Minicourse
Convergence of formal maps I

Bernhard Lamel

Fakultät für Mathematik

universität wien

Nordine Mir

Texas A&M University at Qatar

TEXAS A&M UNIVERSITY at QATAR

Serra Negra, Brazil
August 2017

Formal power series

A formal power series in the variables $x = (x_1, \ldots, x_m)$ is an expression of the form

$$A(x) = \sum_{\alpha \in \mathbb{N}^m} A_\alpha x^\alpha, \quad A_\alpha \in \mathbb{C}.$$

- $A(x)$ is just a notation - not a number for $x \in \mathbb{C}^m$!
- Exception: $A(0) = A_0 \in \mathbb{C}$
- Space of formal power series in x: $\mathbb{C}[\![x]\!]$
- Product: $AB(x) = A(x)B(x) = \sum_{\gamma \in \mathbb{N}^m} \left(\sum_{\alpha+\beta=\gamma} A_\alpha B_\beta \right) x^\gamma$
- $\mathbb{C}[\![x]\!]$ is a local algebra over $\mathbb{C}$ with maximal ideal $\hat{\mathfrak{m}} = \{A \in \mathbb{C}[\![x]\!] : A(0) = 0\}$.

Convergence

A formal power series $A(x)$ is *convergent* if there exists an $x_0 \in \mathbb{C}^m$, $x_0 \neq 0$ such that the series $A(x_0)$ of complex numbers converges.

- Convergence is absolut and uniform on (small) polydiscs
- Equivalent: $\exists C, R > 0\colon |A_\alpha| \leq CR^{|\alpha|}$
- Space of convergent power series: $\mathbb{C}\{x\}$
- $\mathbb{C}\{x\}$ is a (local) subalgebra of $\mathbb{C}[\![x]\!]$
- Closed under taking partial derivatives, composition, etc.

Mapping problems for analytic objects

For a family of analytic objects $\mathcal{F}$ and a family of analytic maps $\mathcal{H}$ the question whether for $F_1, F_2 \in \mathcal{F}$ there exists $H \in \mathcal{H}$ with $H(F_1) = F_2$ translates into a *formal problem* when considered as an equation between formal power series.
If one is able to solve this formal problem with a formal map $\hat{H}$ the question is whether such a solution is really a solution, i.e. whether $\hat{H}$ is (or can be chosen to be) *convergent*.

Example

For $A \in \mathbb{C}[\![x]\!]^m$ with $|A'(0)| \neq 0$, does there exist an inverse map A^{-1}?

Yes: Simple linear algebra on the formal level/estimates to get convergence.

Mapping problems for analytic objects

For a family of analytic objects $\mathcal{F}$ and a family of analytic maps $\mathcal{H}$ the question whether for $F_1, F_2 \in \mathcal{F}$ there exists $H \in \mathcal{H}$ with $H(F_1) = F_2$ translates into a *formal problem* when considered as an equation between formal power series.
If one is able to solve this formal problem with a formal map $\hat{H}$ the question is whether such a solution is really a solution, i.e. whether $\hat{H}$ is (or can be chosen to be) *convergent*.

Example

For $A \in \mathbb{C}[\![x]\!]^m$ with $|A'(0)| \neq 0$, does there exist an inverse map A^{-1}?
Yes: Simple linear algebra on the formal level/estimates to get convergence.

A well-known example: Normal forms for differential equations

The Poincarè-Dulac theorem

One can bring the ordinary differential equation

$$\dot{x} = Mx + O(x^2)$$

into the form

$$\dot{x} = Mx + \text{resonant terms}$$

via a *formal map*.

Convergence is guaranteed for linear parts M which belong to the Poincaré domain, or more generally, the existence of a convergent transformation into normal form can be guaranteed for linear parts whose Eigenvalues satisfy a “small divisor” condition.

Divergence at (irregular) singular points

Consider e.g. the system $\mathcal{E}_\gamma$ for $\gamma \in \mathbb{R}$:

$$t^4 \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \left(\begin{pmatrix} 0 & 0 \\ 0 & 2i \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} t + \begin{pmatrix} 0 & 0 \\ \gamma & -1 \end{pmatrix} t^3 \right) \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

All the systems $\mathcal{E}_\gamma$ are *formally equivalent* to (their normal form) $\mathcal{E}_0$.
But they cannot be *analytically equivalent*: One can show e.g. that $\mathcal{E}_\gamma$, $\gamma \neq 0$ possesses *no* analytic solutions–but $\mathcal{E}_0$ does:

$$x_1(t) = 1, \quad x_2(t) = 0.$$

Theorem (Artin's approximation theorem)

Let $A(x, y) \in \mathbb{C}\{x, y\}^k$*, and* $\hat{y}(x) \in \mathfrak{m}\mathbb{C}[\![x]\!]^n$ *be a formal power series map satisfying* $A(x, \hat{y}(x)) = 0$*. Then for any* $\ell \in \mathbb{N}$ *there exists* $y(x) \in \mathbb{C}\{x\}^n$ *with* $y(x) - \hat{y}(x) \in \hat{\mathfrak{m}}^\ell\mathbb{C}[\![x]\!]^n$ *and* $A(x, y(x)) = 0$*.*

Theorem (Gabrielov's theorem)

Let $H(x) \in \mathbb{C}\{x\}^k$ *be of generic full rank. If* $A(w) \in \mathbb{C}[\![w]\!]$ *satisfies* $A(H(x)) \in \mathbb{C}\{x\}$*, then* $A(w) \in \mathbb{C}\{w\}$*.*

Theorem (Artin's approximation theorem)

Let $A(x,y) \in \mathbb{C}\{x,y\}^k$, and $\hat{y}(x) \in \mathfrak{m}\mathbb{C}[\![x]\!]^n$ be a formal power series map satisfying $A(x,\hat{y}(x)) = 0$. Then for any $\ell \in \mathbb{N}$ there exists $y(x) \in \mathbb{C}\{x\}^n$ with $y(x) - \hat{y}(x) \in \hat{\mathfrak{m}}^\ell\mathbb{C}[\![x]\!]^n$ and $A(x,y(x)) = 0$.

Theorem (Gabrielov's theorem)

Let $H(x) \in \mathbb{C}\{x\}^k$ be of generic full rank. If $A(w) \in \mathbb{C}[\![w]\!]$ satisfies $A(H(x)) \in \mathbb{C}\{x\}$, then $A(w) \in \mathbb{C}\{w\}$.

General goal

Provide convergence for solutions of PDEs, in particular, for solutions to the tangential Cauchy-Riemann equations.
We are going to need to introduce a number of notions making important ODE concepts such as orbits and characteristics accessible to algebraic methods like Artin and Gabrielov.

Our setting

Real (analytic) submanifolds of complex spaces

$M \subset \mathbb{R}^k$ is a real submanifold if it can locally be defined by systems of nonsingular equations:

$$\forall p \in M \quad \exists U(p), \quad \exists \varrho = (\varrho^1, \dots, \varrho^d) \colon U(p) \to \mathbb{R}^d \colon$$

$$M \cap U(p) = \{\varrho = 0\}, \quad d\varrho^1 \wedge \cdots \wedge d\varrho^d \neq 0 \text{ on } U(p).$$

M is a real-analytic submanifold if ϱ can be chosen to be a convergent power series.
d is the *real codimension* of M. Only consider $d = 1$ in this talk.

Ideals and germs of real-analytic submanifolds

Instead of considering a germ (M, p), we can consider its *ideal*: $\mathcal{I}_M = \{f \in \mathbb{C}\{x - p\} \colon f|_M = 0\}$.

Complexification

Consider $Z \in \mathbb{C}^N = \mathbb{R}^{2N}$, set $x = \operatorname{Re} Z$, and $y = \operatorname{Im} Z$.
The rings $\mathbb{C}[\![x, y]\!]$ and $\mathbb{C}[\![Z, \bar{Z}]\!]$ are isomorphic, via the expected map $Z = x + iy$, $\bar{Z} = x - iy$.
Real power series (such as defining equations of real-analytic submanifolds) in $\mathbb{R}[\![x, y]\!]$ get mapped to fixed points of the involution $\iota \colon \mathbb{C}[\![Z, \bar{Z}]\!] \to \mathbb{C}[\![Z, \bar{Z}]\!]$,

$$\iota(\varrho(Z, \bar{Z})) = \bar{\varrho}(\bar{Z}, Z) = \sum_{\alpha,\beta} \bar{\varrho}_{\alpha,\beta} \bar{Z}^\alpha Z^\beta.$$

Complexification: Associate to a function of the *real* variables x, y the function of the complex variables $Z, \bar{Z}$, and consider $\bar{Z} = \zeta$ as an independent variable.

Basic fact

$$\tilde{A}(x, y) = 0 \Leftrightarrow A(Z, \zeta) = \tilde{A}\left(\frac{Z + \zeta}{2}, \frac{Z - \zeta}{2i}\right) = 0.$$

Complexification II

The complexification of a real-analytic submanifold M of $\mathbb{C}^N$ is the complex submanifold $\mathcal{M}$ of $\mathbb{C}^{2N}$ associated to the complexification of the ideal $\mathcal{I}_M$.
More concrete: If $M \subset \mathbb{C}^N_Z$, $Z = x + iy$ is a germ of a real-analytic hypersurface through 0, defined by $\varrho(x + iy, x - iy) = 0$, then $\mathcal{M} \subset \mathbb{C}^{2N}_{Z,\zeta}$, is the complex-analytic hypersurface defined by $\varrho(Z, \zeta) = 0$.
The complexification comes with 2 natural projection operators $\pi_S, \pi_E \colon \mathcal{M} \to \mathbb{C}^N$:

$$\pi_S(Z, \zeta) = Z, \quad \pi_E(Z, \zeta) = \zeta.$$

With $\iota \colon \mathbb{C}^{2N} \to \mathbb{C}^{2N}$, $\iota(Z, \zeta) = (\bar{\zeta}, \bar{Z})$: $\iota(\mathcal{M}) \subset \mathcal{M}$;

$$\pi_S \circ \iota = \overline{\pi_E}.$$

Cauchy-Riemann manifolds

We say that $M \subset \mathbb{C}^N$ is Cauchy-Riemann (CR) if the maximal complex subspaces of the tangent spaces of M,

$$T_p^c M = T_p M \cap i T_p M,$$

form a subbundle $T^c M$ of the tangent bundle of M, i.e. if $\dim T_p^c M$ is independent of p.
The structure bundle of the CR manifold M is the unique subbundle $\mathcal{V} \subset \mathbb{C}TM$ which satisfies $T^c M = \operatorname{Re} \mathcal{V}$. In coordinates, we have $\mathcal{V} = \mathbb{C}TM \cap T^{(0,1)}\mathbb{C}^N$. Sections of $\mathcal{V}$ are called CR vector fields, sections of $\bar{\mathcal{V}}$ are called anti-CR vector fields.

An example

...what else than the ball?

$$\operatorname{Im} w = |z|^2; \qquad w - \bar{w} = 2i(z_1\bar{z}_1 + \cdots + z_n\bar{z}_n)$$

$$\mathcal{V} = \operatorname{span}\left\{ \bar{L}_j = \frac{\partial}{\partial \bar{z}_j} - 2iz_j\frac{\partial}{\partial \bar{w}} : j = 1, \ldots, n \right\}$$

$$T^c M = \operatorname{span}\left\{ X_j = \frac{\partial}{\partial x_j} - 2y_j\frac{\partial}{\partial s} - 2x_j\frac{\partial}{\partial t}, \right.$$

$$\left. Y_j = \frac{\partial}{\partial y_j} + 2x_j\frac{\partial}{\partial s} - 2y_j\frac{\partial}{\partial t} : j = 1, \ldots, n \right\}$$

An example

...what else than the ball?

$$\operatorname{Im} w = |z|^2; \qquad w - \bar{w} = 2i(z_1\bar{z}_1 + \cdots + z_n\bar{z}_n)$$

$$\mathcal{V} = \operatorname{span}\left\{ \bar{L}_j = \frac{\partial}{\partial \bar{z}_j} - 2iz_j\frac{\partial}{\partial \bar{w}} : j = 1, \ldots, n \right\}$$

$$\begin{aligned} T^cM = \operatorname{span}\Big\{ X_j &= \frac{\partial}{\partial x_j} - 2y_j\frac{\partial}{\partial s} - 2x_j\frac{\partial}{\partial t}, \\ Y_j &= \frac{\partial}{\partial y_j} + 2x_j\frac{\partial}{\partial s} - 2y_j\frac{\partial}{\partial t} : j = 1, \ldots, n \Big\} \end{aligned}$$

Coordinates

...sometimes it's good to be normal

The coordinates

$$w = \bar{w} + 2i(z_1\bar{z}_1 + \cdots + z_n\bar{z}_n) = Q(z, \bar{z}, \bar{w})$$

for the ball above have the property

$$Q(z, 0, \bar{w}) = Q(0, \bar{z}, \bar{w}) = 0.$$

Such coordinates are said to be *normal*.
A defining equation for the complexification $\mathcal{M}$ can be written (using $\zeta = (\chi, \tau)$)

$$w = Q(z, \zeta) = Q(z, \chi, \tau).$$

or equivalently

$$\tau = \bar{Q}(\chi, z, w)$$

Complexifications of CR manifolds

Nonintegrable distributions vs. double foliations

Recall: $\pi_S, \pi_E \colon \mathcal{M} \to \mathbb{C}^N$. For $p \in \mathbb{C}^N$, define

$$\mathcal{S}_p^{(1,0)} = \pi_E^{-1}(\{p\}) = \{(Z,p) \in \mathbb{C}^{2N} \colon (Z,p) \in \mathcal{M}\}$$

$$\mathcal{S}_p^{(0,1)} = \pi_S^{-1}(\{p\}) = \{(p,\zeta) \in \mathbb{C}^{2N} \colon (p,\zeta) \in \mathcal{M}\}.$$

These form a double foliation of $\mathcal{M}$ by complex submanifolds, conjugated by the involution ι:

$$\iota(\mathcal{S}_p^{(1,0)}) = \mathcal{S}_{\bar{p}}^{(0,1)}$$

The Segre varieties come up naturally as follows:

$$S_{\bar{p}} = \pi_S(\mathcal{S}_p^{(1,0)}) = \pi_S \circ \pi_E^{-1}(\{p\}).$$

Segre varieties again...

...this time in coordinates

If $w = Q(z, \chi, \tau)$ is defining $\mathcal{M}$, then

$$\mathcal{S}^{(1,0)}_{(\chi,\tau)} = \{(z, Q(z, \chi, \tau), \chi, \tau)\}$$

$$\mathcal{S}^{(0,1)}_{(z,w)} = \{(z, w, \chi, \bar{Q}(\chi, z, w))\}.$$

And so

$$S_p = \{(z, Q(z, \bar{p}))\colon z \in \mathbb{C}^n\}.$$

Complexifications of CR manifolds

Integral manifolds!

The complexifications of (real-analytic) sections of $\mathcal{V}$ (resp. $\bar{\mathcal{V}}$) give rise to holomorphic vector fields on on $\mathcal{M}$ by replacing the differentiation in the barred variable by differentiation in its associated independent complex variable.

$$\bar{L} = \sum_j a_j(Z, \bar{Z}) \frac{\partial}{\partial \bar{Z}_j} \rightsquigarrow \bar{\mathcal{L}} = \sum_j a_j(Z, \zeta) \frac{\partial}{\partial \zeta_j}.$$

If we consider $\mathcal{V}$ and $\bar{\mathcal{V}}$ are considered as bundles on $\mathcal{M}$ in this way, they are *integrable*, with integral manifolds

$$\mathcal{S}_p^{(1,0)}, \quad \text{and } \mathcal{S}_p^{(0,1)}, \text{ respectively.}$$

For each $p \in M$, we can consider the (local) orbit O_P of p, which is the set of points which can be joined with p by a (small) polygonal path which is tangent to T^cM.
We say that M is minimal if the local orbit of p contains a neighbourhood of p in M.

Example

$$\operatorname{Im} w = |z|^2\colon \text{ minimal}$$

$$M\colon\ \operatorname{Im} w = (\operatorname{Re} w)|z|^2 \ldots T^c_{(z,0)} = \{w = 0\} \Rightarrow \text{nonminimal.}$$

Type

We say that M is of finite type at p if the sections of $T^c M$ (or equivalently, of $\mathcal{V}$ and $\bar{\mathcal{V}}$) satisfy the Hörmander condition at p: that is, if the Lie algebra generated by the CR vector fields and the anti-CR vector fields consists locally at p of all complexified tangent vector fields.
If M is of finite type at p, the local orbit of p contains an open subset of M; hence M is minimal at p.

Example

$$\operatorname{Im} w = e^{-\frac{1}{|z|^2}}$$

is minimal but not of finite type at 0.

For real analytic manifolds M, the converse is also true:

$$\text{minimal} \Rightarrow \text{finite type}.$$

Actually, one can say much more, as in this case we have the double foliation and the Segre varieties at our disposal.
Define the *Segre sets*

$$S_p^1 = S_p, \quad S_p^j = \bigcup_{q \in S_p^{j-1}} S_q.$$

The Segre sets come with useful *parametrizations*, the *Segre maps*.

Segre maps

Parametrizing Segre sets

$$S_p^1 = \{(z, Q(z, \bar{p}))\colon z \in \mathbb{C}^n\}$$

$$S_p^2 = \bigcup_{q \in S_p^1} S_q$$
$$= \bigcup_{\chi \in \mathbb{C}^n} S_{(\chi, \bar{Q}(\chi, p))}$$
$$= \{(z, Q(z, \chi, \bar{Q}(\chi, p)) \quad (z, \chi) \in \mathbb{C}^{2n}.\}$$

$$S_p^{2j} = \{(z, Q(z, \chi^1, \bar{Q}(\chi^1, z^1, Q(\ldots, \bar{Q}(\chi^j, p) \ldots))))$$
$$: (z, \chi^1, \ldots, \chi^j) \in \mathbb{C}^{2j}\}$$

Segre maps

Parametrizing Segre sets

$$S_p^1 = \{(z, Q(z, \bar{p}))\colon z \in \mathbb{C}^n\}$$

$$\begin{aligned} S_p^2 &= \bigcup_{q \in S_p^1} S_q \\ &= \bigcup_{\chi \in \mathbb{C}^n} S_{(\chi, \bar{Q}(\chi, p))} \\ &= \{(z, Q(z, \chi, \bar{Q}(\chi, p)) \quad (z, \chi) \in \mathbb{C}^{2n}.\} \end{aligned}$$

$$\begin{aligned} S_p^{2j} = \{(z, Q(z, \chi^1, \bar{Q}(\chi^1, z^1, Q(\dots, \bar{Q}(\chi^j, p) \dots)))) \\ : (z, \chi^1, \dots, \chi^j) \in \mathbb{C}^{2j}\} \end{aligned}$$

Segre maps

Parametrizing Segre sets

$$S_p^1 = \{(z, Q(z, \bar{p}))\colon z \in \mathbb{C}^n\}$$

$$\begin{aligned} S_p^2 &= \bigcup_{q \in S_p^1} S_q \\ &= \bigcup_{\chi \in \mathbb{C}^n} S_{(\chi, \bar{Q}(\chi, p))} \\ &= \{(z, Q(z, \chi, \bar{Q}(\chi, p)) \quad (z, \chi) \in \mathbb{C}^{2n}.\} \end{aligned}$$

$$\begin{aligned} S_p^{2j} = \{(z, Q(z, \chi^1, \bar{Q}(\chi^1, z^1, Q(\dots, \bar{Q}(\chi^j, p) \dots)))) \\ : (z, \chi^1, \dots, \chi^j) \in \mathbb{C}^{2j}\} \end{aligned}$$

Segre maps II

Parametrizing parts of the complexification

One can introduce the Segre maps (associated to a point p) inductively as follows:

$$v^1(t^1) = (t^1, Q(t^1, \bar{p}))$$

$$v^{j+1}(t^1, \ldots, t^{j+1}) = (t^{j+1}, Q(t^{j+1}, \bar{v}^j(t^1, \ldots, t^j))).$$

With this notation we have

$$(v^{j+1}(t^1, \ldots, t^{j+1})), \bar{v}^j(t^1, \ldots, t^j)) \in \mathcal{M}.$$

The image of the "unrestricted" Segre maps $(v^j, \bar{v}^{j-1}, \ldots,)$ (where p is allowed to vary freely) is the j-th iterated complexification $\mathcal{M}^{(j)}$.

Segre maps II

Parametrizing parts of the complexification

One can introduce the Segre maps (associated to a point p) inductively as follows:

$$v^1(t^1) = (t^1, Q(t^1, \bar{p}))$$

$$v^{j+1}(t^1, \ldots, t^{j+1}) = (t^{j+1}, Q(t^{j+1}, \bar{v}^j(t^1, \ldots, t^j))).$$

With this notation we have

$$(v^{j+1}(t^1, \ldots, t^{j+1})), \bar{v}^j(t^1, \ldots, t^j)) \in \mathcal{M}.$$

The image of the “unrestricted” Segre maps $(v^j, \bar{v}^{j-1}, \ldots,)$ (where p is allowed to vary freely) is the j-th iterated complexification $\mathcal{M}^{(j)}$.

One can introduce the Segre maps (associated to a point p) inductively as follows:

$$v^1(t^1) = (t^1, Q(t^1, \bar{p}))$$

$$v^{j+1}(t^1, \ldots, t^{j+1}) = (t^{j+1}, Q(t^{j+1}, \bar{v}^j(t^1, \ldots, t^j))).$$

With this notation we have

$$(v^{j+1}(t^1, \ldots, t^{j+1})), \bar{v}^j(t^1, \ldots, t^j)) \in \mathcal{M}.$$

The image of the "unrestricted" Segre maps $(v^j, \bar{v}^{j-1}, \ldots,)$ (where p is allowed to vary freely) is the j-th iterated complexification $\mathcal{M}^{(j)}$.

Theorem (Baouendi-Ebenfelt-Rothschild)

Assume that M is a (generic) real-analytic CR manifold through p. Then the following are equivalent:

1. *M is of finite type at p*
2. *There exists an integer j, $j \leq d+1$, such that S_p^j contains an open set in $\mathbb{C}^N$*
3. *There exists an integer j, $j \leq d+1$, such that v^j is generically of full rank*
4. *There exists an integer ℓ, $\ell \leq 2(d+1)$, such that S_p^ℓ contains an open neighbourhood of p in $\mathbb{C}^N$*
5. *There exists an integer ℓ, $j \leq 2(d+1)$, and for every neigbhourhood U of the origin in $\mathbb{C}^{2d+1}$ a point $t_0 \in \mathbb{C}^{2d+1}$ such that $v^\ell(t_0) = p$ and v^ℓ is of full rank at t_0.*
6. *M is minimal at p*

Characteristics

Many notions of nondegeneracy

The characteristic bundle of M is $T^0M = (T^cM)^\perp$.
It consists of all characteristic forms θ. Considered as forms on $\mathbb{C}TM$, characteristic forms are real forms which annihilate both CR and anti-CR vector fields.
The holomorphic cotangent bundle is $T'M = \mathcal{V}^\perp$.
Nondegeneracy measures how far from “integrable” the characteristic bundle is, i.e. the failure of a kind of Frobenius condition (always with the understanding that we need to be adapted to the complex coefficients of our PDE and the complex nature of its solutions!).
There are therefore necessarily many notions of nondegeneracy, according to the point at which this condition fails.

Finite nondegeneracy

The Lie derivative of a holomorphic form ω with respect to a CR vector field $\bar{L}$, defined by

$$(\mathcal{L}_{\bar{L}}\omega)(K) = d\theta(\bar{L}, K) = \bar{L}\omega(K) - K\omega(\bar{L}) - \omega([\bar{L}, K])$$

is again a holomorphic form.
One can therefore consider the ascending chain of spaces of forms

$$E_0 = \Gamma(M, T^0M), \quad E_{j+1} = E_j + \{\mathcal{L}_{\bar{L}}\omega \colon \omega \in E_j, \bar{L} \in \Gamma(M, \mathcal{V})\}$$

We say that M is finitely (k-)nondegenerate at p if

$$E_k(p) = T'_pM, \quad E_{k-1}(p) \neq T'_pM.$$

Levi-nondegeneracy

or 1-nondegeneracy

A particular case of finite nondegeneracy occurs when already $E_1(p) = T'_p M$. In that case, for a choice of basis of CR vector fields $\bar{L}_1, \dots, \bar{L}_n$, the forms

$$\theta, \theta([\bar{L}_1, \cdot]), \dots, \theta([\bar{L}_n, \cdot])$$

span $T'_p M$. By dimensional reasons, this means that $\theta([\bar{L}_1, \cdot]), \dots, \theta([\bar{L}_n, \cdot])$ are *linearly independent* at p. Equivalently, the Hermitian form, called the Levi form,

$$\mathcal{L}_p \colon \mathcal{V}_p^2 \to \mathbb{C}, \quad \mathcal{L}_p(L_p, K_p) = i\theta([L, \bar{K}])(p)$$

is *nondegenerate* as a hermitian form. This is the classical definition of *Levi-nondegeneracy*.

Nondegeneracy in the complexification

Yet another important map

In the real-analytic setting, nondegeneracy is measured in terms of properties of the map

$$\pi\colon \mathcal{M} \ni (Z,\zeta) \mapsto (S_{\bar\zeta}, Z)$$

or its variants

$$\pi^{(k)}\colon \mathcal{M} \ni (Z,\zeta) \mapsto j^k_Z S_{\bar\zeta}$$

Theorem

The following are equivalent:

1. *M is finitely at most k-nondegenerate at p;*
2. $\pi^{(k)}|_{\mathcal{S}_p^{(0,1)}}$ *is immersive at* $(p,\bar p)$.

Example

M is Levi nondegenerate $\Leftrightarrow p \mapsto T_p^c M \in \mathbb{PC}^N$ is immersive.

Nondegeneracy in the complexification

Yet another important map

In the real-analytic setting, nondegeneracy is measured in terms of properties of the map

$$\pi\colon \mathcal{M} \ni (Z,\zeta) \mapsto (S_{\bar\zeta}, Z)$$

or its variants

$$\pi^{(k)}\colon \mathcal{M} \ni (Z,\zeta) \mapsto j^k_Z S_{\bar\zeta}$$

Theorem

The following are equivalent:

1. *M is finitely at most k-nondegenerate at p;*
2. $\pi^{(k)}|_{\mathcal{S}_p^{(0,1)}}$ *is immersive at* $(p,\bar p)$.

Example

M is Levi nondegenerate $\Leftrightarrow p \mapsto T^c_p M \in \mathbb{P}\mathbb{C}^N$ is immersive.

Holomorphic nondegeneracy

The global obstruction to being nondegenerate

Definition

We say that M is holomorphically *degenerate* if there exists a holomorphic vector field X tangent to M.

Example

$\operatorname{Im} w = |z_1 z_2|^2$ is holomorphically degenerate.

Generically (at nonsingular points of X) the orbits of X foliate M and give rise to holomorphic coordinates in which M is of the form $M = \hat{M} \times \mathbb{C}$.

Theorem

The following are equivalent:

1. *M is generically finitely nondegenerate*
2. *There exists k such that $\pi^{(k)}$ is generically of full rank on $\mathcal{M}$.*

Holomorphic nondegeneracy

The global obstruction to being nondegenerate

Definition

We say that M is holomorphically *degenerate* if there exists a holomorphic vector field X tangent to M.

Example

$\operatorname{Im} w = |z_1 z_2|^2$ is holomorphically degenerate.

Generically (at nonsingular points of X) the orbits of X foliate M and give rise to holomorphic coordinates in which M is of the form $M = \hat{M} \times \mathbb{C}$.

Theorem

The following are equivalent:

1. *M is generically finitely nondegenerate*
2. *There exists k such that $\pi^{(k)}$ is generically of full rank on $\mathcal{M}$.*

Our nondegeneracy conditions involve the location of "characteristics" with respect to the orbits of the CR equations. One particular way to guarantee nondegeneracy is to completely forbid any characteristics. The notion is important, also for smooth CR manifolds.

Definition (D'Angelo finite type)

We say that M is of D'Angelo finite type at $p \in M$ if the only complex variety X through p sitting completely in M is $X = \{p\}$.

D'Angelo finite type implies a notion slightly stronger than holomorphic nondegeneracy, *essential finiteness*.

Definition (Essential Finiteness)

M is essentially finite if $\pi^{(k)}$ is a finite map for k large enough.

Nondegeneracy and the Segre family

A sufficiently nondegenerate CR manifold gives rise to a “Segre family”

$$\{S_p \colon p \in \mathbb{C}^N\}$$

which actually consists of the integral manifolds of a certain ODE.

Example (The Levi nondegenerate case)

$$S_p = \{w = Q(z, \bar{p})\}$$

Consider $w(z) = Q(z, \bar{p})$; then $w'(z) = Q_z(z, \bar{p})$. Note

$$(Q(z, \bar{p}), Q_z(z, \bar{p})) \sim j^1_{(z, Q(z, \bar{p}))} S_p.$$

Eliminate $\bar{p}$ to obtain a second order ODE.

Segre families in the nonminimal case

If M is *nonminimal* but still nondegenerate enough, Kossovskiy and Shafikov showed that the Segre family can be realized as the integral manifolds of a *singular* second order ODE.

Both in the minimal and in the nonminimal sufficiently nondegenerate case, (analytic/formal) equivalences of the CR manifolds correspond to (analytic/formal) equivalences of the ODEs describing the Segre family.

The details of passing from the ODE side to the CR manifold side are rather intricate; Kossovskiy (and coauthors) provide a sort of “dictionary” and example cases in the form of sphere blowups.

One therefore expects that at least in the nonminimal case, problems will occur–and they do!

CR maps and formal maps

Solutions vs. formal solutions

Given CR submanifolds $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$, a map $H \colon M \to M'$ is said to be CR if $dH \colon \mathcal{V} \to \mathcal{V}'$.
Smooth CR maps between real-analytic submanifolds give rise to *formal power series maps*: If $p \in M$, then

$$T_p H(Z) = \sum_{\alpha \in \mathbb{N}^N} \frac{1}{\alpha!} \frac{\partial^{|\alpha|} H(p)}{\partial Z^\alpha} (Z-p)^\alpha \in \mathbb{C}[\![Z-p]\!]^{N'}$$

satisfies

$$\varrho' \left(T_p H(Z), \overline{T_p H(Z)} \right) = A(Z,\bar{Z}) \varrho(Z,\bar{Z})$$

for some formal power series A.

What to expect...

when expecting divergence

Example (The simplest example: $\mathbb{R} \subset \mathbb{C}$)

Formal power series maps taking $\mathbb{R}$ into itself:

$$H(w) = \sum_{j \geq 1} H_j w^j \colon (\mathbb{R}, 0) \to (\mathbb{R}, 0) \Leftrightarrow H_j \in \mathbb{R} \quad \forall j$$

$$\Leftrightarrow H(w) \in (w)\mathbb{R}[\![w]\!]$$

Example (The second simplest example: $\Gamma \subset \mathbb{C}$)

If Γ is a real-analytic arc, $p \in \Gamma$, then there exists a biholomorphism $\phi \colon (\Gamma, p) \to (\mathbb{R}, 0)$:

$$H \colon (\Gamma, p) \to (\Gamma, p) \Leftrightarrow \varphi \circ H \circ \varphi^{-1} \in (w)\mathbb{R}[\![w]\!]$$

What to expect...

when expecting divergence

Example (The simplest example: $\mathbb{R} \subset \mathbb{C}$)

Formal power series maps taking $\mathbb{R}$ into itself:

$$H(w) = \sum_{j \geq 1} H_j w^j \colon (\mathbb{R}, 0) \to (\mathbb{R}, 0) \Leftrightarrow H_j \in \mathbb{R} \quad \forall j$$

$$\Leftrightarrow H(w) \in (w)\mathbb{R}[\![w]\!]$$

Example (The second simplest example: $\Gamma \subset \mathbb{C}$)

If Γ is a real-analytic arc, $p \in \Gamma$, then there exists a biholomorphism $\phi \colon (\Gamma, p) \to (\mathbb{R}, 0)$:

$$H \colon (\Gamma, p) \to (\Gamma, p) \Leftrightarrow \varphi \circ H \circ \varphi^{-1} \in (w)\mathbb{R}[\![w]\!]$$

How degeneracy leads to divergence

Positive orbit codimension

If $M = \mathbb{R} \times \hat{M}$, no matter what $\hat{M}$ is, there exist divergent maps $H\colon M \to M$.

More generally:

$$M = \{(t, z) \in \mathbb{R} \times \mathbb{C}^n \colon z \in M_t\}$$

possesses divergent self-maps.

Holomorphic degeneracy

X tangent to M...

$$H_\varphi(Z) = e^{\varphi(Z)X} Z \colon M \to M, \quad \varphi \in \mathbb{C}[\![Z]\!]$$

is a divergent self-map for divergent φ. (more details in lecture 2)

Surprisingly there are sufficient conditions-lecture 2.

Formal Equivalence

$$(M,0) \sim_f (M',0) \Leftrightarrow \exists H \in \mathfrak{m}\mathbb{C}[\![Z]\!]^N, |H'(0)| \neq 0,$$
$$H\colon (M,0) \to (M',0)$$

Biholomorphic Equivalence

$$(M,0) \sim_\omega (M',0) \Leftrightarrow \exists H \in \mathfrak{m}\mathbb{C}\{Z\}^N, |H'(0)| \neq 0,$$
$$H\colon (M,0) \to (M',0)$$

CR Equivalence

$$(M,0) \sim_{CR} (M',0) \Leftrightarrow \exists H \in C^\infty_{CR}((M,0),(M',0)), |H'(0)| \neq 0$$

Relationships

Obvious and not so obvious

$$\begin{tikzcd} & \sim_{CR} \arrow[dr, Rightarrow] & \\ \sim_{\omega} \arrow[ur, Rightarrow] \arrow[rr, Rightarrow] & & \sim_{f} \end{tikzcd}$$

1. Biholomorphic $\Rightarrow$ CR $\Rightarrow$ formal: obvious
2. formal / bih. No reason that a given formal automorphism between minimal hypersurfaces converges–tomorrow.
3. Divergent automorphisms can be *approximated* by convergent ones in the minimal setting (Baouendi-Mir-Rothschild)
4. In the nonminimal setting, there exist formally but not biholomorphically equivalent hypersurfaces.

Relationships

Obvious and not so obvious

$$\sim_\omega \quad \Longrightarrow \quad \sim_f$$

1. Biholomorphic $\Rightarrow$ CR $\Rightarrow$ formal: obvious
2. formal $\not\Rightarrow$ bih. No reason that a given formal automorphism between minimal hypersurfaces converges–tomorrow.
3. Divergent automorphisms can be *approximated* by convergent ones in the minimal setting (Baouendi-Mir-Rothschild)
4. In the nonminimal setting, there exist formally but not biholomorphically equivalent hypersurfaces.

Relationships

Obvious and not so obvious

$$\begin{tikzcd} \sim_\omega \arrow[rr, Rightarrow] & & \sim_f \arrow[ll, Rightarrow, bend left, "\text{minimal}"] \end{tikzcd}$$

1. Biholomorphic $\Rightarrow$ CR $\Rightarrow$ formal: obvious
2. formal / bih. No reason that a given formal automorphism between minimal hypersurfaces converges–tomorrow.
3. Divergent automorphisms can be *approximated* by convergent ones in the minimal setting (Baouendi-Mir-Rothschild)
4. In the nonminimal setting, there exist formally but not biholomorphically equivalent hypersurfaces.

Relationships

Obvious and not so obvious

$$\begin{tikzcd} \sim_\omega \arrow[rr, Rightarrow] & & \sim_f \arrow[ll, Rightarrow, bend right, "\text{nonminimal}"', "/" marking] \arrow[ll, Rightarrow, bend left, "\text{minimal}"] \end{tikzcd}$$

1. Biholomorphic $\Rightarrow$ CR $\Rightarrow$ formal: obvious
2. formal / bih. No reason that a given formal automorphism between minimal hypersurfaces converges–tomorrow.
3. Divergent automorphisms can be *approximated* by convergent ones in the minimal setting (Baouendi-Mir-Rothschild)
4. In the nonminimal setting, there exist formally but not biholomorphically equivalent hypersurfaces.

Complete minimal picture

In the minimal setting, the three notions are all equivalent:

$$\sim_{\omega} \Leftrightarrow \sim_{CR} \Leftrightarrow \sim_{f} .$$

This does not mean that necessarily any given formal equivalence converges, nor that every CR diffeomorphism is real-analytic.
There are theorems guaranteeing convergence of formal equivalences (tomorrow).
There are also theorems guaranteeing analyticity of CR diffeomorphisms (Baouendi-Jacobowitz-Treves).

Complete nonminimal picture

Only under nondegeneracy assumptions

Since the notions are *not* all equivalent in the nonminimal setting by the Kossovskiy-Shafikov result. So what about the other notions?

$$
\begin{tikzcd}
 & \sim_{CR} \arrow[dr, Rightarrow] \arrow[dl, Rightarrow, bend right, "/" marking] & \\
\sim_{\omega} \arrow[ur, Rightarrow] \arrow[rr, Rightarrow] & & \sim_{f}
\end{tikzcd}
$$

1. There exist CR equivalent hypersurfaces which are not biholomorphically equivalent : Kossovskiy-L.
2. formal $\Rightarrow$ CR : Kossovskiy-L.-Stolovitch

Complete nonminimal picture

Only under nondegeneracy assumptions

Since the notions are *not* all equivalent in the nonminimal setting by the Kossovskiy-Shafikov result. So what about the other notions?

$$
\begin{tikzcd}
 & \sim_{CR} \arrow[dl, Rightarrow, bend right, "/" marking, red] \arrow[dr, Rightarrow] & \\
\sim_{\omega} \arrow[ur, Rightarrow] \arrow[rr, Rightarrow] & & \sim_{f} \arrow[ll, Rightarrow, bend left, "/" marking, red]
\end{tikzcd}
$$

1. There exist CR equivalent hypersurfaces which are not biholomorphically equivalent : Kossovskiy-L.
2. formal $\Rightarrow$ CR : Kossovskiy-L.-Stolovitch

Complete nonminimal picture

Only under nondegeneracy assumptions

Since the notions are *not* all equivalent in the nonminimal setting by the Kossovskiy-Shafikov result. So what about the other notions?

$$
\begin{tikzcd}
& \sim_{CR} \arrow[dl, Rightarrow, bend right, "/" marking] \arrow[dr, Leftrightarrow] & \\
\sim_{\omega} \arrow[ur, Rightarrow] \arrow[rr, Rightarrow] & & \sim_{f} \arrow[ll, Rightarrow, bend left, "/" marking]
\end{tikzcd}
$$

1. There exist CR equivalent hypersurfaces which are not biholomorphically equivalent : Kossovskiy-L.
2. formal $\Rightarrow$ CR : Kossovskiy-L.-Stolovitch