Minicourse
Convergence of formal maps II

Bernhard Lamel

Fakultät für Mathematik

universität wien

Nordine Mir

Texas A&M University at Qatar

TEXAS A&M UNIVERSITY at QATAR

Serra Negra, Brazil
August 2017

Formal CR maps

Suppose that $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ are germs through the origin of real-analytic generic submanifolds.
A formal holomorphic power series mapping $H\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ is called a **formal CR map** (or sends M into M') if for any germ of a real-analytic function $\delta\colon (\mathbb{C}^{N'}, 0) \to \mathbb{R}$ vanishing on M' near 0 and any real-analytic parametrization $\psi\colon (\mathbb{R}_x^{\dim M}, 0) \to (M, 0)$ the power series identity

$$\delta((H \circ \psi)(x), \overline{(H \circ \psi)}(x)) = 0$$

holds in the ring $\mathbb{C}[[x]]$.
Remark: If H is a convergent power series, it defines a local holomorphic map sending $(M, 0)$ into $(M', 0)$ in the usual sense.

Formal CR maps

Suppose that $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ are germs through the origin of real-analytic generic submanifolds.
A formal holomorphic power series mapping $H\colon (\mathbb{C}^N, 0) \to (\mathbb{C}^{N'}, 0)$ is called a **formal CR map** (or sends M into M') if for any germ of a real-analytic function $\delta\colon (\mathbb{C}^{N'}, 0) \to \mathbb{R}$ vanishing on M' near 0 and any real-analytic parametrization $\psi\colon (\mathbb{R}_x^{\dim M}, 0) \to (M, 0)$ the power series identity

$$\delta((H \circ \psi)(x), \overline{(H \circ \psi)}(x)) = 0$$

holds in the ring $\mathbb{C}[[x]]$.
Remark: If H is a convergent power series, it defines a local holomorphic map sending $(M, 0)$ into $(M', 0)$ in the usual sense.

Convergence problem

When $N = N'$ and H is an invertible formal CR map, we call H a formal CR **equivalence**.

Main question today: When does a formal CR equivalence converge?

Example (The simplest example: $M = \mathbb{R} \subset \mathbb{C}$)

Formal CR equivalences taking $\mathbb{R}$ into itself:

$$H(w) = \sum_{j \geq 1} H_j w^j : (\mathbb{R}, 0) \to (\mathbb{R}, 0) \Leftrightarrow H_j \in \mathbb{R} \quad \forall j \quad \text{and } H_1 \neq 0$$

So plenty of divergent formal CR equivalences...

Convergence problem

When $N = N'$ and H is an invertible formal CR map, we call H a formal CR **equivalence.**

Main question today: When does a formal CR equivalence converge?

Example (The simplest example: $M = \mathbb{R} \subset \mathbb{C}$)

Formal CR equivalences taking $\mathbb{R}$ into itself:

$$H(w) = \sum_{j \geq 1} H_j w^j : (\mathbb{R}, 0) \to (\mathbb{R}, 0) \Leftrightarrow H_j \in \mathbb{R} \quad \forall j \quad \text{and } H_1 \neq 0$$

So plenty of divergent formal CR equivalences...

Convergence problem

When $N = N'$ and H is an invertible formal CR map, we call H a formal CR **equivalence**.

Main question today: When does a formal CR equivalence converge?

Example (The simplest example: $M = \mathbb{R} \subset \mathbb{C}$)

Formal CR equivalences taking $\mathbb{R}$ into itself:

$$H(w) = \sum_{j \geq 1} H_j w^j \colon (\mathbb{R}, 0) \to (\mathbb{R}, 0) \Leftrightarrow H_j \in \mathbb{R} \quad \forall j \quad \text{and } H_1 \neq 0$$

So plenty of divergent formal CR equivalences...

Convergence problem

When $N = N'$ and H is an invertible formal CR map, we call H a formal CR **equivalence**.

Main question today: When does a formal CR equivalence converge?

Example (The simplest example: $M = \mathbb{R} \subset \mathbb{C}$)

Formal CR equivalences taking $\mathbb{R}$ into itself:

$$H(w) = \sum_{j \geq 1} H_j w^j \colon (\mathbb{R}, 0) \to (\mathbb{R}, 0) \Leftrightarrow H_j \in \mathbb{R} \quad \forall j \quad \text{and } H_1 \neq 0$$

So plenty of divergent formal CR equivalences...

Example (The second simplest example: $M = \Gamma \subset \mathbb{C}$)

If Γ is a real-analytic arc, $p \in \Gamma$, then there exists a biholomorphism $\phi\colon (\Gamma, p) \to (\mathbb{R}, 0)$:

$$H\colon (\Gamma, p) \to (\Gamma, p) \Leftrightarrow \varphi \circ H \circ \varphi^{-1} \in (w)\mathbb{R}[[w]]$$

Example

When M and M' are maximally real real-analytic submanifolds in $\mathbb{C}^N$, the same kind of argument as before can be used to construct plenty of divergent formal CR equivalences.

Example

If M is the real hyperplane in $\mathbb{C}^N$ given by $\mathrm{Im}\, z_N = 0$, then any formal map of the form $\mathbb{C}^N \ni (z', z_N) \mapsto (h(z'), z_n)$ with $h\colon (\mathbb{C}^{N-1}, 0) \to (\mathbb{C}^{N-1}, 0)$ formal (divergent) biholomorphism is a formal CR self-map of M.

Example (The second simplest example: $M = \Gamma \subset \mathbb{C}$)

If Γ is a real-analytic arc, $p \in \Gamma$, then there exists a biholomorphism $\phi\colon (\Gamma, p) \to (\mathbb{R}, 0)$:

$$H\colon (\Gamma, p) \to (\Gamma, p) \Leftrightarrow \varphi \circ H \circ \varphi^{-1} \in (w)\mathbb{R}[[w]]$$

Example

When M and M' are maximally real real-analytic submanifolds in $\mathbb{C}^N$, the same kind of argument as before can be used to construct plenty of divergent formal CR equivalences.

Example

If M is the real hyperplane in $\mathbb{C}^N$ given by $\operatorname{Im} z_N = 0$, then any formal map of the form $\mathbb{C}^N \ni (z', z_N) \mapsto (h(z'), z_n)$ with $h\colon (\mathbb{C}^{N-1}, 0) \to (\mathbb{C}^{N-1}, 0)$ formal (divergent) biholomorphism is a formal CR self-map of M.

Example (The second simplest example: $M = \Gamma \subset \mathbb{C}$)

If Γ is a real-analytic arc, $p \in \Gamma$, then there exists a biholomorphism $\phi\colon (\Gamma, p) \to (\mathbb{R}, 0)$:

$$H\colon (\Gamma, p) \to (\Gamma, p) \Leftrightarrow \varphi \circ H \circ \varphi^{-1} \in (w)\mathbb{R}[[w]]$$

Example

When M and M' are maximally real real-analytic submanifolds in $\mathbb{C}^N$, the same kind of argument as before can be used to construct plenty of divergent formal CR equivalences.

Example

If M is the real hyperplane in $\mathbb{C}^N$ given by $\operatorname{Im} z_N = 0$, then any formal map of the form $\mathbb{C}^N \ni (z', z_N) \mapsto (h(z'), z_n)$ with $h\colon (\mathbb{C}^{N-1}, 0) \to (\mathbb{C}^{N-1}, 0)$ formal (divergent) biholomorphism is a formal CR self-map of M.

Chern-Moser convergence result

Despite all these, Chern-Moser proved the first striking convergence result for formal CR equivalences.

Theorem (Chern-Moser, 1974)

Let $M, M' \subset \mathbb{C}^N$ be germs through the origin of real-analytic Levi-nondegenerate hypersurfaces. Then any formal CR equivalence $H\colon (M,0) \to (M',0)$ necessarily converges.

Levi-nondegenerate case

- Let $M, M' \subset \mathbb{C}^N$ be germs through the origin of real-analytic hypersurfaces with $N \geq 2$, and $H\colon (M,0) \to (M',0)$ a formal invertible CR map.
  Our main assumptions on the germs at the origin of M and M' are the following:
  M is of **finite type** and M' is **Levi-nondegenerate.**
- The proof of the convergence in the Levi-nondegenerate case involves two steps:
  1) Derivation of the reflection identity (using that M' is Levi-nondegenerate)
  2) Iteration along the iterated Segre sets (using the finite type assumption)

Levi-nondegenerate case

- Let $M, M' \subset \mathbb{C}^N$ be germs through the origin of real-analytic hypersurfaces with $N \geq 2$, and $H\colon (M,0) \to (M',0)$ a formal invertible CR map.
  Our main assumptions on the germs at the origin of M and M' are the following:
  M is of **finite type** and M' is **Levi-nondegenerate.**
- The proof of the convergence in the Levi-nondegenerate case involves two steps:
  1) Derivation of the reflection identity (using that M' is Levi-nondegenerate)
  2) Iteration along the iterated Segre sets (using the finite type assumption)

Levi-nondegenerate case

- Let $M, M' \subset \mathbb{C}^N$ be germs through the origin of real-analytic hypersurfaces with $N \geq 2$, and $H\colon (M,0) \to (M',0)$ a formal invertible CR map.
  Our main assumptions on the germs at the origin of M and M' are the following:
  M is of **finite type** and M' is **Levi-nondegenerate.**
- The proof of the convergence in the Levi-nondegenerate case involves two steps:
  1) Derivation of the reflection identity (using that M' is Levi-nondegenerate)
  2) Iteration along the iterated Segre sets (using the finite type assumption)

Levi-nondegenerate case

- Let $M, M' \subset \mathbb{C}^N$ be germs through the origin of real-analytic hypersurfaces with $N \geq 2$, and $H\colon (M,0) \to (M',0)$ a formal invertible CR map.
  Our main assumptions on the germs at the origin of M and M' are the following:
  M is of **finite type** and M' is **Levi-nondegenerate.**
- The proof of the convergence in the Levi-nondegenerate case involves two steps:
  1) Derivation of the reflection identity (using that M' is Levi-nondegenerate)
  2) Iteration along the iterated Segre sets (using the finite type assumption)

Derivation of the reflection identity

We may assume that M' is given near 0 by a real-analytic defining function $\rho' = \rho'(w, \bar{w})$ satisfying $d\rho'(0) \neq 0$.We also pick a basis $\bar{L}_1, \ldots, \bar{L}_{N-1}$ of real-analytic CR vector fields for M near 0.

H sends M into M' reads as

$$\rho'(H(z), \overline{H(z)})|_M = 0. \tag{1}$$

Applying the CR vector fields $\bar{L}_k$ to (1), we get

$$\sum_{j=1}^{N} \left\{ \bar{L}_k \overline{H(z)}\, \rho'_{\bar{w}_j}(H(z), \overline{H(z)}) \right\} \Big|_M = 0. \tag{2}$$

Derivation of the reflection identity

We may assume that M' is given near 0 by a real-analytic defining function $\rho' = \rho'(w, \bar{w})$ satisfying $d\rho'(0) \neq 0$.We also pick a basis $\bar{L}_1, \ldots, \bar{L}_{N-1}$ of real-analytic CR vector fields for M near 0.

H sends M into M' reads as

$$\rho'(H(z), \overline{H(z)})|_M = 0. \tag{1}$$

Applying the CR vector fields $\bar{L}_k$ to (1), we get

$$\sum_{j=1}^{N} \left\{ \bar{L}_k \overline{H(z)} \, \rho'_{\bar{w}_j}(H(z), \overline{H(z)}) \right\} \Big|_M = 0. \tag{2}$$

Derivation of the reflection identity

We may assume that M' is given near 0 by a real-analytic defining function $\rho' = \rho'(w, \bar{w})$ satisfying $d\rho'(0) \neq 0$.We also pick a basis $\bar{L}_1, \ldots, \bar{L}_{N-1}$ of real-analytic CR vector fields for M near 0.
H sends M into M' reads as

$$\rho'(H(z), \overline{H(z)})|_M = 0. \tag{1}$$

Applying the CR vector fields $\bar{L}_k$ to (1), we get

$$\sum_{j=1}^{N} \left\{ \bar{L}_k \overline{H(z)}\, \rho'_{\bar{w}_j}(H(z), \overline{H(z)}) \right\} \Big|_M = 0. \tag{2}$$

Derivation of the reflection identity

We view the preceding system as follows:

$$\begin{cases} \rho'(w, \overline{H(z)})|_M = 0 \\ \sum_{j=1}^{N} \left\{ \bar{L}_k \overline{H(z)}\, \rho'_{\bar{w}_j}(w, \overline{H(z)}) \right\} \Big|_M = 0, \ k = 1, \ldots, N-1, \end{cases} \tag{3}$$

where $w = H(z)|_M$ is a formal solution.

Using the **Levi-nondegeneracy** of M' at 0 and the **invertibility** of the mapping H, one can easily check that the assumptions of the implicit function theorem are fullfilled so that

$$H(z) = \Psi(\overline{H(z)}, (\bar{L}_k \overline{H(z)})_{1 \le k \le N-1}) \quad \text{on } M, \tag{4}$$

for some (convergent) holomorphic mapping Ψ (in all its arguments) near $(0, (\bar{L}_k \overline{H}|_0)_{1 \le k \le N-1})$.

Derivation of the reflection identity

We view the preceding system as follows:

$$\begin{cases} \rho'(w, \overline{H(z)})|_M = 0 \\ \sum_{j=1}^N \left\{ \bar{L}_k \overline{H(z)}\, \rho'_{\bar{w}_j}(w, \overline{H(z)}) \right\} \Big|_M = 0, \; k = 1, \ldots, N-1, \end{cases} \tag{3}$$

where $w = H(z)|_M$ is a formal solution.
Using the **Levi-nondegeneracy** of M' at 0 and the **invertibility** of the mapping H, one can easily check that the assumptions of the implicit function theorem are fullfilled so that

$$H(z) = \Psi(\overline{H(z)}, (\bar{L}_k \overline{H(z)})_{1 \le k \le N-1}) \quad \text{on } M, \tag{4}$$

for some (convergent) holomorphic mapping Ψ (in all its arguments) near $(0, (\bar{L}_k \overline{H}|_0)_{1 \le k \le N-1}))$.

Iteration along iterated Segre sets

It is convenient to rewrite the basic reflection identity (4) as follows

$$H(z) = \Phi(z, \bar{z}, (\overline{\partial^\alpha H(z)})_{|\alpha| \le 1}) \quad \text{on } M, \tag{5}$$

for some holomorphic mapping Φ (in all its arguments) near $(0, 0, (\overline{\partial^\alpha H(0)})_{|\alpha| \le 1})$. Complexifying (8) we get

$$H(z) = \Phi(z, \zeta, (\partial^\alpha \bar{H}(\zeta))_{|\alpha| \le 1}) \quad \text{for } (z, \zeta) \in \mathcal{M}, \tag{6}$$

where $\mathcal{M}$ is the complexification of M.

Iteration along iterated Segre sets

It is convenient to rewrite the basic reflection identity (4) as follows

$$H(z) = \Phi(z, \bar{z}, (\overline{\partial^\alpha H(z)})_{|\alpha|\leq 1}) \quad \text{on } M, \tag{5}$$

for some holomorphic mapping Φ (in all its arguments) near $(0, 0, (\overline{\partial^\alpha H(0)})_{|\alpha|\leq 1})$. Complexifying (8) we get

$$H(z) = \Phi(z, \zeta, (\partial^\alpha \bar{H}(\zeta))_{|\alpha|\leq 1}) \quad \text{for } (z, \zeta) \in \mathcal{M}, \tag{6}$$

where $\mathcal{M}$ is the complexification of M.

Iteration along iterated Segre sets

We note that (6) implies that for every $\beta \in \mathbb{N}^N$, one has

$$\partial^\beta H(z) = \Phi_\beta(z, \zeta, (\partial^\alpha \bar{H}(\zeta))_{|\alpha| \le |\beta|+1}) \quad \text{for } (z, \zeta) \in \mathcal{M}, \tag{7}$$

for some holomorphic mapping Φ_β (in all its arguments) near $(0, 0, (\overline{\partial^\alpha H(0)})_{|\alpha| \le |\beta|+1})$.

Consider $t^1 \in \mathbb{C}^{N-1}$, then $((t^1, 0), 0) = (v^1(t^1), 0) \in \mathcal{M}$. Using this in (6), we obtain

$$(H \circ v^1)(t^1) = H(t^1, 0) = \Phi((t^1, 0), 0, (\partial^\alpha \bar{H}(0))_{|\alpha| \le 1}). \tag{8}$$

Hence we in particular obtain that $(H \circ v^1)(t^1) = H(t^1, 0)$ is **convergent**!

Iteration along iterated Segre sets

We note that (6) implies that for every $\beta \in \mathbb{N}^N$, one has

$$\partial^\beta H(z) = \Phi_\beta(z, \zeta, (\partial^\alpha \bar{H}(\zeta))_{|\alpha| \le |\beta|+1}) \quad \text{for } (z, \zeta) \in \mathcal{M}, \tag{7}$$

for some holomorphic mapping Φ_β (in all its arguments) near $(0, 0, (\overline{\partial^\alpha H(0)})_{|\alpha| \le |\beta|+1})$.
Consider $t^1 \in \mathbb{C}^{N-1}$, then $((t^1, 0), 0) = (v^1(t^1), 0) \in \mathcal{M}$. Using this in (6), we obtain

$$(H \circ v^1)(t^1) = H(t^1, 0) = \Phi((t^1, 0), 0, (\partial^\alpha \bar{H}(0))_{|\alpha| \le 1}). \tag{8}$$

Hence we in particular obtain that $(H \circ v^1)(t^1) = H(t^1, 0)$ is **convergent**!

Iteration along iterated Segre sets

We note that (6) implies that for every $\beta \in \mathbb{N}^N$, one has

$$\partial^\beta H(z) = \Phi_\beta(z, \zeta, (\partial^\alpha \bar{H}(\zeta))_{|\alpha| \le |\beta|+1}) \quad \text{for } (z, \zeta) \in \mathcal{M}, \tag{7}$$

for some holomorphic mapping Φ_β (in all its arguments) near $(0, 0, (\overline{\partial^\alpha H(0)})_{|\alpha| \le |\beta|+1})$.
Consider $t^1 \in \mathbb{C}^{N-1}$, then $((t^1, 0), 0) = (v^1(t^1), 0) \in \mathcal{M}$. Using this in (6), we obtain

$$(H \circ v^1)(t^1) = H(t^1, 0) = \Phi((t^1, 0), 0, (\partial^\alpha \bar{H}(0))_{|\alpha| \le 1}). \tag{8}$$

Hence we in particular obtain that $(H \circ v^1)(t^1) = H(t^1, 0)$ is **convergent**!

Iteration along iterated Segre sets

Similarly using (7), we obtain for every multiindex $\beta \in \mathbb{N}^N$,

$$\begin{aligned}((\partial^\beta H)\circ v^1)(t^1) =&(\partial^\beta H)(t^1,0)\\ =&\Phi_\beta((t^1,0),0,(\partial^\alpha \bar{H}(0))_{|\alpha|\le|\beta|+1}),\end{aligned} \tag{9}$$

and hence $(\partial^\beta H)\circ v^1$ is convergent for every multiindex β.
Now starts the iteration procedure. By the definition of the iterated Segre mappings, we have

$$(v^2(t^1,t^2),\bar{v}^1(t^1)) \in \mathcal{M}$$

for $t^1, t^2 \in \mathbb{C}^{N-1}$ near the origin. Hence (7) yields for every $\beta \in \mathbb{N}^N$

$$((\partial^\beta H)\circ v^2)(t^1,t^2) = \Phi_\beta(v^2(t^1,t^2),\bar{v}^1(t^1),\underbrace{((\partial^\alpha \bar{H})\circ \bar{v}^1(t^1))_{|\alpha|\le|\beta|+1}}_{\text{convergent from previous step}}), \tag{10}$$

Iteration along iterated Segre sets

Similarly using (7), we obtain for every multiindex $\beta \in \mathbb{N}^N$,

$$\begin{aligned}((\partial^\beta H) \circ v^1)(t^1) =& (\partial^\beta H)(t^1, 0) \\ =& \Phi_\beta((t^1, 0), 0, (\partial^\alpha \bar{H}(0))_{|\alpha| \le |\beta|+1}),\end{aligned} \tag{9}$$

and hence $(\partial^\beta H) \circ v^1$ is convergent for every multiindex β. Now starts the iteration procedure. By the definition of the iterated Segre mappings, we have

$$(v^2(t^1, t^2), \bar{v}^1(t^1)) \in \mathcal{M}$$

for $t^1, t^2 \in \mathbb{C}^{N-1}$ near the origin. Hence (7) yields for every $\beta \in \mathbb{N}^N$

$$((\partial^\beta H) \circ v^2)(t^1, t^2) = \Phi_\beta(v^2(t^1, t^2), \bar{v}^1(t^1), \underbrace{((\partial^\alpha \bar{H}) \circ \bar{v}^1(t^1))_{|\alpha| \le |\beta|+1}}_{\text{convergent from previous step}}), \tag{10}$$

Iteration along iterated Segre sets

Hence for every multiindex $\beta \in \mathbb{N}^N$, $((\partial^\beta H) \circ v^2)(t^1, t^2)$ is a convergent mapping.

Iterating this procedure to higher order, we easily see that for every integer j and every multiindex $\beta \in \mathbb{N}^N$, $((\partial^\beta H) \circ v^j)(t^{[j]})$, $t^{[j]} = (t^1, \dots, t^j)$, is a convergent holomorphic map from $\mathbb{C}^{(N-1)j}$ to $\mathbb{C}^N$.

Now we recall the following from the first lecture (finite type criterion for hypersurfaces):

Theorem

The real hypersurface $M \subset \mathbb{C}^N$ is of finite type at 0 if and only if there exists a positive integer ℓ, $2 \le \ell \le 4$, such that in any neighborhood U of 0 in $\mathbb{C}^{(N-1)\ell}$ there exists $t_0 \in U$ such that

$$\operatorname{rk} \frac{\partial v^\ell}{\partial t^{[\ell]}}(t_0) = N, \quad v^\ell(t_0) = 0. \tag{11}$$

Iteration along iterated Segre sets

Hence for every multiindex $\beta \in \mathbb{N}^N$, $((\partial^\beta H) \circ v^2)(t^1, t^2)$ is a convergent mapping.

Iterating this procedure to higher order, we easily see that for every integer j and every multiindex $\beta \in \mathbb{N}^N$, $((\partial^\beta H) \circ v^j)(t^{[j]})$, $t^{[j]} = (t^1, \ldots, t^j)$, is a convergent holomorphic map from $\mathbb{C}^{(N-1)j}$ to $\mathbb{C}^N$.

Now we recall the following from the first lecture (finite type criterion for hypersurfaces):

Theorem

The real hypersurface $M \subset \mathbb{C}^N$ is of finite type at 0 if and only if there exists a positive integer ℓ, $2 \leq \ell \leq 4$, such that in any neighborhood U of 0 in $\mathbb{C}^{(N-1)\ell}$ there exists $t_0 \in U$ such that

$$\operatorname{rk} \frac{\partial v^\ell}{\partial t^{[\ell]}}(t_0) = N, \quad v^\ell(t_0) = 0. \tag{11}$$

Iteration along iterated Segre sets

Hence for every multiindex $\beta \in \mathbb{N}^N$, $((\partial^\beta H) \circ v^2)(t^1, t^2)$ is a convergent mapping.
Iterating this procedure to higher order, we easily see that for every integer j and every multiindex $\beta \in \mathbb{N}^N$, $((\partial^\beta H) \circ v^j)(t^{[j]})$, $t^{[j]} = (t^1, \dots, t^j)$, is a convergent holomorphic map from $\mathbb{C}^{(N-1)j}$ to $\mathbb{C}^N$.
Now we recall the following from the first lecture (finite type criterion for hypersurfaces):

Theorem

The real hypersurface $M \subset \mathbb{C}^N$ is of finite type at 0 if and only if there exists a positive integer ℓ, $2 \le \ell \le 4$, such that in any neighborhood U of 0 in $\mathbb{C}^{(N-1)\ell}$ there exists $t_0 \in U$ such that

$$\operatorname{rk} \frac{\partial v^\ell}{\partial t^{[\ell]}}(t_0) = N, \quad v^\ell(t_0) = 0. \tag{11}$$

Iteration along iterated Segre sets

It is now easy to conclude the convergence of the mapping as follows.

Choose ℓ as in the theorem and a neighborhood U of 0 in $\mathbb{C}^{(N-1)\ell}$ such that $H \circ v^\ell$ is holomorphic in U. Pick $t_0 \in U$ as in the theorem. By the rank theorem there exists a convergent holomorphic map $\Theta\colon (\mathbb{C}^N, 0) \to \mathbb{C}^{(N-1)\ell}$ with $\Theta(0) = t_0$ and satisfying $v^\ell \circ \Theta = \mathrm{Id}_{\mathbb{C}^N}$. Hence

$$H = (H \circ v^\ell) \circ \Theta$$

is a convergent holomorphic map.

Iteration along iterated Segre sets

It is now easy to conclude the convergence of the mapping as follows.

Choose ℓ as in the theorem and a neighborhood U of 0 in $\mathbb{C}^{(N-1)\ell}$ such that $H \circ v^\ell$ is holomorphic in U. Pick $t_0 \in U$ as in the theorem. By the rank theorem there exists a convergent holomorphic map $\Theta \colon (\mathbb{C}^N, 0) \to \mathbb{C}^{(N-1)\ell}$ with $\Theta(0) = t_0$ and satisfying $v^\ell \circ \Theta = \mathrm{Id}_{\mathbb{C}^N}$. Hence

$$H = (H \circ v^\ell) \circ \Theta$$

is a convergent holomorphic map.

Iteration along iterated Segre sets

It is now easy to conclude the convergence of the mapping as follows.

Choose ℓ as in the theorem and a neighborhood U of 0 in $\mathbb{C}^{(N-1)\ell}$ such that $H \circ v^\ell$ is holomorphic in U. Pick $t_0 \in U$ as in the theorem. By the rank theorem there exists a convergent holomorphic map $\Theta\colon (\mathbb{C}^N, 0) \to \mathbb{C}^{(N-1)\ell}$ with $\Theta(0) = t_0$ and satisfying $v^\ell \circ \Theta = \mathrm{Id}_{\mathbb{C}^N}$. Hence

$$H = (H \circ v^\ell) \circ \Theta$$

is a convergent holomorphic map.

Beyond Levi-nondegeneracy

Relaxing the Levi-nondegeneracy condition on M' is the natural next step.

- (a) If M' is k_0-nondegenerate as defined in the first lecture, the previous arguments all go through (with a very simple little change in the basic reflection identity).
- (b)The situation becomes technically more difficult if one assumes that M' does not contain any complex-analytic subvariety. This situation has been dealt with by Baouendi, Ebenfelt and Rothschild in 2000 (even for so-called *essentially finite* CR manifolds).
- Both for (a) and (b), the convergence of formal invertible CR maps with such targets (and finite type sources) has been shown to be true (Baouendi-Ebenfelt-Rothschild), though situations in (a) and (b) can be very different. Indeed, case (a) allows for targets entirely foliated by complex-analytic curves. And case (b) allows for Levi-degeneracies that can be "worse" than those in (a).

Beyond Levi-nondegeneracy

Relaxing the Levi-nondegeneracy condition on M' is the natural next step.

- (a) If M' is k_0-nondegenerate as defined in the first lecture, the previous arguments all go through (with a very simple little change in the basic reflection identity).
- (b)The situation becomes technically more difficult if one assumes that M' does not contain any complex-analytic subvariety. This situation has been dealt with by Baouendi, Ebenfelt and Rothschild in 2000 (even for so-called *essentially finite* CR manifolds).
- Both for (a) and (b), the convergence of formal invertible CR maps with such targets (and finite type sources) has been shown to be true (Baouendi-Ebenfelt-Rothschild), though situations in (a) and (b) can be very different. Indeed, case (a) allows for targets entirely foliated by complex-analytic curves. And case (b) allows for Levi-degeneracies that can be "worse" than those in (a).

Beyond Levi-nondegeneracy

Relaxing the Levi-nondegeneracy condition on M' is the natural next step.

- (a) If M' is k_0-nondegenerate as defined in the first lecture, the previous arguments all go through (with a very simple little change in the basic reflection identity).
- (b)The situation becomes technically more difficult if one assumes that M' does not contain any complex-analytic subvariety. This situation has been dealt with by Baouendi, Ebenfelt and Rothschild in 2000 (even for so-called *essentially finite* CR manifolds).
- Both for (a) and (b), the convergence of formal invertible CR maps with such targets (and finite type sources) has been shown to be true (Baouendi-Ebenfelt-Rothschild), though situations in (a) and (b) can be very different. Indeed, case (a) allows for targets entirely foliated by complex-analytic curves. And case (b) allows for Levi-degeneracies that can be "worse" than those in (a).

Beyond Levi-nondegeneracy

Relaxing the Levi-nondegeneracy condition on M' is the natural next step.

- (a) If M' is k_0-nondegenerate as defined in the first lecture, the previous arguments all go through (with a very simple little change in the basic reflection identity).
- (b)The situation becomes technically more difficult if one assumes that M' does not contain any complex-analytic subvariety. This situation has been dealt with by Baouendi, Ebenfelt and Rothschild in 2000 (even for so-called *essentially finite* CR manifolds).
- Both for (a) and (b), the convergence of formal invertible CR maps with such targets (and finite type sources) has been shown to be true (Baouendi-Ebenfelt-Rothschild), though situations in (a) and (b) can be very different. Indeed, case (a) allows for targets entirely foliated by complex-analytic curves. And case (b) allows for Levi-degeneracies that can be "worse" than those in (a).

Beyond Levi-nondegeneracy

Relaxing the Levi-nondegeneracy condition on M' is the natural next step.

- (a) If M' is k_0-nondegenerate as defined in the first lecture, the previous arguments all go through (with a very simple little change in the basic reflection identity).
- (b)The situation becomes technically more difficult if one assumes that M' does not contain any complex-analytic subvariety. This situation has been dealt with by Baouendi, Ebenfelt and Rothschild in 2000 (even for so-called *essentially finite* CR manifolds).
- Both for (a) and (b), the convergence of formal invertible CR maps with such targets (and finite type sources) has been shown to be true (Baouendi-Ebenfelt-Rothschild), though situations in (a) and (b) can be very different. Indeed, case (a) allows for targets entirely foliated by complex-analytic curves. And case (b) allows for Levi-degeneracies that can be "worse" than those in (a).

Beyond Levi-nondegeneracy: holomorphic nondegeneracy

Hence it is natural to ask if there exists some more general geometric condition that would take care of both previous type of Levi-degeneracies at the same time and that would, if possible, be also necessary to obtain the convergence of formal equivalences between real-analytic CR manifolds.
This is what **holomorphic nondegeneracy** is here for!

Definition

(Stanton) Let $M \subset \mathbb{C}^N$ be a generic real-analytic submanifold and $p \in M$. It is said to be holomorphically degenerate at p if there exists a nontrivial holomorphic vector field X (i.e. a (1,0) holomorphic vector field with holomorphic coefficients) near p such that X is tangent to M near p.

Beyond Levi-nondegeneracy: holomorphic nondegeneracy

Hence it is natural to ask if there exists some more general geometric condition that would take care of both previous type of Levi-degeneracies at the same time and that would, if possible, be also necessary to obtain the convergence of formal equivalences between real-analytic CR manifolds.
This is what **holomorphic nondegeneracy** is here for!

Definition

(Stanton) Let $M \subset \mathbb{C}^N$ be a generic real-analytic submanifold and $p \in M$. It is said to be holomorphically degenerate at p if there exists a nontrivial holomorphic vector field X (i.e. a (1,0) holomorphic vector field with holomorphic coefficients) near p such that X is tangent to M near p.

Beyond Levi-nondegeneracy: holomorphic nondegeneracy

Hence it is natural to ask if there exists some more general geometric condition that would take care of both previous type of Levi-degeneracies at the same time and that would, if possible, be also necessary to obtain the convergence of formal equivalences between real-analytic CR manifolds. This is what **holomorphic nondegeneracy** is here for!

Definition

(Stanton) Let $M \subset \mathbb{C}^N$ be a generic real-analytic submanifold and $p \in M$. It is said to be holomorphically degenerate at p if there exists a nontrivial holomorphic vector field X (i.e. a (1,0) holomorphic vector field with holomorphic coefficients) near p such that X is tangent to M near p.

Some properties of hol. nondeg.

- Propagation : if M is connected, then M is holomorphically degenerate at one point of M if and only if it is holomorphically degenerate at all its points.
- If M is connected, M is holomorphically degenerate if and only if for a generic point $q \in M$, $(M, q) \sim_{bihol} (\widetilde{M} \times \mathbb{C}, 0)$ for some generic submanifold $\widetilde{M} \subset \mathbb{C}^{N-1}$ through 0.
- A germ of a k_0-nondegenerate real-analytic generic submanifold is holomorphically nondegenerate. Conversely, any (connected) holomorphically nondeg. real-analytic generic submanifold is k-nondegenerate (for some k) on a Zariski open subset.
- Any real-analytic generic submanifold of D'Angelo finite type is holomorphically nondegenerate. The converse is not true. In fact, there exists holomorphically nondegenerate real-analytic hypersurfaces that are entirely foliated by complex curves (Tube over the light cone).

Some properties of hol. nondeg.

- Propagation : if M is connected, then M is holomorphically degenerate at one point of M if and only if it is holomorphically degenerate at all its points.
- If M is connected, M is holomorphically degenerate if and only if for a generic point $q \in M$, $(M, q) \sim_{bihol} (\widetilde{M} \times \mathbb{C}, 0)$ for some generic submanifold $\widetilde{M} \subset \mathbb{C}^{N-1}$ through 0.
- A germ of a k_0-nondegenerate real-analytic generic submanifold is holomorphically nondegenerate. Conversely, any (connected) holomorphically nondeg. real-analytic generic submanifold is k-nondegenerate (for some k) on a Zariski open subset.
- Any real-analytic generic submanifold of D'Angelo finite type is holomorphically nondegenerate. The converse is not true. In fact, there exists holomorphically nondegenerate real-analytic hypersurfaces that are entirely foliated by complex curves (Tube over the light cone).

Some properties of hol. nondeg.

- Propagation : if M is connected, then M is holomorphically degenerate at one point of M if and only if it is holomorphically degenerate at all its points.
- If M is connected, M is holomorphically degenerate if and only if for a generic point $q \in M$, $(M, q) \sim_{bihol} (\widetilde{M} \times \mathbb{C}, 0)$ for some generic submanifold $\widetilde{M} \subset \mathbb{C}^{N-1}$ through 0.
- A germ of a k_0-nondegenerate real-analytic generic submanifold is holomorphically nondegenerate. Conversely, any (connected) holomorphically nondeg. real-analytic generic submanifold is k-nondegenerate (for some k) on a Zariski open subset.
- Any real-analytic generic submanifold of D'Angelo finite type is holomorphically nondegenerate. The converse is not true. In fact, there exists holomorphically nondegenerate real-analytic hypersurfaces that are entirely foliated by complex curves (Tube over the light cone).

Some properties of hol. nondeg.

- Propagation : if M is connected, then M is holomorphically degenerate at one point of M if and only if it is holomorphically degenerate at all its points.
- If M is connected, M is holomorphically degenerate if and only if for a generic point $q \in M$, $(M, q) \sim_{bihol} (\widetilde{M} \times \mathbb{C}, 0)$ for some generic submanifold $\widetilde{M} \subset \mathbb{C}^{N-1}$ through 0.
- A germ of a k_0-nondegenerate real-analytic generic submanifold is holomorphically nondegenerate. Conversely, any (connected) holomorphically nondeg. real-analytic generic submanifold is k-nondegenerate (for some k) on a Zariski open subset.
- Any real-analytic generic submanifold of D'Angelo finite type is holomorphically nondegenerate. The converse is not true. In fact, there exists holomorphically nondegenerate real-analytic hypersurfaces that are entirely foliated by complex curves (Tube over the light cone).

Hol. nondeg. and convergence

Most, if not all, previous properties were proven by Baouendi, Ebenfelt and Rothschild. They also observed the following:

Proposition

Let $M \subset \mathbb{C}^N$ be a connected generic real-analytic holomorphically degenerate submanifold. Then for every $p \in M$, there exists a divergent formal CR equivalence $H \colon (M, p) \to (M, p)$.

Proof.

Let $p \in M$ and let X be a nontrivial holomorphic vector field tangent to M near p. Let $\varphi(t, z)$ be the holomorphic flow of X for complex time t, for $|t| < \epsilon$, $|z - p| < \epsilon$. Because X is tangent to M, $\varphi(t, z)$ is a one-complex parameter family of local biholomorphisms of $\mathbb{C}^N$ fixing p and M. Let $\delta(z)$ be any divergent formal holomorphic power series such that $\delta(z) = O(|z - p|^2)$. Then $H(z) := \varphi(\delta(z), z)$ is a formal divergent CR equivalence sending (M, p) into itself. $\square$

Hol. nondeg. and convergence

Most, if not all, previous properties were proven by Baouendi, Ebenfelt and Rothschild. They also observed the following:

Proposition

Let $M \subset \mathbb{C}^N$ be a connected generic real-analytic holomorphically degenerate submanifold. Then for every $p \in M$, there exists a divergent formal CR equivalence $H\colon (M,p) \to (M,p)$.

Proof.

Let $p \in M$ and let X be a nontrivial holomorphic vector field tangent to M near p. Let $\varphi(t,z)$ be the holomorphic flow of X for complex time t, for $|t| < \epsilon$, $|z-p| < \epsilon$. Because X is tangent to M, $\varphi(t,z)$ is a one-complex parameter family of local biholomorphisms of $\mathbb{C}^N$ fixing p and M. Let $\delta(z)$ be any divergent formal holomorphic power series such that $\delta(z) = O(|z-p|^2)$. Then $H(z) := \varphi(\delta(z),z)$ is a formal divergent CR equivalence sending (M,p) into itself. $\square$

Hol. nondeg. and convergence

Most, if not all, previous properties were proven by Baouendi, Ebenfelt and Rothschild. They also observed the following:

Proposition

Let $M \subset \mathbb{C}^N$ be a connected generic real-analytic holomorphically degenerate submanifold. Then for every $p \in M$, there exists a divergent formal CR equivalence $H \colon (M, p) \to (M, p)$.

Proof.

Let $p \in M$ and let X be a nontrivial holomorphic vector field tangent to M near p. Let $\varphi(t, z)$ be the holomorphic flow of X for complex time t, for $|t| < \epsilon$, $|z - p| < \epsilon$. Because X is tangent to M, $\varphi(t, z)$ is a one-complex parameter family of local biholomorphisms of $\mathbb{C}^N$ fixing p and M. Let $\delta(z)$ be any divergent formal holomorphic power series such that $\delta(z) = O(|z - p|^2)$. Then $H(z) := \varphi(\delta(z), z)$ is a formal divergent CR equivalence sending (M, p) into itself. $\square$

The general convergence result

It was therefore asked whether holomorphic nondegeneracy is a sufficient condition that guarantees the convergence of all formal CR equivalences. This was answered by the affirmative in 2002 for finite type generic submanifolds:

Theorem (Baouendi, M., Rothschild, 2002 – convergence theorem)

Let $M, M' \subset \mathbb{C}^N$ be (connected) generic real-analytic submanifolds with M holomorphically nondegenerate and of finite type. Then for every $p \in M$, any formal CR equivalence $H\colon (M, p) \to M'$ necessarily converges.

A few words regarding the finite type assumption :

- The result can not hold for generic submanifolds that are everywhere of infinite type, (consider e.g. $M = M_1 \times \mathbb{R}^d$).
- There was a left open question of whether relaxing the condition "finite type" in the previous theorem to "generically of finite type" was possible.

The general convergence result

It was therefore asked whether holomorphic nondegeneracy is a sufficient condition that guarantees the convergence of all formal CR equivalences. This was answered by the affirmative in 2002 for finite type generic submanifolds:

Theorem (Baouendi, M., Rothschild, 2002 – convergence theorem)

Let $M, M' \subset \mathbb{C}^N$ be (connected) generic real-analytic submanifolds with M holomorphically nondegenerate and of finite type. Then for every $p \in M$, any formal CR equivalence $H\colon (M,p) \to M'$ necessarily converges.

A few words regarding the finite type assumption :

- The result can not hold for generic submanifolds that are everywhere of infinite type, (consider e.g. $M = M_1 \times \mathbb{R}^d$).
- There was a left open question of whether relaxing the condition "finite type" in the previous theorem to "generically of finite type" was possible.

The general convergence result

It was therefore asked whether holomorphic nondegeneracy is a sufficient condition that guarantees the convergence of all formal CR equivalences. This was answered by the affirmative in 2002 for finite type generic submanifolds:

Theorem (Baouendi, M., Rothschild, 2002 – convergence theorem)

Let $M, M' \subset \mathbb{C}^N$ be (connected) generic real-analytic submanifolds with M holomorphically nondegenerate and of finite type. Then for every $p \in M$, any formal CR equivalence $H\colon (M,p) \to M'$ necessarily converges.

A few words regarding the finite type assumption :

- The result can not hold for generic submanifolds that are everywhere of infinite type, (consider e.g. $M = M_1 \times \mathbb{R}^d$).
- There was a left open question of whether relaxing the condition "finite type" in the previous theorem to "generically of finite type" was possible.

The general convergence result

It was therefore asked whether holomorphic nondegeneracy is a sufficient condition that guarantees the convergence of all formal CR equivalences. This was answered by the affirmative in 2002 for finite type generic submanifolds:

Theorem (Baouendi, M., Rothschild, 2002 – convergence theorem)

Let $M, M' \subset \mathbb{C}^N$ be (connected) generic real-analytic submanifolds with M holomorphically nondegenerate and of finite type. Then for every $p \in M$, any formal CR equivalence $H\colon (M,p) \to M'$ necessarily converges.

A few words regarding the finite type assumption :

- The result can not hold for generic submanifolds that are everywhere of infinite type, (consider e.g. $M = M_1 \times \mathbb{R}^d$).
- There was a left open question of whether relaxing the condition "finite type" in the previous theorem to "generically of finite type" was possible.

The general convergence result

It was therefore asked whether holomorphic nondegeneracy is a sufficient condition that guarantees the convergence of all formal CR equivalences. This was answered by the affirmative in 2002 for finite type generic submanifolds:

Theorem (Baouendi, M., Rothschild, 2002 – convergence theorem)

Let $M, M' \subset \mathbb{C}^N$ be (connected) generic real-analytic submanifolds with M holomorphically nondegenerate and of finite type. Then for every $p \in M$, any formal CR equivalence $H\colon (M,p) \to M'$ necessarily converges.

A few words regarding the finite type assumption :

- The result can not hold for generic submanifolds that are everywhere of infinite type, (consider e.g. $M = M_1 \times \mathbb{R}^d$).
- There was a left open question of whether relaxing the condition "finite type" in the previous theorem to "generically of finite type" was possible.

The divergence theorem

This has been answered very recently by the negative by Kossovskiy and Shafikov (2016).

Theorem (Kossovskiy-Shafikov, 2016 – divergence theorem)

There exists real-analytic hypersurfaces $M \subset \mathbb{C}^N$ that are holomorphically nondegenerate (and hence automatically generically of finite type) and (infinite type) points $p \in M$ where (M, p) admits formal divergent self CR maps.

Main lines of the proof

The (modern but not original!) proof of the general convergence theorem for formal CR equivalences is done mainly through 2 steps:

1) Derivation of singular systems of reflections identities (using that M, M' are holomorphically nondegenerate): these singular systems are basically the substitute of the "nice" reflection identities we had in the Levi-nondegenerate.

2) A general convergence result for formal power series mappings with formal parameters ("the hammer"), valid only for finite type generic submanifolds.

Main lines of the proof

The (modern but not original!) proof of the general convergence theorem for formal CR equivalences is done mainly through 2 steps:

1) Derivation of singular systems of reflections identities (using that M, M' are holomorphically nondegenerate): these singular systems are basically the substitute of the "nice" reflection identities we had in the Levi-nondegenerate.

2) A general convergence result for formal power series mappings with formal parameters ("the hammer"), valid only for finite type generic submanifolds.

Main lines of the proof

The (modern but not original!) proof of the general convergence theorem for formal CR equivalences is done mainly through 2 steps:

1) Derivation of singular systems of reflections identities (using that M, M' are holomorphically nondegenerate): these singular systems are basically the substitute of the "nice" reflection identities we had in the Levi-nondegenerate.

2) A general convergence result for formal power series mappings with formal parameters ("the hammer"), valid only for finite type generic submanifolds.

Singular systems of reflection identities

Proposition

Let $M, M' \subset \mathbb{C}^N$ be (connected) generic real-analytic submanifolds through the origin. Let $H\colon (M,0) \to (M',0)$ be a formal CR equivalence and assume that M is holomorphically nondegenerate. Then there exists an integer ℓ (depending only on M) and convergent holomorphic power series $\Theta_1, \ldots, \Theta_N$ in all their arguments such that

$$\Theta_i(z, \bar{z}, (\overline{(\partial^\alpha H)(z)})_{|\alpha|\leq \ell}, H(z))|_M = 0, \; i = 1, \ldots, N \tag{12}$$

$$\det \left(\frac{\partial \Theta_i}{\partial w_j} \left(z, \bar{z}, (\overline{(\partial^\alpha H)(z)})_{|\alpha|\leq \ell}, H(z) \right) \right)_{i,j} \Big|_M \not\equiv 0 \tag{13}$$

Singular systems of reflection identities

Sketch of proof.

One may derive such singular systems by applying repeatedly the CR vector fields to the fundamental identity

$$\rho'(H(z), \overline{H(z)})|_M = 0, \tag{14}$$

and using the invertibility of the map H as well as a useful criterion due to Stanton detecting holomorphic nondegeneracy from a given defining function of M' (written in the so-called complex form). □

The hammer

Proposition

Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold through the origin and $\Theta = (\Theta_1, \ldots, \Theta_{N'})$ be a convergent power series mapping with components in $\mathbb{C}\{z, \bar{z}, \lambda, w\}$ where $z \in \mathbb{C}^N$, $w \in \mathbb{C}^{N'}$, $\lambda \in \mathbb{C}^r$, $N', N, r \geq 1$. Let $h\colon (\mathbb{C}^N, 0) \to \mathbb{C}^{N'}$, $g\colon (\mathbb{C}^N, 0) \to \mathbb{C}^r$ be formal holomorphic power series mappings, vanishing at 0, satisfying

$$\Theta(z, \bar{z}, \overline{g(z)}, h(z))|_M = 0, \quad \text{and}$$

$$\det \frac{\partial \Theta}{\partial w}\left(z, \bar{z}, \overline{g(z)}, h(z)\right)\Big|_M \not\equiv 0.$$

If M is of finite type at 0, then h is a convergent holomorphic map.

The proof of the general convergence theorem follows immediately from the previous proposition and the "hammer".

The hammer

Proposition

Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold through the origin and $\Theta = (\Theta_1, \ldots, \Theta_{N'})$ be a convergent power series mapping with components in $\mathbb{C}\{z, \bar{z}, \lambda, w\}$ where $z \in \mathbb{C}^N$, $w \in \mathbb{C}^{N'}$, $\lambda \in \mathbb{C}^r$, $N', N, r \geq 1$. Let $h\colon (\mathbb{C}^N, 0) \to \mathbb{C}^{N'}$, $g\colon (\mathbb{C}^N, 0) \to \mathbb{C}^r$ be formal holomorphic power series mappings, vanishing at 0, satisfying

$$\Theta(z, \bar{z}, \overline{g(z)}, h(z))|_M = 0, \quad \text{and}$$

$$\det \frac{\partial \Theta}{\partial w} \left(z, \bar{z}, \overline{g(z)}, h(z) \right) \Big|_M \not\equiv 0.$$

If M is of finite type at 0, then h is a convergent holomorphic map.

The proof of the general convergence theorem follows immediately from the previous proposition and the "hammer".

Specific features of the hammer

The previous convergence result possesses some specific features setting it apart from known results in the literature and making it impossible to deduce its conclusion from these existing classical convergence results for systems of (singular) analytic equations.

- The system of equations is valid when restricted to a certain finite type generic submanifold instead of being valid in the ambient euclidean space.
- The system allows the appearance of a formal power series mapping g that is not related to the solution mapping h and that can be even divergent. The conclusion is nevertheless that the formal mapping h has to converge.
- Even in the nonsingular case, i.e. the determinant does not vanish at 0, the result can not be obtained by using only the implicit function theorem and the iterated Segre technique mentioned in the Levi-nondegenerate case.

Specific features of the hammer

The previous convergence result possesses some specific features setting it apart from known results in the literature and making it impossible to deduce its conclusion from these existing classical convergence results for systems of (singular) analytic equations.

- The system of equations is valid when restricted to a certain finite type generic submanifold instead of being valid in the ambient euclidean space.
- The system allows the appearance of a formal power series mapping g that is not related to the solution mapping h and that can be even divergent. The conclusion is nevertheless that the formal mapping h has to converge.
- Even in the nonsingular case, i.e. the determinant does not vanish at 0, the result can not be obtained by using only the implicit function theorem and the iterated Segre technique mentioned in the Levi-nondegenerate case.

Specific features of the hammer

The previous convergence result possesses some specific features setting it apart from known results in the literature and making it impossible to deduce its conclusion from these existing classical convergence results for systems of (singular) analytic equations.

- The system of equations is valid when restricted to a certain finite type generic submanifold instead of being valid in the ambient euclidean space.
- The system allows the appearance of a formal power series mapping g that is not related to the solution mapping h and that can be even divergent. The conclusion is nevertheless that the formal mapping h has to converge.
- Even in the nonsingular case, i.e. the determinant does not vanish at 0, the result can not be obtained by using only the implicit function theorem and the iterated Segre technique mentioned in the Levi-nondegenerate case.

Specific features of the hammer

The previous convergence result possesses some specific features setting it apart from known results in the literature and making it impossible to deduce its conclusion from these existing classical convergence results for systems of (singular) analytic equations.

- The system of equations is valid when restricted to a certain finite type generic submanifold instead of being valid in the ambient euclidean space.
- The system allows the appearance of a formal power series mapping g that is not related to the solution mapping h and that can be even divergent. The conclusion is nevertheless that the formal mapping h has to converge.
- Even in the nonsingular case, i.e. the determinant does not vanish at 0, the result can not be obtained by using only the implicit function theorem and the iterated Segre technique mentioned in the Levi-nondegenerate case.

Specific features of the hammer

The previous convergence result possesses some specific features setting it apart from known results in the literature and making it impossible to deduce its conclusion from these existing classical convergence results for systems of (singular) analytic equations.

- The system of equations is valid when restricted to a certain finite type generic submanifold instead of being valid in the ambient euclidean space.
- The system allows the appearance of a formal power series mapping g that is not related to the solution mapping h and that can be even divergent. The conclusion is nevertheless that the formal mapping h has to converge.
- Even in the nonsingular case, i.e. the determinant does not vanish at 0, the result can not be obtained by using only the implicit function theorem and the iterated Segre technique mentioned in the Levi-nondegenerate case.

Ingredients in the proof of the hammer

The proof of the proposition consists of :

- A result of the partial convergence properties of formal power series mappings solutions of certain analytic equations containing formal parameters – in particular, a propagation convergence property along subspaces (in the spirit of previous works from M., and Meylan, M. and Zaitsev).
- Application of the above partial convergence result in conjunction with the iterated Segre mapping technique (with parameters) previously discussed.

Ingredients in the proof of the hammer

The proof of the proposition consists of :

- A result of the partial convergence properties of formal power series mappings solutions of certain analytic equations containing formal parameters – in particular, a propagation convergence property along subspaces (in the spirit of previous works from M., and Meylan, M. and Zaitsev).
- Application of the above partial convergence result in conjunction with the iterated Segre mapping technique (with parameters) previously discussed.

Ingredients in the proof of the hammer

The proof of the proposition consists of :

- A result of the partial convergence properties of formal power series mappings solutions of certain analytic equations containing formal parameters – in particular, a propagation convergence property along subspaces (in the spirit of previous works from M., and Meylan, M. and Zaitsev).
- Application of the above partial convergence result in conjunction with the iterated Segre mapping technique (with parameters) previously discussed.

Ingredients in the proof of the hammer

The proof of the proposition consists of :

- A result of the partial convergence properties of formal power series mappings solutions of certain analytic equations containing formal parameters – in particular, a propagation convergence property along subspaces (in the spirit of previous works from M., and Meylan, M. and Zaitsev).
- Application of the above partial convergence result in conjunction with the iterated Segre mapping technique (with parameters) previously discussed.

Propagation of partial convergence

Lemma

Let $R = (R_1, \ldots, R_\ell)$ be a convergent power series with components in $\mathbb{C}\{s, t, x, \lambda, Y\}$ with $s \in \mathbb{C}^{k_1}$, $t \in \mathbb{C}^{k_2}$, $x \in \mathbb{C}^{k_3}$, $\lambda \in \mathbb{C}^{k_4}$, $Y \in \mathbb{C}^\ell$, $\ell, k_i \geq 1$. Let $\Delta(s, x)$ and $\psi(s, t, x)$ be respectively $\mathbb{C}^{k_4}$-valued and $\mathbb{C}^\ell$-valued formal power series mappings, vanishing at the origin, satisfying

$$\begin{cases} R\,(s, t, x, \Delta(s, x), \psi(s, t, x)) = 0, \\ \eta(s, t, x) := \det \frac{\partial R}{\partial Y}\,(s, t, x, \Delta(s, x), \psi(s, t, x)) \not\equiv 0. \end{cases} \tag{15}$$

Assume that all partial derivatives of ψ are convergent along the subspace $F := \{t = 0, x = 0\}$. Then for every $\gamma \in \mathbb{N}^{k_3}$, $\partial_x^\gamma \psi$ is convergent along the subspace $E := \{x = 0\}$.

The proof uses the fact that the power series Δ, though possibly divergent, does not depend on t, Artin's approximation theorem and some ideas from Meylan, M., Zaitsev and from Juhlin.

Propagation of partial convergence

Lemma

Let $R = (R_1, \ldots, R_\ell)$ be a convergent power series with components in $\mathbb{C}\{s, t, x, \lambda, Y\}$ with $s \in \mathbb{C}^{k_1}$, $t \in \mathbb{C}^{k_2}$, $x \in \mathbb{C}^{k_3}$, $\lambda \in \mathbb{C}^{k_4}$, $Y \in \mathbb{C}^\ell$, $\ell, k_i \geq 1$. Let $\Delta(s, x)$ and $\psi(s, t, x)$ be respectively $\mathbb{C}^{k_4}$-valued and $\mathbb{C}^\ell$-valued formal power series mappings, vanishing at the origin, satisfying

$$\begin{cases} R\,(s, t, x, \Delta(s, x), \psi(s, t, x)) = 0, \\ \eta(s, t, x) := \det \frac{\partial R}{\partial Y}\,(s, t, x, \Delta(s, x), \psi(s, t, x)) \not\equiv 0. \end{cases} \tag{15}$$

Assume that all partial derivatives of ψ are convergent along the subspace $F := \{t = 0, x = 0\}$. Then for every $\gamma \in \mathbb{N}^{k_3}$, $\partial_x^\gamma \psi$ is convergent along the subspace $E := \{x = 0\}$.

The proof uses the fact that the power series Δ, though possibly divergent, does not depend on t, Artin's approximation theorem and some ideas from Meylan, M., Zaitsev and from Juhlin.

Iterated Segre mappings are back

The final step in the proof of the hammer is to use the previous lemma applied to the iterated Segre mappings with parameters.

We choose a real-analytic submanifold $\Sigma \subset M$ of real dimension d passing through 0 transverse to $T_0^c M$. We parametrize such a curve by $(\mathbb{R}_x^d, 0) \ni x \mapsto \psi(x)$.

Let $n = \dim_{CR} M$. Consider the iterated Segre mappings attached to every point of Σ i.e. :

$$v^j : \mathbb{C}^{nj} \times \mathbb{R}^d \to \mathbb{C}^N, \; v^j(t^{[j]}, x) := v^j(t^{[j]}; \psi(x)).$$

We then use the lemma to prove the following propagation property of partial convergence : If $j \geq 0$ and if for every multiindex $\alpha \in \mathbb{N}^N$, $(\partial^\alpha h) \circ v^j(t^{[j]}, x)$ is convergent along the subspace $S = \{x = 0\}$, then for every multiindex $\alpha \in \mathbb{N}^N$, $(\partial^\alpha h) \circ v^{j+2}(t^{[j+2]}, x)$ is also convergent along S.

Iterated Segre mappings are back

The final step in the proof of the hammer is to use the previous lemma applied to the iterated Segre mappings with parameters. We choose a real-analytic submanifold $\Sigma \subset M$ of real dimension d passing through 0 transverse to $T_0^c M$. We parametrize such a curve by $(\mathbb{R}_x^d, 0) \ni x \mapsto \psi(x)$.

Let $n = \dim_{CR} M$. Consider the iterated Segre mappings attached to every point of Σ i.e. :

$$v^j : \mathbb{C}^{nj} \times \mathbb{R}^d \to \mathbb{C}^N, \ v^j(t^{[j]}, x) := v^j(t^{[j]}; \psi(x)).$$

We then use the lemma to prove the following propagation property of partial convergence : If $j \geq 0$ and if for every multiindex $\alpha \in \mathbb{N}^N$, $(\partial^\alpha h) \circ v^j(t^{[j]}, x)$ is convergent along the subspace $S = \{x = 0\}$, then for every multiindex $\alpha \in \mathbb{N}^N$, $(\partial^\alpha h) \circ v^{j+2}(t^{[j+2]}, x)$ is also convergent along S.

Iterated Segre mappings are back

The final step in the proof of the hammer is to use the previous lemma applied to the iterated Segre mappings with parameters. We choose a real-analytic submanifold $\Sigma \subset M$ of real dimension d passing through 0 transverse to $T_0^c M$. We parametrize such a curve by $(\mathbb{R}_x^d, 0) \ni x \mapsto \psi(x)$.
Let $n = \dim_{CR} M$. Consider the iterated Segre mappings attached to every point of Σ i.e. :

$$v^j \colon \mathbb{C}^{nj} \times \mathbb{R}^d \to \mathbb{C}^N, \; v^j(t^{[j]}, x) := v^j(t^{[j]}; \psi(x)).$$

We then use the lemma to prove the following propagation property of partial convergence : If $j \geq 0$ and if for every multiindex $\alpha \in \mathbb{N}^N$, $(\partial^\alpha h) \circ v^j(t^{[j]}, x)$ is convergent along the subspace $S = \{x = 0\}$, then for every multiindex $\alpha \in \mathbb{N}^N$, $(\partial^\alpha h) \circ v^{j+2}(t^{[j+2]}, x)$ is also convergent along S.

Iterated Segre mappings are back

The final step in the proof of the hammer is to use the previous lemma applied to the iterated Segre mappings with parameters. We choose a real-analytic submanifold $\Sigma \subset M$ of real dimension d passing through 0 transverse to $T_0^c M$. We parametrize such a curve by $(\mathbb{R}_x^d, 0) \ni x \mapsto \psi(x)$.
Let $n = \dim_{CR} M$. Consider the iterated Segre mappings attached to every point of Σ i.e. :

$$v^j \colon \mathbb{C}^{nj} \times \mathbb{R}^d \to \mathbb{C}^N, \; v^j(t^{[j]}, x) := v^j(t^{[j]}; \psi(x)).$$

We then use the lemma to prove the following propagation property of partial convergence : If $j \geq 0$ and if for every multiindex $\alpha \in \mathbb{N}^N$, $(\partial^\alpha h) \circ v^j(t^{[j]}, x)$ is convergent along the subspace $S = \{x = 0\}$, then for every multiindex $\alpha \in \mathbb{N}^N$, $(\partial^\alpha h) \circ v^{j+2}(t^{[j+2]}, x)$ is also convergent along S.

End of the proof of the hammer

The proof of the hammer is completed by noticing that for every multiindex $\alpha \in \mathbb{N}^N$, $(\partial^\alpha h) \circ v^0$ is convergent along S. Hence the same property holds for $(\partial^\alpha h) \circ v^j(t^{[j]}, x)$ for every j. We conclude the proof using the submersivity property of the iterated Segre mappings $v^j(t^{[j]}, 0)$ for j large enough (coming from the finite type criterion already discussed).