounded domain $\Omega \subseteq \mathbb{C}^N$ s.t. at each $p \in \Omega$ there exists a biholomorphic involution with p as isolated fixed point.

Irreducible bounded symmetric domains fall into four series of *classical symmetric domains* and two exceptional domains (E. Cartan 1935).

Computing ν on the boundary of classical symmetric domains

$D_I^{m,n} \subseteq \mathbb{C}^{m\times n}$:	$n_+ = m+n-2$	$\nu = m+n-4$
$D_{II}^m \subseteq \mathbb{C}^{m(m-1)/2}$:	$n_+ = 2m-4$	$\nu = 2m-8$
$D_{III}^m \subseteq \mathbb{C}^{m(m+1)/2}$:	$n_+ = m-1$	$\nu = m-1$
$D_{IV}^m \subseteq \mathbb{C}^m$:	$n_+ = m-2$	$\nu = 0$