# On a Theorem of Métivier

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# Preliminaries

$$U \subseteq \mathbb{R}^n \qquad \ldots \text{open set}$$

$$P(x, D) = \sum_{|\alpha| \leq d} a_\alpha(x) D^\alpha \qquad \ldots \text{PDO with analytic coeff. } a_\alpha \in \mathcal{A}(U)$$

$$p(x, \xi) = \sum_{|\alpha| \leq d} a_\alpha(x) \xi^\alpha \qquad \ldots \text{symbol of } P$$

$$p_d(x, \xi) = \sum_{|\alpha| = d} a_\alpha(x) \xi^\alpha \qquad \ldots \text{principal symbol of } P$$

# Analytic vectors

A distribution $u \in \mathcal{D}'(U)$ is an analytic vector of $P$ if

$$\forall\, V \Subset U \;\exists\, C, h > 0 : \;\; \left\|P^k u\right\|_{L^2(V)} \leq C h^k (dk)! \qquad \forall\, k \in \mathbb{N}_0.$$

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**Theorem (Kotake, Narasimhan 1962, Komatsu 1962)**

*Let $P$ be an elliptic operator with analytic coefficients. Then*

$$\mathcal{A}(U; P) = \mathcal{A}(U).$$

# Gevrey vectors

Let $s \geq 1$. The Gevrey class $\mathcal{G}^s(U)$ is defined by

$$f \in \mathcal{G}^s(U) :\Longleftrightarrow \ \forall\, V \Subset U \ \exists\, C, h > 0 \ \forall\, \alpha \in \mathbb{N}_0^n : \\ \sup_{x \in V} |D^\alpha f(x)| \leq C h^{|\alpha|} |\alpha|!^s.$$

The space of Gevrey vectors is given by

$$u \in \mathcal{G}^s(U; P) :\Longleftrightarrow \ \forall\, V \Subset U \ \exists\, C, h > 0 \ \forall\, k \in \mathbb{N}_0 : \\ \left\| P^k u \right\|_{L^2(V)} \leq C h^k (dk)!^s.$$

# Ultradifferentiable vectors

## Weight sequences

$\mathbf{M} = (M_k)_k$ is a weight sequence if $M_0 = 1$,

$$M_k^2 \leq M_{k-1}M_{k+1} \quad \text{and} \quad \sqrt[k]{M_k} \longrightarrow \infty.$$

The Denjoy-Carleman class $\mathcal{E}^{\{\mathbf{M}\}}(U)$ is defined by

$$f \in \mathcal{E}^{\{\mathbf{M}\}}(U) :\Longleftrightarrow \ \forall\, V \Subset U \ \exists\, C, h > 0 \ \forall\, \alpha \in \mathbb{N}_0^n : \sup_{x \in V} |D^\alpha f(x)| \leq Ch^{|\alpha|} M_{|\alpha|}.$$

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## Semiregular weight sequences

A weight sequence **M** is semiregular if

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## Theorem (Bolley-Camus-Mattera 1979)

*Let $\mathbf{M}$ be a semiregular weight sequence and $P$ be an elliptic differential operator with analytic coefficients. Then*

$$\mathcal{E}^{\{\mathbf{M}\}}(U;P) = \mathcal{E}^{\{\mathbf{M}\}}(U).$$

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**Goal:** Generalize Métivier's Theorem to other weight sequences.

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- Choose $\varphi \in \mathcal{G}^{s_0}(\mathbb{R}^n)$ with $\operatorname{supp}\varphi \subseteq \{x \in \mathbb{R}^n :\ |x| < 2\delta\}$ and $\varphi(x) = 1$ for $|x| < \delta$.

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## Ansatz

$$u(x) = \int_1^\infty \varphi\big(t^\varepsilon(x - x_0)\big)\, e^{-t^{1/s'}}\, e^{it\xi_0(x-x_0)}\, dt.$$

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- Hence $D^k_{\xi_0} u(x_0) = s'\Gamma(s'(k+1)) - b_k$, where $b_k = \int_0^1 t^k e^{1/s'}\,dt \to 0$ for $k \to \infty$.

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Question: How to define $u$ in the $q$-Gevrey case?
Hint: $s'\Gamma(s'k) = \int_0^\infty t^{k-1} e^{-t^{1/s'}}$ is the *Mellin transform* of the function $t \mapsto e^{-t^{1/s'}}$.

# The $q$-Gevrey scale

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Hence

$$\theta(z, \lambda) = \int_0^\infty t^{z-1} \Theta(t, \lambda)\, dt,$$

$$N_k^\lambda = \int_0^\infty t^{k-1} \Theta(t, \lambda)\, dt.$$

# Proof of Main Theorem: Part 1

Let $x_0 \in U$, $\xi_0 \in S^{n-1}$ be such that $p_d(x_0, \xi_0) = 0$ and define $\delta > 0$ and $B_0$ as before.

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- $\psi \in \mathcal{E}^{\{\mathbf{N}^{\lambda_0}\}}(\mathbb{R}^n)$ such that $\operatorname{supp} \psi \subseteq \{x \in \mathbb{R}^n : \ |x| < 2\delta\}$ and $\psi(x) = 1$ for $|x| < \delta$.

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$$D^k_{\xi_0} u(x_0) = \int_1^\infty t^k \Theta(t, \lambda')\, dt = N^{\lambda'}_{k+1} - \int_0^1 t^k \Theta(t, \lambda')\, dt.$$

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$\rightsquigarrow$ $u \notin \mathcal{E}^{\{\mathbf{N}^\tau\}}(U)$ for all $\tau < \lambda'$.

# The Iterates of P

$$P^k u(x) = \int_1^\infty Q_k(x,t)\Theta(t,\lambda')e^{-it(x-x_0)\xi_0}\,dt$$

where

$$Q_0(x,t) = \psi\left(t^\varepsilon(x-x_0)\right)$$

and

$$Q_{k+1}(x,t) = \sum_{|\alpha|\le d} \frac{1}{\alpha!}\partial_\xi^\alpha p(x,t\xi_0)D_x^\alpha Q_k(x,t), \quad k\in\mathbb{N}_0.$$

# Estimates for $Q_k$

## Theorem

$\exists\, A > 0$ *s.t.* $\forall\, k \in \mathbb{N}_0 \;\; \forall\, \nu \in \mathbb{N}_0^n, \;\; \forall\, x \in B_0 \;\; \forall\, t \geq 1$:

$$|D_x^\nu Q_k(x,t)| \leq C_0 \left(h_0 t^\varepsilon\right)^{|\nu|} A^k \left[t^{(d-\varepsilon)k} N_{|\nu|}^{\lambda_0} + t^{(2d-1)k\varepsilon} N_{|\nu|+dk}^{\lambda_0}\right].$$

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# Associated weight functions

If $\mathbf{M}$ is a weight sequence then the associated weight function $\omega_{\mathbf{M}}$ is defined by

$$\omega_{\mathbf{M}}(t) = \sup_{k \in \mathbb{N}_0} \log \frac{t^k}{M_k}.$$

In particular

$$t^k \leq M_k e^{\omega_{\mathbf{M}}(t)} \tag{$\star$}$$

for all $k \in \mathbb{N}_0$ and $t \geq 0$.

# Weight functions

A weight function in the sense of Braun-Meise-Taylor is a continuous and increasing function $\omega : [0,\infty) \to [0,\infty)$ satisfying

- $\omega(t) = 0$ for $t \in [0,1]$,
- $\omega(2t) = O(\omega(t))$ for $t \to \infty$,
- $\log t = O(\omega(t))$ for $t \to \infty$,
- $\varphi_\omega = \omega \circ \exp$ is convex.

Let $\varphi^*_\omega(t) = \sup_{s\geq 0}(st - \varphi_\omega(s))$ be the conjugate function of $\varphi$. The ultradifferentiable class associated to $\omega$ is given by

$$f \in \mathcal{E}^{\{\omega\}}(U) :\Longleftrightarrow \ \forall\, V \Subset U \ \exists\, C, h > 0 \ \forall\, \alpha \in \mathbb{N}_0^n :$$
$$\sup_{x\in V} |D^\alpha f(x)| \leq C e^{h^{-1}\varphi^*_\omega(h|\alpha|)}$$

# Weight matrices

A weight matrix $\mathfrak{M}$ is a family of weight sequences such that for all $\mathbf{M}, \mathbf{N} \in \mathfrak{M}$ we have either $M_k \leq N_k$ for all $k$ or $N_k \leq M_k$ for all $k$. The ultradifferentiable associated to $\mathfrak{M}$ is defined by

$$f \in \mathcal{E}^{\{\mathfrak{M}\}}(U) :\Longleftrightarrow \ \forall\, V \Subset U \ \exists\, \mathbf{M} \in \mathfrak{M} \ \exists\, C, h > 0 \ \forall\, \alpha \in \mathbb{N}_0^n : \sup_{x \in V} |D^\alpha f(x)| \leq C h^{|\alpha|} M_{|\alpha|}$$

# Weight functions vs. Weight matrices

Let $\omega$ be a weight function. The weight matrix $\mathfrak{W} = \{\mathbf{W}^\rho : \rho > 0\}$ associated to $\omega$ is defined by

$$W_k^\rho = \exp\left[\rho^{-1}\varphi_\omega^*(\rho k)\right].$$

Then $\mathcal{E}^{\{\omega\}}(U) = \mathcal{E}^{\{\mathfrak{W}\}}(U)$ as topological vector spaces.

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## Lemma

*Let $\omega$ be a weight function and $\mathfrak{W} = \{\mathbf{W}^\rho : \rho > 0\}$ the weight matrix associated to $\omega$. Then*

$$\omega_{\mathbf{W}^\rho}(t) \leq \frac{\omega(t)}{\rho} \qquad \forall\, t > 0, \;\; \forall\, \rho > 0.$$

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$$\mathbf{N}^\lambda = \mathbf{W}^{2,4\lambda} \qquad \forall\, \lambda > 0.$$

Hence

$$\omega_{\mathbf{N}^\lambda}(t) \leq \frac{(\log t)^2}{4\lambda}, \qquad \forall\, t \geq 1, \quad \forall\, \lambda > 0. \qquad (\diamond)$$

# Final estimates I

Set $r = t^{1-\varepsilon/d}$, $R = t^{\varepsilon(2-1/d)}$ for $t \geq 1$ and $\lambda_1 = \lambda - \lambda_0 > 0$.

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$\rightsquigarrow$ $(\star)$ and $(\diamond)$ $\Rightarrow$

$$t^{(d-\varepsilon)k} = r^{dk} \leq N_{dk}^{\lambda} \exp\left[\frac{(d-\varepsilon)^2}{d^2\lambda}\frac{(\log t)^2}{4}\right]$$
$$t^{(2d-1)k\varepsilon} = R^{dk} \leq N_{dk}^{\lambda_1} \exp\left[\frac{\varepsilon^2(2d-1)^2}{d^2\lambda_1}\frac{(\log t)^2}{4}\right]$$

# Fixing the constants

For fixed $\lambda > 0$ choose $\varepsilon$ and $\lambda_0 < \lambda$ such that

$$0 < \varepsilon \leq \frac{d\sqrt{\lambda - \lambda_0}}{\sqrt{\lambda - \lambda_0} + \sqrt{\lambda}(2d - 1)} < \frac{1}{2}$$

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$$\implies \qquad \frac{\varepsilon^2(2d-1)^2}{(\lambda - \lambda_0)d^2} \leq \frac{(d-\varepsilon)^2}{\lambda d^2}$$

# Final estimates II

The integral

$$\int_1^\infty \left(t^{(d-\varepsilon)k} + t^{(2d-1)k\varepsilon} N_{dk}^{\lambda_0}\right) \exp\left[-\frac{(\log t)^2}{4\lambda'}\right] dt$$
$$\leq N_{dk}^{\lambda} \int_1^\infty \exp\left[\frac{(\log t)^2}{4}\left(\frac{(d-\varepsilon)^2}{\lambda d^2} - \frac{1}{\lambda'}\right)\right] dt$$

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$$\Longrightarrow \qquad \left\| P^k u \right\|_{L^2(B_0)} \leq C A^k N_{dk}^{\lambda}$$

Since $u \in \mathcal{D}(B_0)$ we have shown that $u \in \mathcal{E}^{\{\mathbf{N}^\lambda\}}(U;P)$.