Convergence and divergence of CR maps

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1. The problem
2. Results
3. The proof

Motivation

- $M \subset \mathbb{C}_z^N$ real-analytic submanifold, $p \in M$ (*Source*)
- $M' \subset \mathbb{C}_w^{N'}$ real-analytic subvariety (*Target*)
- $H \in \mathbb{C}[\![z-p]\!]^{N'}$, $H(M) \subset M'$

The convergence problem

Under which conditions on M and M' can we guarantee that H converges?

What to expect

- $M = M' = \mathbb{R} \subset \mathbb{C}$, $H \in \mathbb{R}[\![z]\!]$: $H(M) \subset M'$
- $M = \mathbb{C}^N$, $M' = \mathbb{C}$, $H \in \mathbb{C}[\![z]\!]$: $H(M) \subset M'$.

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A positive example

Selfmaps of spheres

$M = M' = \mathbb{S}^{2n-1}$, $p \in \mathbb{S}^{2n-1}$, $H(\mathbb{S}^{2n-1}) \subset \mathbb{S}^{2n-1}$.

$$\Rightarrow H(z) = U\frac{z - L_a z}{1 - z \cdot \bar{a}}$$

is linear fractional, hence convergent (Alexander, 1974).

The main takeaway

Obstructions to (automatic) convergence of formal maps:

- M misses "bad" directions.
- Complex varieties in M'.

There are examples of settings in which every formal map converges.

Why should I care?

Formal vs. holomorphic equivalence

If there exists a formal equivalence H with $H(M) \subset M'$, does there exist a convergent one?

Formal vs. holomorphic embeddability

Assuming that a real-analytic manifold $M \subset \mathbb{C}^N$ can be imbedded formally into $M' \subset \mathbb{C}^{N'}$, does there exist a holomorphic embedding?

Relates to e.g. nonembeddability of (some) strictly pseudoconvex domains into balls (Forstneric, 1986).

Real analytic varieties in $\mathbb{C}^N$

- $M \subset (\mathbb{C}^N_z, p)$ germ of generic real-analytic submanifold
- $M' \subset \mathbb{C}^{N'}_w$ real analytic subset

$$\mathcal{I}_p(M) \subset \mathbb{C}\{z-p, \overline{z-p}\} \text{ ideal of } M$$

$$\hat{\mathcal{I}}_p(M) \subset \mathbb{C}[\![z-p, \overline{z-p}]\!] \text{ formal ideal of } M$$

$$\mathbb{C}\{M\} = {\mathbb{C}\{z-p, \overline{z-p}\}}/{\mathcal{I}_p(M)} \text{ real-analytic functions on } M$$

$$\mathbb{C}[\![M]\!] = {\mathbb{C}[\![z-p, \overline{z-p}]\!]}/{\hat{\mathcal{I}}_p(M)} \text{ formal functions on } M$$

$$\mathbb{C}(\!(M)\!): \text{ quotient field of } \mathbb{C}[\![M]\!]$$

$$\mathrm{CR}(\!(M)\!) \subset \mathbb{C}(\!(M)\!) \text{ CR elements}$$

$$A(z,\bar{z})|_M = A(z,\bar{z}) + \hat{\mathcal{I}}(M) \in \mathbb{C}[\![M]\!], \quad A(z,\bar{z}) \in \mathbb{C}[\![z-p, \overline{z-p}]\!]$$

Formal maps

- $H = (H_1, \dots, H_{N'}) \in \mathbb{C}[\![z - p]\!]$ formal map centered at p
- $p \in X \subset \mathbb{C}^N$, $X' \subset \mathbb{C}^{N'}$

Definition

We say that $H(X) \subset X'$ if for every $k \in \mathbb{N}$ there exists a germ of a real-analytic map $h_k(z, \bar{z}) \in \mathbb{C}\{z - p, \overline{z - p}\}^{N'}$ such that

i) $h_k(z, \bar{z}) - H(z) = O(|z - p|^{k+1})$, and
ii) $h_k(X \cap U_k) \subset X'$ for some neighbourhood U_k of p

(Important) Remark

If X and X' are real-analytic subvarieties, then

$$H(X) \subset X' \Leftrightarrow H^* \left(\mathcal{I}_{H(p)}(X')\right) \subset \hat{\mathcal{I}}_p(X).$$

We also write $H \colon (M, p) \to M'$ in that case.

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Commutator Type

Definition

We say that M is of finite (commutator) type at p if

$$\mathrm{Lie}\left(\Gamma_p(T^{(1,0)}M) \cup \Gamma_p(T^{(0,1)}M)\right)(p) = \mathbb{C}T_pM.$$

Finite vs. infinite type: hypersurface case

If M is a real-analytic hypersurface, then M is of infinite type at p if and only if there exists a complex analytic hyperplane X through p which is fully contained in M.

D'Angelo Type

Definition

We say that M' is of finite D'Angelo (DA) type at p' if there is no nontrivial holomorphic disc

$$A\colon \Delta = \{\zeta \in \mathbb{C} : |\zeta| < 1\} \to \mathbb{C}^N, A(0) = p', A(\Delta) \subset M.$$

Points of infinite DA type

$$\mathcal{E}_{M'} = \left\{ p' \in M' : \exists \text{ holomorphic disc } A, A(0) = p', A(\Delta) \subset M \right\}$$

Divergence revisited

$$A(\Delta) \subset \mathcal{E}_{M'}, \quad H(z) = A \circ \varphi(z), \quad \varphi(z) \in \mathbb{C}[\![z - p]\!],$$

is a formal map taking $\mathbb{C}^N_z$ into M', diverges if φ does.

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Convergence of all formal maps

Theorem (L.-Mir 2017 [2])

Assume that M is of finite type at p, and $H\colon (M,p) \to M'$ is a formal map. If H is divergent, then $H(M) \subset \mathcal{E}_{M'}$.

Corollary

If M is of finite type, then every formal map $H\colon (M,p) \to M'$ converges if and only if $\mathcal{E}_{M'} = \emptyset$.

Corollary

Let κ denote the maximum dimension of real submanifolds of $\mathcal{E}_{M'}$. If the formal map $H\colon (M,0) \to M'$ is of rank $> \kappa$, then H converges.

Earlier results

- Baouendi, Ebenfelt, Rothschild (1998) : formal biholomorphisms of finitely nondegenerate hypersurfaces.
- Baouendi, Ebenfelt, Rothschild (2000) : relaxed geometrical conditions.
- L. (2001) : strongly pseudoconvex targets + additional stringent conditions on the maps.
- Mir (2002) : strongly pseudoconvex target, $N' = N + 1$
- Baouendi, Mir, Rothschild (2002) : equidimensional case
- Meylan, Mir, Zaitsev (2003) : real-algebraic case
- L, Mir (2016) : strongly pseudoconvex targets in general

The role of commutator type

The convergence results deal with sources M which are of finite type at the reference point p. Manifolds which are everywhere of infinite type don't work because of examples of divergent maps. What about the generically finite type case?

- Kossovskiy-Shafikov (2013): There exist infinite type hypersurfaces which are formally, but not biholomorphically equivalent.
- L.-Kossovskiy (2014): There exist infinite type hypersurfaces which are $\mathcal{C}^\infty$ CR equivalent, but not biholomorphically equivalent. Fuchsian type condition.
- L.-Kossovskiy-Stolovitch (2016): If $M, M' \subset \mathbb{C}^2$ are infinite type hypersurfaces, and $H\colon M \to M'$ is a formal map, then H is the Taylor series of a smooth CR diffeomorphism $h\colon M \to M'$.

From now on: M of finite type at p.

Approximate deformations

k-approximate formal deformations

A formal map $B^k(z,t) \in \mathbb{C}[\![z-p_0,t]\!]^{N'}$ is a *k-approximate formal deformation* for (M, M') at p ($t \in \mathbb{C}^r$, $k \in \mathbb{N}$) if

(i) $\operatorname{rk} \frac{\partial B^k}{\partial t}(z,0) = r$;

(ii) For every $\varrho' \in \mathcal{I}_{M'}(p')$,

$$\varrho'(B^k(z,t), \overline{B^k(z,t)})|_{z\in M} = O(|t|^{k+1}).$$

Formal maps admitting approximate deformations

$H\colon (M,p) \to M'$ admits a k-approximate formal deformation if there exists a k-approximate formal deformation for (M, M') with $B(z,0) = H(z)$.

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Existence of approximate deformations

Theorem (Divergent maps have deformations)

If $H\colon (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p')$ *is divergent,* $\exists 1 \le r \le N'$*, and* $\forall k \in \mathbb{N}$*, a formal holomorphic map* $B^k\colon (\mathbb{C}^N \times \mathbb{C}^r, (p, 0)) \to (\mathbb{C}^{N'}, p')$ *such that for every real-analytic set* $M' \subset \mathbb{C}^{N'}$ *passing through* p'*, if* $H(M) \subset M'$ *then* H *admits* B^k *as a* k*-approximate formal deformation of* (M, M')*.*

Corollary

If $H\colon (M, p) \to M'$ *is divergent,* $\exists 1 \le r \le N'$*, and* $\forall k \in \mathbb{N}$*, a neighbhorhood* U_k *of* p *in* $\mathbb{C}^N$ *and a real-analytic map* $h_k\colon U_k \to \mathbb{C}^{N'}$ *such that:*

(a) $h_k(M \cap U_k) \subset M'$ *and* h_k *agrees with* H *at* p *up to order* k*;*

(b) *there exists a Zariski open subset* Ω_k *of* $M \cap U_k$ *such that* $h_k(\Omega_k) \subset \widetilde{\mathcal{E}}^r_{M'} = \{p' : \exists p \in V \subset M', \dim V = r\}$*.*

In particular it holds that $h_k(M \cap U_k) \subset \mathcal{E}_{M'}$ *for every positive integer* k*.*

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Corollary

If $H\colon (M, p) \to M'$ is divergent, $\exists 1 \le r \le N'$, and $\forall k \in \mathbb{N}$, a neighbhorhood U_k of p in $\mathbb{C}^N$ and a real-analytic map $h_k\colon U_k \to \mathbb{C}^{N'}$ such that:

(a) *$h_k(M \cap U_k) \subset M'$ and h_k agrees with H at p up to order k;*

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Further Consequences

The nonexistence of formal deformations can also be detected in cases where $\mathcal{E}_{M'} \neq \emptyset$:

Corollary

Let $M \subset \mathbb{C}^N$ and $M' \subset \mathbb{C}^{N'}$ be (connected) real-analytic Levi-nondegenerate hypersurfaces, of signature ℓ and ℓ'. Assume that $\ell = \ell'$ or $N - \ell = N' - \ell'$. If $H\colon (M, p) \to M'$ is a formal holomorphic map which is CR transversal at p, then H is convergent.

$\mathbb{T}^{N'}$: tube over light cone; everywhere Levi degenerate, foliated by complex lines.

Corollary

If $H\colon (M, p) \to \mathbb{T}^{N'}$ is a formal holomorphic map with $\operatorname{rk} H \geq 2$, then H is convergent.

Tool: Convergence Proposition

Proposition [L.-Mir 2016 [1]]

- $\Theta(z,\bar{z},\lambda,w) \in \mathbb{C}\{z,\bar{z},\lambda,w\}^{N'}$, $\lambda \in \mathbb{C}^m$: *convergent* map
- $H(z) \in \mathbb{C}[\![z-p]\!]^{N'}$ *formal* map
- $G(z) \in \mathbb{C}[\![z-p]\!]^{m}$ *formal* map

Assume that

i) $\Theta(z,\bar{z},\overline{G(z)},H(z))|_M = 0$

ii) $\frac{\partial\Theta}{\partial w}\left(z,\bar{z},\overline{G(z)},H(z)\right)\Big|_M \not\equiv 0$

Then H is convergent.

Aside: The typical strategy

Fix M and M'; $\bar{L}_j$ CR vector fields on M.

$$\varrho'(H(z), \overline{H(z)})|_M = 0$$

$$\Rightarrow 0 = L^{\bar{\alpha}} \varrho'(H(z), \overline{H(z)})|_M = \Theta_\alpha \left(z, \bar{z}, H(z), \overline{\frac{\partial^{|\alpha|} H}{\partial z^\alpha}(z)} \colon |\beta| \le |\alpha| \right).$$

The convergence proposition does not apply if

$$\dim \left\{ L^{\bar{\alpha}} \varrho'_w(H(z), \overline{H(z)})|_M \colon \alpha \in \mathbb{N}^n, \varrho' \in \mathcal{I}_{p'}(M') \right\} < N'.$$

In that case, one can hope for getting the *missing equations* in a different way then from prolongation. So the main question is: What happens if we don't get any additional equations?

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Divergence rank

$$\mathcal{A}_H = \left\{ (\Delta, S) \colon \Delta \in \mathbb{C}[\![z]\!]^m,\ S = S(z, \bar{z}, \lambda, w) \in \mathbb{C}\{z, \bar{z}, \lambda - \overline{\Delta(0)}, w\} \right\}$$

$$S^{\Delta} := S(z, \bar{z}, \overline{\Delta(z)}, H(z))|_M \in \mathbb{C}[\![M]\!], \quad S^{\Delta}_w := \left(S^{\Delta}_{w_1}, \dots, S^{\Delta}_{w_{N'}} \right)$$

$$\mathcal{S}_H(M) = \left\{ \psi \in \mathbb{C}[\![M]\!] \colon \psi = S^{\Delta}, (\Delta, S) \in \mathcal{A}_H \right\}, \quad \mathbb{K}^M_H \dots \text{ quotient field}$$

$$\mathcal{A}^0_H(M) = \left\{ (\Delta, S) \colon S^{\Delta} = 0 \right\}$$

$$\operatorname{rank} \mathcal{A}^0_H(M) := \dim_{\mathbb{K}^M_H} \operatorname{span} \left\{ S^{\Delta}_w \colon (\Delta, S) \in \mathcal{A}^0_H(M) \right\},$$

The divergence rank

$$\operatorname{divrk}_M H = N' - \operatorname{rank}_{\mathbb{K}^M_H} \mathcal{A}^0_H(M).$$

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Some linear algebra

Convergence and divergence rank

$$H \text{ is convergent} \Leftrightarrow \mathrm{divrk} H = 0.$$

If H is divergent, then

$$\mathcal{V}_H^M := \left\{ V = (V_1, \ldots, V_{N'}) \in (\mathbb{K}_H^M)^{N'} : V \cdot S_w^\Delta = 0, \forall (\Delta, S) \in \mathcal{A}_H^0(M) \right\}$$

is not trivial ($\dim_{\mathbb{K}_H^M} \mathcal{V}_H^M = \mathrm{divrk} H = r > 0$).

Important Fact

$\mathcal{V}_H^M$ can be generated by CR vectors $V^1, \ldots, V^r \in \mathrm{CR}((M))^{N'}$, (since $\mathcal{A}_H^0(M)$ closed under the applications of CR vector fields).

Divergence forces deformations

The main idea: "Exponential map"

$$D^1(t) := t \cdot \mathbb{V} = t_1 V^1 + \cdots + t_r V^r, \quad D^{\ell+1}(t) = \frac{1}{\ell+1}(t \cdot \mathbb{V}) \cdot D^\ell_w(t),$$

$$D(t) = \sum_{\ell=1}^{\infty} D^\ell(t) = e^{t \cdot \mathbb{V}} \in (\mathbb{K}^M_H[\![t]\!])^{N'}.$$

Main properties

(i) $D(t) \in (\mathrm{CR}((M))[\![t]\!])^{N'}$;

(ii) If $\rho \in \mathbb{C}\{w, \bar{w}\}$ satisfies $\rho(H(z), \overline{H(z)})|_M = 0$ then

$$\rho\left(H + D(t), \overline{H + D(t)}\right) = 0 \quad \text{in} \quad \mathbb{C}((M))[\![t, \bar{t}]\!].$$

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Approximate deformations

$$\rho\left(H+D(t),\overline{H+D(t)}\right)=0 \quad \text{in} \quad \mathbb{C}(\!(M)\!)[\![t,\bar{t}]\!].$$

Truncate and clear denominators:

$$B^k(z,t)=H+\sum_{\ell=1}^{k} D^\ell\left(\frac{t}{E(z)}\right)$$

Existence of approximate formal deformations

$$\rho\left(B^k(t),\overline{B^k(t)}\right)=O(k+1)\in\mathbb{C}[\![M]\!][\![t,\bar{t}]\!]$$

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Deformations give varieties

Key property: Can think of

$$\rho\left(B^k(t), \overline{B^k(t)}\right) = O(k+1)$$

as describing an approximate solution $(B^k, \overline{B^k})$ to a real-analytic system of equations.

Theorem (Parameter version of Hickel-Rond)

$R_1, \ldots, R_m \in \mathbb{C}\{u-q, \bar{u}-\bar{q}, t, \bar{t}, \zeta, \bar{\zeta}\}$, $u \in \mathbb{C}^{n_1}$, $t \in \mathbb{C}^{n_2}$, $\zeta \in \mathbb{C}^{n_3}$, $q \in \mathbb{C}^{n_1}$. $\exists$ an open neighbourhood V of q in $\mathbb{C}^{n_1}$ and $\exists \mathcal{L} \colon \mathbb{N} \to \mathbb{N}$ such that: For every $u \in V$, if $S(t) \in (\mathbb{C}\{t\})^{n_3}$ satisfies $S(0) = 0$ and

$$R_j(u, \bar{u}, t, \bar{t}, S(t), \overline{S(t)}) = O(|t|^{\mathcal{L}(k)+1}), \; j = 1, \ldots, m,$$

for some $k \in \mathbb{N}$, then there exists $\widetilde{S}(t) \in (\mathbb{C}\{t\})^{n_3}$ such that

$$R_j(u, \bar{u}, t, \bar{t}, \widetilde{S}(t), \overline{\widetilde{S}(t)}) = 0, \; j = 1, \ldots, m,$$

Application of Hickel-Rond

Pick a real-analytic function ρ with $M' = \{\rho = 0\}$. Apply the theorem and get real-analytic $\hat{B}_0^k$, U_k and for each $z \in U_k$ a $\widetilde{S}_z^k(t)$ such that B_0 and $\hat{B}_0^k$ agree up to order k

$$\rho\left(\widehat{B}_0^k(z,\bar{z}) + \widetilde{S}_z^k(t), \overline{\widehat{B}_0^k(z,\bar{z}) + \widetilde{S}_z^k(t)}\right)\Big|_{z\in M\cap U_k} = 0.$$

This proves that $t \mapsto \widetilde{S}_z^k(t)$ parametrizes a holomorphic submanifold of dimension r completely contained in M', passing through $\hat{B}_0^k(z,\bar{z})$, and therefore the main result.

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