Minicourse
Convergence of formal maps III

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Recalling the setting ... again

Driving in some nails

- $M \subset \mathbb{C}_z^N$, $M \subset \mathbb{C}_w^{N'}$ germs of real-analytic hypersurfaces (or more generally CR submanifolds), through 0
- $M = \{\varrho(z,\bar{z}) = 0\}$, $M' = \{\varrho'(w,\bar{w}) = 0\}$
- $H \in \mathbb{C}[\![z]\!]^{N'}$ formal map with $H(0) = 0$
- $H \colon M \to M' :\Leftrightarrow \varrho'(H(z), \overline{H(z)}) = a(z,\bar{z})\varrho(z,\bar{z})$ for some $a \in \mathbb{C}[\![z,\bar{z}]\!]$.

Theorem (L.-Mir 2016)

If M is minimal at 0, M' is strictly pseudoconvex, and $H \in \mathbb{C}[\![z]\!]^{N'}$ is a formal map with $H \colon M \to M'$, then H is convergent.

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About those assumptions... again

Minimality

Recall that there exist examples of nonminimal M which allow divergent formal automorphisms (Kossovskiy-Shafikov). Therefore minimality is necessary.

Strict pseudoconvexity

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Why should I care?

We already know that there is no a priori reason why a formal map should converge. But let us also recall again:

Theorem (Chern and Moser 74)

If $M, M' \subset \mathbb{C}^N$ are strictly pseudoconvex, and $H\colon M \to M'$ is a formal map, then H converges.

We have discussed “the” result for convergence in the equidimensional case on tuesday. The positive codimension case offers new and particular challenges.

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Issues to overcome

Recall the strategy

For *invertible* maps: Prolongation of

$$\varrho'(H, \bar{H}) = 0,$$

(i.e. application of CR vector fields) yields a “singular reflection identity”

$$\Theta(z, \bar{z}, \overline{j_z^k H}, H(z)) = 0$$

to which we can apply the hammer.

Issues to overcome

Prolongation does not suffice

Additional information about the location of the image of the “characteristics of the source” with respect to the “characteristics of the image” is needed.

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Before we discuss this, let's first review some earlier results.

Earlier and related work

- L. (2001) : strongly pseudoconvex hypersurfaces + additional stringent conditions on the maps.
- Mir (2002) : Corollary in the case $N' = N + 1$ (codimensional one case).
- Meylan, Mir, Zaitsev (2003) : main result +additional assumption that M' is real-algebraic (instead of real-analytic).
- Ebenfelt, L. (2004) : Finite determination of embeddings (again rather stringent conditions)
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Recall the Set-up...yet again

Let M, M' be real-analytic generic CR submanifolds in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$, through the origin, and $H\colon M \to M'$, $H(0) = 0$, be a formal holomorphic map.

- ϱ' defining function for M'
- $\bar{L}_1, \ldots, \bar{L}_n$ local basis of real-analytic CR vector fields for M
- $\mathbb{C}[\![M]\!]$ be the formal coordinate ring of M
- $\mathbb{C}(\!(M)\!)$ quotient field of $\mathbb{C}[\![M]\!]$

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Covariant derivatives

We consider "Lie derivatives" of the characteristic form of M':

$$E^\alpha := \left(\bar{L}^\alpha \varrho'_{w'_1}(H,\overline{H})\big|_M, \ldots, \bar{L}^\alpha \varrho'_{w'_N}(H,\overline{H})\big|_M\right) \in \mathbb{C}[\![M]\!]^{N'}.$$

For $k \in \mathbb{N}$, we define a vector space over $\mathbb{C}(\!(M)\!)^{N'}$:

$$\mathcal{E}_k(H) := \mathrm{Span}_{\mathbb{C}(\!(M)\!)}\{E^\alpha : \alpha \in \mathbb{N}^n,\ |\alpha| \leq k,\ \} \subset \mathbb{C}(\!(M)\!)^{N'},$$

- $\mu_k^H := \dim_{\mathbb{C}(\!(M)\!)} \mathcal{E}_k(H)$.
- $\mathcal{E}_k(H)$ is independent of all of the choices.
- $1 = \mu_0^H < \mu_1^H < \cdots < \mu_{k_0}^H = \mu_{k_0+1}^H = \ldots$
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Generic degeneracy

Definition

Let H, M, M' be as above.

(a) We define the generic degeneracy of H as $\kappa^H := N' - \mu^H$.

(b) We say that H is holomorphically nondegenerate if $\kappa^H = 0$.

- $0 \leq \kappa^H \leq N' - 1$
- M is holomorphically nondegenerate in the sense of Stanton if and only if the identity mapping is holomorphically nondegenerate.
- The more stringent conditions alluded to above, appearing in earlier work, can be expressed by saying that

$$\dim \operatorname{Span}_{\mathbb{C}}\{E^{\alpha}\big|_0 : \alpha \in \mathbb{N}^n\} = \mu^H.$$

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Looking for a nail?

Here is one.

Theorem (Nail 1)

Let $H\colon M \to M'$ be a holomorphically nondegenerate formal map. Then H is convergent.

Proof.

There exists $\alpha^1, \ldots, \alpha^{N'}$, $|\alpha^j| \le k_0$ s.t. $E_{\alpha^1}, \ldots, E_{\alpha^{N'}}$ are linearly independent.

Set $\Theta_j(z, \bar{z}, (\overline{\partial^\beta H(z)})_{|\beta| \le k_0}, H(z)) = \bar{L}^{\alpha_j} \varrho'(H(z), \overline{H(z)})$.

Note that $\frac{\partial \Theta_j}{\partial w} = E_{\alpha^j}$. So $\Theta = (\Theta_1, \ldots, \Theta_{N'})$ satisfies

$$\Theta((z, \bar{z}, (\overline{\partial^\beta H(z)})_{|\beta| \le k_0}, H(z)))\Big|_M = 0$$

$$\det \frac{\partial \Theta}{\partial w}((z, \bar{z}, (\overline{\partial^\beta H(z)})_{|\beta| \le k_0}, H(z)))\Big|_M \not\equiv 0.$$

□

Recall the hammer...

...it finishes the proof

Proposition

Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold through the origin and $\Theta = (\Theta_1, \dots, \Theta_{N'})$ be a convergent power series mapping with components in $\mathbb{C}\{z, \bar{z}, \lambda, w\}$ where $z \in \mathbb{C}^N$, $w \in \mathbb{C}^{N'}$, $\lambda \in \mathbb{C}^r$, $N', N, r \geq 1$. Let $h\colon (\mathbb{C}^N, 0) \to \mathbb{C}^{N'}$, $g\colon (\mathbb{C}^N, 0) \to \mathbb{C}^r$ be formal holomorphic power series mappings, vanishing at 0, satisfying

$$\Theta(z, \bar{z}, \overline{g(z)}, h(z))|_M = 0, \quad \text{and}$$

$$\det \frac{\partial \Theta}{\partial w} \left(z, \bar{z}, \overline{g(z)}, h(z) \right) \Big|_M \not\equiv 0.$$

If M is of finite type at 0, then h is a convergent holomorphic map.

Found some equations?

We are missing them.

In the general case, simple prolongation only yields μ^H generically independent equations.
So in order to apply the hammer, we need to find κ^H additional or “missing” equations.
In order to find them, we must look into the curvature condition of M', which we did not need to touch yet.
But first some simple yet important linear algebra.

Definition

Let H, M, M' be as above. Let $V = (V_1, \ldots, V_{N'}) \in \mathbb{C}(\!|M|\!)^{N'}$. We say that V is a *formal meromorphic infinitesimal deformation of* H if V is tangent to M' along $H(M)$ i.e. if

$$\sum_{r=1}^{N'} (V_r\, \varrho'_{w'_r}(H, \overline{H}))\big|_M = 0 \ \text{ in } \ \mathbb{C}(\!|M|\!).$$

- If $M = M'$ and $H = \mathrm{id}$ then a formal meromorphic infinitesimal deformation of H corresponds to a formal meromorphic vector field tangent to M.

Deformations

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Degeneracy vs deformations

Proposition

Let H, M, M' be as above. Then the following conditions are equivalent :

(i) *H is a holomorphically degenerate map of generic degeneracy κ;*

(ii) *The space of formal meromorphic infinitesimal deformations of H is a vector space of dimension κ over $\mathbb{C}(\!|M|\!)$.*

- When $M = M'$ and $H = \mathrm{id}$, then the proposition is just Stanton's criterion for holomorphic nondegeneracy.

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Degeneracy and deformations through the years

- For (constantly degenerate) smooth CR maps between smooth CR manifolds, this is due to Berhanu-Xiao (2015)
- For (constantly degenerate) formal maps between real-analytic CR manifolds, this appears in (L. 2001)
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The Leviform

...yes, again recalled

The Levi form associated to the defining function ϱ' of M' at p' is defined for $u, v \in \mathbb{C}^{N'}$

$$\mathcal{L}_{p'}^{\varrho'}(u, \bar{v}) = \sum_{j,k=1}^{N'} \frac{\partial \varrho'(p, \bar{p})}{\partial w_j \bar{w}_k} u_j \bar{v}_k.$$

When restricted to $u, v \in T_p^c M$, it corresponds to the Levi form introduced in monday's lecture when measured with respect to the characteristic form $i\bar{\partial}\varrho'$.

Maps with "nondegenerate deformations"

Theorem (Nail 2)

Assume that M is of finite type, M' is a hypersurface (for simplicity), and that H is a holomorphically degenerate map of generic degeneracy $\kappa > 0$. Assume that for every κ-tuple $(V^1, \dots, V^\kappa)$ of $\mathbb{C}(\!|M|\!)$-linearly independent formal meromorphic infinitesimal deformations of H, $V^j = (V^j_1, \dots, V^j_{N'}) \in (\mathbb{C}(\!|M|\!))^{N'}$, the Gram matrix

$$\begin{pmatrix} \mathcal{L}^{\varrho'}_{H(z)}(V^1, \bar{V}^1) & \dots & \mathcal{L}^{\varrho'}_{H(z)}(V^1, \bar{V}^\kappa) \\ \vdots & & \vdots \\ \mathcal{L}^{\varrho'}_{H(z)}(V^\kappa, \bar{V}^1) & \dots & \mathcal{L}^{\varrho'}_{H(z)}(V^\kappa, \bar{V}^\kappa) \end{pmatrix}$$

is nonsingular. Then H is convergent.

The proof is inspired by the work of Berhanu-Ming, combining our convergence proposition with the tool of the meromorphic infinitesimal deformations. As before we consider the system

$$0 = \bar{L}^{\alpha_k} \varrho'(H(z), \overline{H(z)}) = \Theta_k(z, \bar{z}, \overline{G(z)}, H(z)), \quad k = 1, \ldots, N' - \kappa,$$

complemented with the “missing equations”

$$0 = \sum_{\ell=1}^{N'} \bar{V}_k^j \varrho'_{\bar{w}_k}(H(z), \overline{H(z)}) = \Theta_{N'-\kappa+j}, \quad j = 1, \ldots, \kappa$$

(where we clear the denominators).
The assumption on the Levi form allows us to apply the convergence proposition.

The main result...

follows because in the strictly pseudoconvex case, if H is degenerate, the matrix

$$\begin{pmatrix} \mathcal{L}^{\varrho'}_{H(z)}(V^1, \bar{V}^1) & \dots & \mathcal{L}^{\varrho'}_{H(z)}(V^1, \bar{V}^\kappa) \\ \vdots & & \vdots \\ \mathcal{L}^{\varrho'}_{H(z)}(V^\kappa, \bar{V}^1) & \dots & \mathcal{L}^{\varrho'}_{H(z)}(V^\kappa, \bar{V}^\kappa) \end{pmatrix}$$

is of full rank (because $\mathcal{L}$ is positive definite).