Some remarks on the global distribution of the points of finite D'Angelo type

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joint work in progress with

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Warnings

This talk contains:

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- Background material

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Warnings

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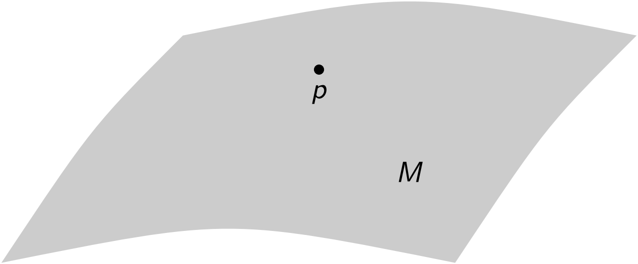
- More questions

D'Angelo type

Let M be a smooth real hypersurface in $\mathbb{C}^n$.

Recall the definition of **type** of M at a point p given by D'Angelo.



It measures the maximum order of contact of M at p with (possibly singular) holomorphic curves.

D'Angelo type

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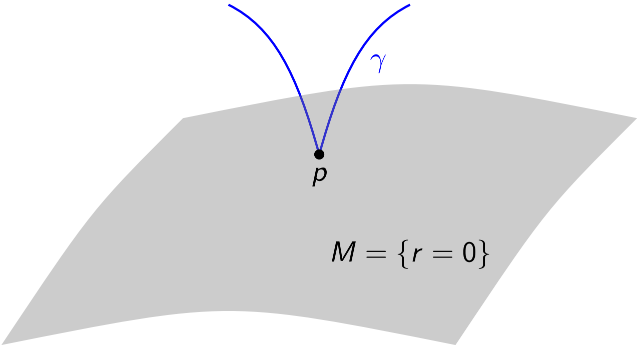
For every germ of a holomorphic curve γ at p consider $\frac{\nu(r \circ \gamma)}{\nu(\gamma)}$.



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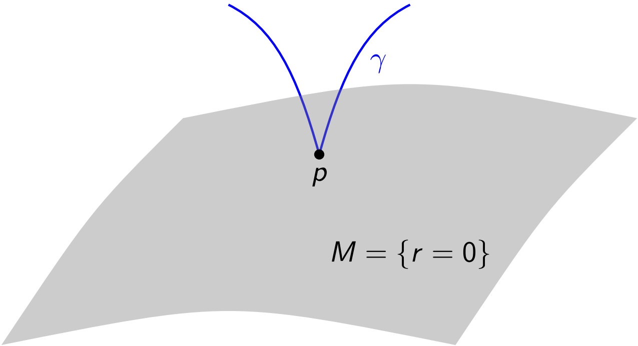

$$\mathbf{T}(M,p) = \sup_{\gamma} \frac{\nu(r\circ\gamma)}{\nu(\gamma)}$$

The local perspective

Work on **local normal forms** at a point of finite/infinite type: Ebenfelt, Lamel, Kolář, Kossovskiy, Meylan, Stanton, Wong, Zaitsev, ...

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Work on **local normal forms** at a point of finite/infinite type: Ebenfelt, Lamel, Kolář, Kossovskiy, Meylan, Stanton, Wong, Zaitsev, ...

Finite type was introduced to characterize **local properties** (subellipticity of $\overline{\partial}$): Catlin, D'Angelo, Kohn, ...

The global perspective

Theorem 1 (D'Angelo, 1982)

Let M be a smooth real hypersurface in $\mathbb{C}^n$. Then the set

$$\{p \in M \,|\, \mathbf{T}(M, p) \neq \infty\}$$

is an open subset of M.

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We consider the set of Levi degenerate finite type points

$$\mathcal{F}_M := \{p \in M \mid 2 < \mathbf{T}(M,p) < \infty\}.$$

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- How arbitrary is the set $\mathcal{F}_M$?
- Does $\mathcal{F}_M$ have any structure?

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Same questions for $\mathcal{I}_M := \{p \in M \mid \mathbf{T}(M,p) = \infty\}$.

Smooth vs Real Analytic setting

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Proposition 2 (D'Angelo)

Let M be a **real analytic** hypersurface in $\mathbb{C}^n$, and let $p \in M$ be such that $\mathbf{T}(M, p) = \infty$. Then M contains the germ of a holomorphic curve γ through p.

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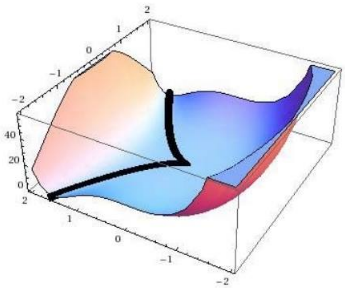
Let M be defined in $\mathbb{C}^3$ by

$$\operatorname{Re} z_3 + |z_1^2 - z_2^3|^2 = 0.$$

$\mathbf{T}(M,0) = \infty$, since M contains $\gamma(t) = (t^3, t^2, 0)$.



Smooth vs Real Analytic setting

Let M be defined in $\mathbb{C}^2$ by

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The complex line $\gamma(t) = (t, 0)$ is such that $\nu(r \circ \gamma) = \infty$.

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Theorem 3 (Kim-Thu, 2015)

There exist smooth real hypersurfaces M of infinite type for which the supremum in

$$\mathbf{T}(M, p) = \sup_{\gamma} \frac{\nu(r \circ \gamma)}{\nu(\gamma)}$$

is not achieved.

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Such M can be taken to be pseudoconvex (Fornæss-Thu, 2018).

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Let $A \subset \mathbb{R}^n$ be a closed set. There exists a function $f \in C^\infty(\mathbb{R}^n)$ such that $f^{-1}(0) = A$ and f is **flat** at every point of A.

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The function f can be taken to be

- bounded,
- non-negative,
- real analytic outside A.

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Theorem 5

Let $\mathcal{H}$ be a real hyperplane in $\mathbb{C}^n$, and A any closed subset in $\mathcal{H}$. There exists a smooth real hypersurface M such that $\mathcal{I}_M = A$.

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Proof.

Take $\mathcal{H}\colon\ \operatorname{Im} z_n = 0$.
Consider a smooth $f(z_1, \bar{z}_1, \ldots, z_{n-1}, \bar{z}_{n-1}, \operatorname{Re} z_n)$ such that

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Let M be defined by $\operatorname{Im} z_n = f(z_1, \bar{z}_1, \ldots, z_{n-1}, \bar{z}_{n-1}, \operatorname{Re} z_n)$. Then

- $M \cap \mathcal{H} = A$.

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- $M \cap \mathcal{H} = A$.
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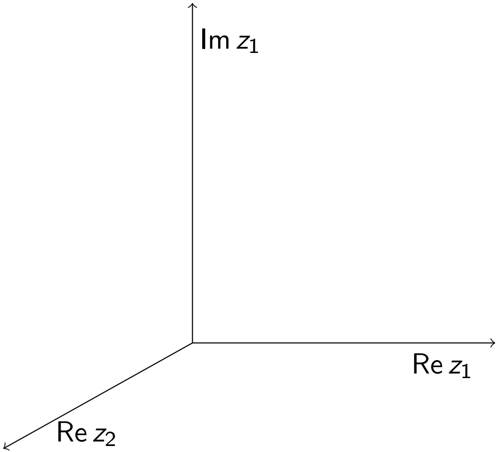
Let M be defined by $\operatorname{Im} z_n = f(z_1, \bar{z}_1, \ldots, z_{n-1}, \bar{z}_{n-1}, \operatorname{Re} z_n)$. Then

- $M \cap \mathcal{H} = A$.
- $\mathbf{T}(M,p) = \infty$ for every $p \in A$.
- $\mathbf{T}(M,p) < \infty$ for every $p \in M \setminus \mathcal{H}$.

$\square$

The set $\mathcal{I}_M$: Examples in $\mathbb{C}^2$



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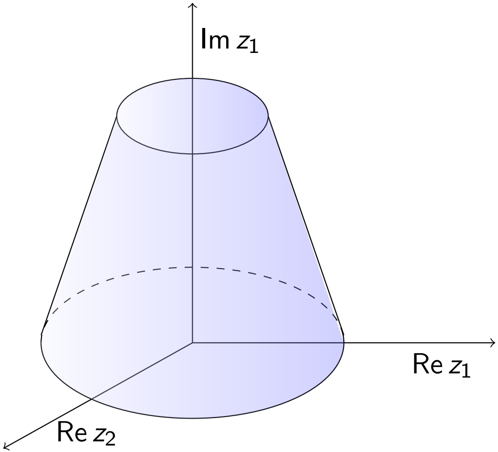
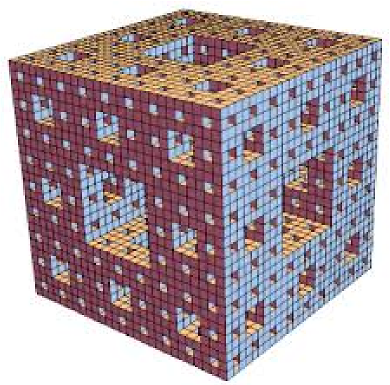


The truncated cone in $\mathbb{R}^3$ represents the intersection of a real hypersurface M in $\mathbb{C}^2$ with the hyperplane $\operatorname{Im} z_2 = 0$. It consists of all the points of infinite type for M.

The set $\mathcal{I}_M$: Examples in $\mathbb{C}^2$



The sponge is the intersection of a real hypersurface M in $\mathbb{C}^2$ with the hyperplane $\operatorname{Im} z_2 = 0$. It consists of all the points of infinite type for M.

Detecting finite type in $\mathbb{C}^2$: the algorithm

Let M be defined by

$$2\,\mathrm{Re}\, w + f(z,\bar{z}) + (2\,\mathrm{Im}\, w)\, g(z,\bar{z},\mathrm{Im}\, w) = 0,$$

where f and g are smooth functions and $f(0) = df(0) = g(0) = 0$.

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- If $\nu(f) = \infty$, then $\mathbf{T}(M,0) = \infty$.

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- If $\nu(f) = \infty$, then $\mathbf{T}(M,0) = \infty$.
- If $\nu(f) = m \neq \infty$, consider the homogeneous term f_m in the Taylor expansion of f at 0.

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  - If f_m is not harmonic, then $\mathbf{T}(M,0) = m$.

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- If $\nu(f) = m \neq \infty$, consider the homogeneous term f_m in the Taylor expansion of f at 0.
  - If f_m is not harmonic, then $\mathbf{T}(M,0) = m$.
  - If f_m is harmonic, $f_m = 2\,\mathrm{Re}\, F_m$, perform the change of coordinates

    $$\begin{cases} \tilde{z} = z \\ \tilde{w} = w + F_m, \end{cases}$$

    and repeat the analysis on the new defining function.

Detecting finite type in $\mathbb{C}^2$: Example

Let M be defined by

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Perform the change of coordinates

$$\begin{cases} \tilde{z} = z \\ \tilde{w} = w + z^2. \end{cases}$$

In the new coordinates M is defined by

$$2\,\mathrm{Re}\,\tilde{w} + i(\tilde{z}^2 - \overline{\tilde{z}}^2)(\tilde{z} + \overline{\tilde{z}}) + (2\,\mathrm{Im}\,\tilde{w})(\tilde{z} + \overline{\tilde{z}}) = 0.$$

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$\mathbf{T}(M, 0) = 3$ and is realized by the curve $(t, 0)$.

Detecting finite type in $\mathbb{C}^2$

Theorem 6 (Folklore)

Let M be a smooth real hypersurface in $\mathbb{C}^2$ and $p \in M$. Then $\mathbf{T}(M,p) = m$ **if and only if** *there exist coordinates $\{z, w\}$ for which p is the origin and M has a local defining function at p of the form*

$$2\,\mathrm{Re}\,w + f(z,\bar{z}) + (2\,\mathrm{Im}\,w)\,g(z,\bar{z},\mathrm{Im}\,w) = 0.$$

Here

1. *The functions f, g are smooth and $f(0) = df(0) = g(0) = 0$.*
2. *The function f vanishes to order m at 0.*
3. *The homogeneous term f_m of order m in the Taylor series of f at 0 is not harmonic.*

Detecting finite type in $\mathbb{C}^2$: the rigid case

Definition 7

A smooth function f is said to be *formally harmonic* at p if the Taylor expansion of f at p contains only harmonic terms.

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Corollary 8

Let M be a smooth real hypersurface in $\mathbb{C}^2$ globally defined by

$$\mathrm{Im}(w) = f(z, \overline{z}),$$

where $f(z, \overline{z})$ is a smooth function. Then

1. $\mathbf{T}(M, p) = \infty$ **if and only if** *f is formally harmonic at p.*
2. $\mathbf{T}(M, p) = m$ **if and only if** $\nu_p\Big(\Delta f(z, \overline{z})\Big) = m - 2.$

The set $\mathcal{F}_M = \{p \in M \mid 2 < \mathbf{T}(M, p) < \infty\}$

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Theorem 9

Let $A \subset \mathbb{R}$ be a closed set. There exists a smooth pseudoconvex real hypersurface M in $\mathbb{C}^2$ such that $\mathcal{F}_M \simeq A \times \mathbb{R}$.

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Proof.

Let $z = x + iy$. Take $h(x)$ non-negative smooth such that $h^{-1}(0) = A$, with h being flat at every point of A.

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$$f(z, \bar{z}) = y^{2m+2} + \int_0^x \int_0^t h(\tau)\, d\tau\, dt.$$

Define M by $\operatorname{Im} w = f(z, \bar{z})$.

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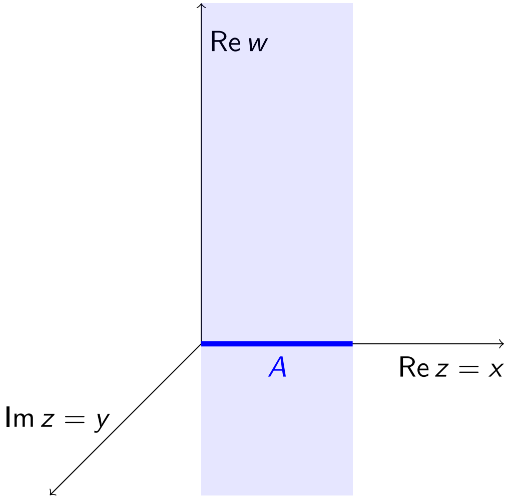
Define M by $\operatorname{Im} w = f(z,\overline{z})$. Then

$$\Delta f(z,\overline{z}) = \frac{y^{2m}}{(2m+2)(2m+1)} + h(x).$$

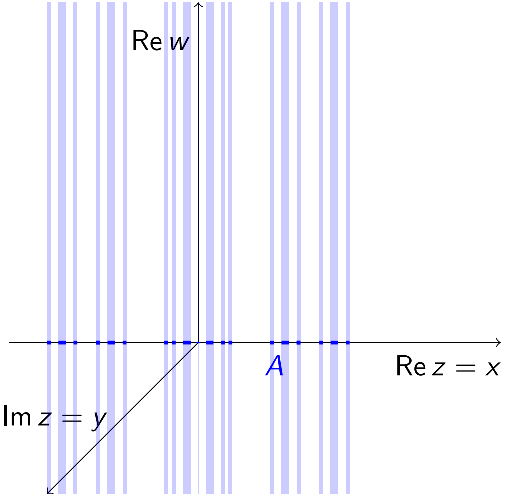
$\square$

The set $\mathcal{F}_M$: Example



The shaded region is the projection onto $\operatorname{Im} w = 0$ of the set $\mathcal{F}_M$.

The set $\mathcal{F}_M$: Cantor trees of finite type points



In $\mathbb{C}^2$ every "tree" sits at a potentially different (Im w)-coordinate.

The set $\mathcal{F}_M$: More exotic examples

One can prescribe directly Δf to be some smooth function u with the wanted vanishing properties, then solve globally

$$\Delta f = u \quad \text{in } \mathbb{C}$$

(by applying Hörmander $\overline{\partial}$-solution twice). Finally, define M by

$$\operatorname{Im} w = f(z, \bar{z}).$$

The set $\mathcal{F}_M$: Cantor forest of finite type points

Let $\theta_1, \ldots, \theta_s$ be in $[0, \pi]$, and define

$$\begin{bmatrix} x_j \\ y_j \end{bmatrix} = \begin{bmatrix} \cos\theta_j & \sin\theta_j \\ -\sin\theta_j & \cos\theta_j \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$u = \prod_{j=1}^{s} (y_j^{2m_j} + h(x_j)).$$

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Solve

$$\Delta f = u \quad \text{ in } \mathbb{C},$$

and let M be defined by

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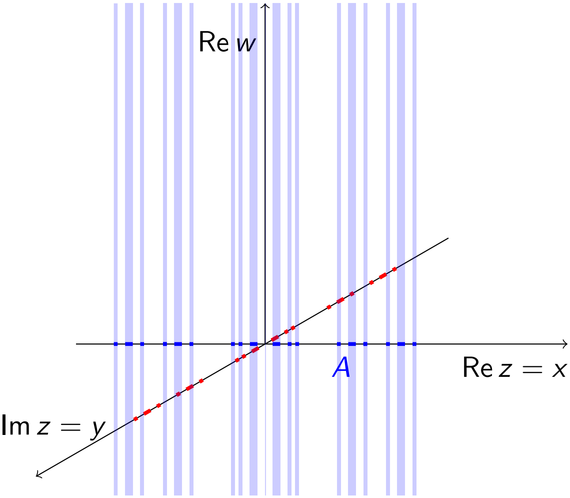
The set $\mathcal{F}_M$: Cantor forest of finite type points



The "forest" is the projection onto $\mathrm{Im}\, w = 0$ of the set $\mathcal{F}_M$.

The set $\mathcal{F}_M$: Prison of finite type points

Solve

$$\Delta f = y^2 - \sin\left(\frac{\pi}{x}\right) e^{-\frac{1}{x^2}} \quad \text{in } \mathbb{C},$$

Let M be the rigid real hypersurface defined in $\mathbb{C}^2$ by

$$\operatorname{Im} w = f(z, \bar{z}).$$

Recall that the points in $\mathcal{F}_M$ are the ones at which Δf vanishes to finite order.

A stratification result

Theorem 10 (Bär, 1999)

Let U be an open neighborhood of 0 in $\mathbb{R}^n$. Let $f : U \to \mathbb{R}$ be a smooth function vanishing to finite order at 0. Then for sufficiently small $r > 0$ the set $f^{-1}(0) \cap B(0, r)$ is countably $(n-1) - C^\infty -$rectifiable.

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Corollary 11

Let M be a smooth rigid real hypersurface in $\mathbb{C}^2$. Then the set $\mathcal{F}_M$ is contained in the countable union of smooth codimension 1 submanifolds. In particular, $\mathcal{F}_M$ is of measure zero.

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Corollary 11

Let M be a smooth rigid real hypersurface in $\mathbb{C}^2$. Then the set $\mathcal{F}_M$ is contained in the countable union of smooth codimension 1 submanifolds. In particular, $\mathcal{F}_M$ is of measure zero.

To generalize examples and results to rigid real hypersurfaces in $\mathbb{C}^n$ one needs to replace the Laplace operator with the Monge-Ampère operator.

The non-rigid case

(Open?) Question

Is it true that $\mathbf{T}(M, p) < \infty$ implies that the Levi determinant vanishes to finite order at p (along the tangential directions)?

Thank you for your attention!