EQUIVALENCE OF CAUCHY-RIEMANN MANIFOLDS AND MULTISUMMABILITY THEORY

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Abstract. We prove that if two real-analytic hypersurfaces in $\mathbb{C}^2$ are equivalent formally, then they are also C^∞ CR-equivalent at the respective point. As a corollary, we prove that all formal equivalences between real-algebraic Levi-nonflat hypersurfaces in $\mathbb{C}^2$ are algebraic (in particular are convergent). The result is obtained by using the recent *CR - DS technique*, connecting degenerate CR-manifolds and Dynamical Systems, and employing subsequently the *multisummability theory* of divergent power series used in the Dynamical Systems theory.

Contents

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1. Introduction

1.1. **Overview.** The main goal of this paper is to obtain a general result on a geometric realization of formal power series maps between real hypersurfaces in complex space. We are doing so by further developing a new technique in Complex Analysis, when a real submanifold M in complex Euclidean space with a given degeneracy of its induced *CR-structure* is replaced by an appropriate dynamical system $\mathcal{E}(M)$. The procedure

$$M \longrightarrow \mathcal{E}(M)$$

allows to reduce numerous CR-geometric problems to a classical setting in Dynamics. Using this approach (sometimes referred to as the *CR - DS technique*), several significant problems in Complex Analysis were solved recently ([KS16a, KS16b, KL14]). In this paper, we develop the CR - DS approach further by extending the procedure $M \longrightarrow \mathcal{E}(M)$ to *arbitrary* real-analytic hypersurfaces in complex two-space. The key dynamical ingredient that we use for proving our results is the *multisummability theory* in Dynamical Systems. We give below a historic outline, and describe our results in detail. In the end of this Introduction, we provide an overview of our approach.

1.2. **Historic outline.** Let M, M^* be two real-analytic hypersurfaces in complex Euclidean space $\mathbb{C}^N$, $N \geq 2$, and $p \in M$, $p^* \in M^*$ distinguished points in them. A lot of work in Complex Analysis in the last 40 years has been dedicated to studying of the following general question.

Problem 1. Assume that there exists an invertible formal power series transformation $\hat{H} : (\mathbb{C}^N, p) \mapsto (\mathbb{C}^N, p^*)$ mapping M into M^*. Can $\widehat{H}$ be realized by a (smooth or analytic) CR-map $H : M \mapsto M^*$? More generally, does the existence of a formal transformation $\widehat{H}$ under discussion implies the existence of a (smooth of analytic) CR-map H of M into M^* with $H(p) = p^*$?

The problem of a geometric realization of formal CR-maps is, first of all, closely related to the holomorphic classification problem for real-analytic submanifolds in complex space. A beautiful example here is given by the class of finite type hypersurfaces in $\mathbb{C}^2$. For such hypersurfaces, Kolar constructed a *formal* normal form [Ko05], which is in general *divergent* [Ko12]. However, the convergence result of Baouendi-Ebenfelt-Rothschild [BER00] implies that all formal invertible transformations within the class of real-analytic finite type hypersurfaces are *convergent.* Hence two hypersurfaces with the same normal form become equivalent holomorphically, and Kolar's *formal* normal form solves the *holomorphic* equivalence problem for finite type hypersurfaces in $\mathbb{C}^2$.

Besides problems of holomorphic classification, Problem 1 is strongly motivated by the study of boundary regularity of holomorphic maps between domains in $\mathbb{C}^N$ (see, e.g., the survey of Forstnerić on the subject [Fo93]). We shall also emphasize the connections of Problem 1 with the *Artin Approximation problem* for solutions of PDEs (see, e.g., the survey of Mir [Mir14]).

The first results toward solving Problem 1 shall be attributed to E. Cartan [Ca32], Tanaka [Ta62] and Chern-Moser [CM74]. The main results of the cited papers lead to the *convergence* of all formal invertible power series maps between Levi-nondegenerate hypersurfaces in $\mathbb{C}^N$, $N \geq 2$. For example, the convergence in this setting can be seen from the convergence of Moser's normal form in [CM74]. In contrast, in the *Levi-degenerate* case, no such normal form is available in general (nor a canonical frame construction in the spirit of [Ca32, Ta62, CM74] is available). In this way, the problem has to be studied by different methods. This was done by various authors in the 1990's and 2000's, who were able to extend the convergence phenomenon for formal power series maps to a wide range of Levi-degenerate real hypersurfaces. We shall particularly emphasize here the work of Baouendi, Ebenfelt and Rothschild [BER00] who proved the convergence of formal invertible power series maps between *finite D'Angelo type* [D'A82] real-analytic hypersurfaces (i.e. hypersurfaces not containing non-trivial complex curves through the reference point), and the work of Mir [Mir00] who proved the convergence in the case of *holomorphically nondegenerate* and *minimal* real hypersurface. (We recall that minimality, notion due to Tumanov [Tu89], means the non-existence of a complex hypersurface $X \subset M$ through the reference point, and holomorphic nondegeneracy, notion due to Stanton [Sta96] and Baouendi-Ebenfelt-Rothschild [BER96], means the non-existence of non-zero holomorphic local sections of the $(1,0)$ bundle $T^{1,0}M$ of M). The key ingredient for the cited works was the jet parametrization technique due to Baouendi, Ebenfelt and Rothschild introduced in [BER97].

A lot of effort has been done in the 2000's for extending the convergence results under discussion to *nonminimal* (or *infinite Bloom-Graham type* [BER99]) hypersurfaces, and thus solving Problem 1 in its full generality. For an overview of this work we refer to the survey of Mir [Mir14] on the subject. The commonly conjectured outcome was solving Problem 1 in its full generality positively:

Conjecture 1. Two real-analytic hypersurfaces in $\mathbb{C}^N$, $N \geq 2$, which are *formally* equivalent at their reference points $p \in M$, $p^* \in M^*$, are also equivalent *holomorphically* at the respective points (e.g., Baouendi-Mir-Rothschild [BMR02], Mir[Mir14]).

However, somewhat surprisingly, Conjecture 1 was solved *negatively* by Shafikov and the first author in 2013.

Divergence Theorem (see [KS16a]). *For any $N \geq 2$, there exist real-analytic hypersurfaces $M, M^* \subset \mathbb{C}^N$ which are equivalent at the origin formally but are inequivalent there holomorphically.*

The Divergence Theorem was recently strengthened by the first two authors as follows.

Nonanalyticity Theorem (see [KL14]). *For any $N \geq 2$, there exist real-analytic hypersurfaces $M, M^* \subset \mathbb{C}^N$ which are C^∞ CR-equivalent at the origin, but are inequivalent there holomorphically.*

Both the Divergence and the Nonanalyticity Theorems were proved by employing the CR - DS technique mentioned above, and by subsequently using the *Stokes phenomenon* as in Dynamical Systems theory.

As discussed above, the Divergence Theorem answers the above Conjecture 1 negatively; however, Problem 1 in its *smooth* version remained open, and the following was conjectured in [KL14].

Conjecture 1A. Two real-analytic hypersurfaces in $\mathbb{C}^N$, $N \geq 2$, which are equivalent at their reference points $p \in M$, $p^* \in M^*$ *formally*, are also C^∞ *CR-equivalent* at the respective points.

1.3. **Main results.** The main result of this paper resolves Conjecture 1A *in the affirmative* in complex dimension 2.

Theorem 1. *Let $M, M^* \subset \mathbb{C}^2$ be two real-analytic hypersurfaces. Assume that M and M^* are formally equivalent at their reference points $p \in M$, $p^* \in M^*$. Then M and M^* are C^∞ CR-equivalent at the respective points p, p^*. Moreover, in case M, M^* are Levi-nonflat, the given formal equivalence $\widehat{H}$ between them can be realized by a C^∞ CR-map $H : M \mapsto M^*$, $H(p) = p^*$, formal expansion of which at p is $\widehat{H}$.*

It was observed by Nordine Mir that Theorem 1 implies the following remarkable corollary.

Theorem 2. *Let $M, M^* \subset \mathbb{C}^2$ be two real-algebraic Levi-nonflat hypersurfaces, and $\widehat{H} : (M, p) \mapsto (M^*, p^*)$ a formal invertible CR-map. Then $\widehat{H}$ is necessarily algebraic (in particular, it is convergent).*

We recall that a map is called *algebraic*, if its graph is contained in a real-algebraic set.

Proof of Theorem 2. According to Theorem 1, we can find a C^∞ CR-map H, transforming the germs (M, p) and (M^*, p^*) into each other, the Taylor expansion of which at p coincides with $\widehat{H}$. By the theorem of Baouendi, Huang and Rothschild [BHR96], H is algebraic (in particular, it is holomorphic). Since, again, $\widehat{H}$ is the Taylor expansion of H at p, this implies the assertion of the theorem. ☐

The assertion of Theorem 2, being of a significant interest within the CR-geometry community (see, e.g., [Mir14, MW16]), should be considered in the light of numerous existing algebraicity results for CR-mappings (see, e.g., the well known paper [Za99] of Zaitsev on the subject and references therein). We shall emphasize that most of these results rely on some given *initial smoothness* of a CR-map under consideration, which allows for transferring the problem to a *generic point* in a CR-manifold (this is the case in e.g. the above cited work of Baouendi-Huang-Rothschild). Such a transfer is, however, *not possible for formal CR-maps* (as, for example, illuminated by the Divergence Theorem, where the real hypersurfaces giving the counter-examples are all biholomorphically equivalent at their generic points, but are inequivalent at points lying in the infinite type locus). That is why *one does have to consider* formal CR-maps defined at points, where the behaviour of the CR-structure can be really exotic. Theorem 2 is probably the first result where this kind of difficulty has been successfully overcome in the most general case. For further discussion here we refer to [Mir14].

1.4. **Further results.** It can be shown that the C^∞ CR-map in Theorem 1 possesses some further properties, and in fact has much stronger regularity than the one stated in Theorem 1. We formulate the respective theorem below (in the most relevant nonminimal case). We recall that a formal power series

$$\widehat{H}(z, w) = \sum_{k,l \geq 0} c_{kl} z^k w^l$$

is said to be *of the (r, s) multi Gevrey class*, $r, s > 0$, if there exist appropriate constants $A, B, C > 0$ such that the Taylor coefficients c_{kl}, $k, l \geq 0$ satisfy the bounds:

$$|c_{kl}| \leq A \cdot B^k \cdot C^l \cdot (k!)^r (l!)^s. \tag{1.1}$$

For the more technical concept of *Gevrey asymptotic expansion* we refer to Section 2.6 below.

Theorem 3. *Let $M, M^* \subset \mathbb{C}^2$ be two real-analytic nonminimal at the origin Levi-nonflat hypersurfaces. Then there exist a constant $s > 0$ and appropriate local holomorphic coordinates (z, w) for M, M^* at 0 at which the complex locus X is $\{w = 0\}$, such that the following holds: any formal invertible CR-map $\widehat{H} : (M, 0) \mapsto (M^*, 0)$ is the $(0, s)$ multi Gevrey asymptotic expansion* *in appropriate domains $\Delta \times S^{\pm}$ of a holomorphic map H, where $\Delta \subset \mathbb{C}$ is a disc centered at 0, and $S^{\pm} \subset (\mathbb{C}, 0)$ are sectors with the vertex at 0 containing the directions $\mathbb{R}^{\pm}$, respectively. The restriction of H onto M defines a C^{∞} CR-map of M onto M^*.*

As a consequence, the given formal power series map $\widehat{H}(z,w)$ belongs to the $(0,s)$ multi Gevrey class, and thus satisfies (1.1) with $r = 0$.

Remark 1.1. In fact, it can be seen from the proof of Theorem 3 that the formal map $\widehat{H}$ in Theorem 3 has the *multisummability* property (see Section 2.6 for details).

Remark 1.2. As follows from the counter-examples given in [KS16a, KL14], the properties of formal CR-maps stated in Theorem 3 and Remark 1.1 are in general optimal and can't be strengthened further.

Remark 1.3. It can be verified from the proof of the main theorem that, for $m \geq 2$, the opening of the sectors $S^{\pm}$ in Theorem 3 can be chosen to be $\frac{\pi}{m-1}$ for a generic hypersurface M under consideration, and the Gevrey order s can be chosen to be $s = \frac{1}{m-1}$. For $m = 1$ one can take $s = 0$ (i.e., $\widehat{H}$ is *convergent*), as follows from the result of Juhlin and the second author [JL13].

1.5. **Principal method.** The main tool of the paper is the recent CR $\longrightarrow$ DS (Cauchy-Riemann manifolds $\longrightarrow$ Dynamical Systems) technique developed by Shafikov and the first two authors in the recent work [KS16a, KS16b, KL14, KL16]. The technique suggests to replace a given CR-submanifold M with a CR-degeneracy (such as nonminimality) by an appropriate holomorphic dynamical system $\mathcal{E}(M)$, and then study mappings of CR-submanifolds accordingly. The possibility to replace a real-analytic CR-manifold by a complex dynamical system is based on the fundamental parallel between CR-geometry and the geometry of completely integrable PDE systems. This parallel was first observed by E. Cartan and Segre [Ca32, Se32] (see also Webster [We77]), and was revisited and further developed in the important series of publications by Sukhov [Su01, Su03]. The "mediator" between a CR-manifold and the associated PDE system is the Segre family of the CR-manifold. Unlike the nondegenerate setting in the cited work [Ca32, Se32, Su01, Su03], the CR - DS technique deals systematically with the *degenerate* setting, providing sort of a dictionary between CR-geometry and Dynamical Systems.

In this paper, we develop the CR - DS technique further, extending it for the *entire* class of real-analytic hypersurfaces in $\mathbb{C}^2$.

Our proof the main theorem (and further results) at a glance goes as follows. In Section 3, we consider the case of infinite type hypersurfaces satisfying a certain nondegeneracy assumption (generic infinite type case). In this case, we follow the approach in [KS16b, KL16] and consider complex meromorphic differential equations, associated with real hypersurfaces. Any formal map between real hypersurfaces has to transform the associated ODEs into each other, and working out the latter condition gives a certain *singular Cauchy problem* for components of the map. We then apply the multisummability theory for formal power series solutions of nonlinear systems of ODE's at an irregular singularity [Bra92, RS94] to show that, first, the latter problem has solutions, holomorphic in certain sectorial domains and having there Gevrey asymptotic expansion, and second, that the solutions have certain uniqueness properties giving the condition $H(M) \subset M^*$ for the arising CR-map defined on M.

In Section 4, we have to extend the scheme in Section 3 to the exceptional (non-generic) case. For doing so, we introduce a new tool which is *associated differential equations of high order*. In turns out that *any* Levi-nonflat real-analytic hypersurface M (including finite type hypersurfaces!) can be associated, in appropriate local holomorphic coordinates, a system of singular ODEs of the kind (4.20). In is achieved by a sequence of coordinate changes and appropriate *blow-ups* (both in the initial space and in the space of parameters for Segre families). The initial formal CR-map is shown to be, again, a transformation between the associated systems of singular ODEs. Working out the transformation rule here brings new significant difficulties, since one has to deals with jet prolongations of arbitrarily high order. After overcoming these difficulties, we are again able to apply the multisummability theory and obtain the desired regularity property for the formal CR-map.

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2. Preliminaries

2.1. **Segre varieties.** Let M be a smooth real-analytic submanifold in $\mathbb{C}^{n+k}$ of CR-dimension n and CR-codimension k, $n, k > 0$, $0 \in M$, and U a neighbourhood of the origin where $M \cap U$ admits a real-analytic defining function $\phi(Z, \overline{Z})$ with the property that $\phi(Z, \zeta)$ is a holomorphic function for for $(Z, \zeta) \in U \times \bar{U}$. For every point $\zeta \in U$ we associate its Segre variety in U by

$$Q_\zeta = \{Z \in U : \phi(Z, \overline{\zeta}) = 0\}.$$

Segre varieties depend holomorphically on the variable $\overline{\zeta}$, and for small enough neighbourhoods U of 0, they are actually holomorphic submanifolds of U of codimension k.

One can choose coordinates $Z = (z, w) \in \mathbb{C}^n \times \mathbb{C}^k$ and a neighbourhood $U = U^z \times U^w \subset \mathbb{C}^n \times \mathbb{C}^k$ such that, for any $\zeta \in U$,

$$Q_\zeta = \left\{(z, w) \in U^z \times U^w : w = h(z, \overline{\zeta})\right\}$$

is a closed complex analytic graph. h is a holomorphic function on $U^z \times \bar{U}$. The antiholomorphic $(n+k)$-parameter family of complex submanifolds $\{Q_\zeta\}_{\zeta \in U_1}$ is called *the Segre family* of M at the origin. The following basic properties of Segre varieties follow from the definition and the reality condition on the defining function:

$$\begin{aligned} Z \in Q_\zeta &\Leftrightarrow \zeta \in Q_Z, \\ Z \in Q_Z &\Leftrightarrow Z \in M, \\ \zeta \in M &\Leftrightarrow \{Z \in U \colon Q_\zeta = Q_Z\} \subset M. \end{aligned} \tag{2.1}$$

The fundamental role of Segre varieties for holomorphic maps is due to their *invariance property*: If $f : U \to U'$ is a holomorphic map which sends a smooth real-analytic submanifold $M \subset U$ into another such submanifold $M' \subset U'$, and U is chosen as above (with the analogous choices and notations for M'), then

$$f(Q_Z) \subset Q'_{f(Z)}.$$

For more details and other properties of Segre varieties we refer the reader to e.g. [We77], [DP03], or [BER99].

The space of Segre varieties $\{Q_Z : Z \in U\}$, for appropriately chosen U, can be identified with a subset of $\mathbb{C}^K$ for some $K > 0$ in such a way that the so-called *Segre map* $\lambda : Z \to Q_Z$ is antiholomorphic. This can be seen from the fact that if we write

$$h(z, \bar{\zeta}) = \sum_{\alpha \in \mathbb{N}^n} h_\alpha(\bar{\zeta}) z^\alpha,$$

then $\lambda(Z)$ can be identified with $\big(h_\alpha(\bar Z)\big)_{\alpha\in\mathbb{N}^n}$. After that the desired fact follows from the Noetherian property.

If M is a hypersurface, then its Segre map is one-to-one in a neighbourhood of every point p where M is Levi nondegenerate. When such a real hypersurface M contains a complex hypersurface X, for any point $p \in X$ we have $Q_p = X$ and $Q_p \cap X \neq \emptyset \Leftrightarrow p \in X$, so that the Segre map λ sends the entire X to a unique point in $\mathbb{C}^N$ and, accordingly, λ is not even finite-to-one near each $p \in X$ (i.e., M is *not essentially finite* at points $p \in X$).

2.2. **Nonminimal real hypersurfaces.** We recall that given a real-analytic Levi-nonflat hypersurface $M \subset \mathbb{C}^2$, for every $p \in M$ there exist so-called *normal coordinates* (z, w) centered at p, i.e. a local holomorphic coordinate system near p in which $p = 0$ and near 0, M is defined by an equation of the form

$$v = F(z, \bar z, u)$$

for some germ F of a holomorphic function on $\mathbb{C}^3$ which satisfies

$$F(z, 0, u) = F(0, \bar z, u) = 0$$

and the *reality condition* $F(z, \bar z, u) \in \mathbb{R}$ for $(z, u) \in \mathbb{C} \times \mathbb{R}$ close to 0 (see e.g. [BER99]).

We say that M is *nonminimal* at p if there exists a germ of a nontrivial complex curve $X \subset M$ through p. It turns out that in normal coordinates, such a curve X is necessarily defined by $w = 0$; in particular, any such X is nonsingular.

Thus a Levi-nonflat hypersurface M is nonminimal if and only if with normal coordinates (z, w) and a defining function F as above, we have that $F(z, \bar z, 0) = 0$, or equivalently, if M can defined by an equation of the form

$$v = u^m \psi(z, \bar z, u), \text{ with } \psi(z, 0, u) = \psi(0, \bar z, u) = 0 \text{ and } \psi(z, \bar z, 0) \not\equiv 0, \tag{2.2}$$

where $m \geq 1$.

It turns out that the integer $m \geq 1$ is independent of the choice of normal coordinates (see [Me95]), and actually also of the choice of $p \in X$; we refer to m as the *nonminimality order* of a Levi-nonflat hypersurface M on X (or at p) and say that M is *m-nonminimal* along X (or at p).

Several other variants of defining functions for M are useful. Throughout this paper, we use the *complex defining function* Θ in which M is defined by

$$w = \Theta(z, \bar z, \bar w);$$

it is obtained from F by solving the equation

$$\frac{w - \bar w}{2i} = F\left(z, \bar z, \frac{w + \bar w}{2}\right)$$

for w. The complex defining function satisfies the conditions

$$\Theta(z, 0, \eta) = \Theta(0, \xi, \eta) = \tau, \quad \Theta(z, \xi, \bar\Theta(\xi, z, w)) = w.$$

If M is m-nonminimal at p, then $\Theta(z, \xi, \eta) = \eta\theta(z, \xi, \eta)$ and thus M is defined by

$$w = \bar w \theta(z, \bar z, \bar w) = \bar w(1 + \bar w^{m-1}\tilde\theta(z, \bar z, \bar w)), \text{ where } \tilde\theta(z, 0, \eta) = \tilde\theta(0, \xi, \eta) = 0 \text{ and } \tilde\theta(z, \xi, 0) \neq 0.$$

The Segre family of M, where M is given in normal coordinates as above, with the complex defining function $\Theta\colon U_z \times \bar U_z \times \bar U_w = U_z \times \bar U \to U_w$ consists of the complex hypersurfaces $Q_\zeta \subset U$, defined for $\zeta \in U$ by

$$Q_\zeta = \{(z, w)\colon w = \Theta(z, \bar\zeta)\}.$$

The real line

$$\Gamma = \{(z,w) \in M \colon z = 0\} = \{(0,u) \in M \colon u \in \mathbb{R}\} \subset M \tag{2.3}$$

has the property that

$$Q_{(0,u)} = \{w = u\}, \quad (0,u) \in \Gamma$$

for $u \in \mathbb{R}$, a property which actually is equivalent to the normality of the coordinates (z,w). More exactly, for any real-analytic curve γ through p one can find normal coordinates (z,w) in which γ corresponds to Γ in (4.7) (see e.g. [LM07]).

We finally have to point out that a real-analytic Levi-nonflat hypersurface $M \subset \mathbb{C}^2$ can exhibit nonminimal points of two kinds, which can be referred to as *generic* and *exceptional* nonminimal points, respectively. A generic point $p \in M$ is characterized by the condition that *the minimality locus $M \setminus X$ of M is Levi-nondegenerate locally near p.* At a generic nonminimal point, (2.2) is supplemented by the condition $\psi_{z\bar z}(0,0,0,) \neq 0$. In terms of the complex defining function, it gives the following useful representation for M:

$$w = \Theta(z,\bar z,\bar w) = \bar w + \bar w^m \sum_{k,l \geq 1} \Theta_{kl}(\bar w) z^k \bar z^l, \quad \Theta_{11}(0) \neq 0 \tag{2.4}$$

(see, e.g., [KS16b]).

If, otherwise, the intersection of the minimal locus $M \setminus X$ of M with any neighborhood of p in M contains Levi-degenerate points, then such a point p is referred to as *exceptional*.

2.3. **Real hypersurfaces and second order differential equations.** To every Levi nondegenerate real hypersurface $M \subset \mathbb{C}^N$ we can associate a system of second order holomorphic PDEs with 1 dependent and $N-1$ independent variables, using the Segre family of the hypersurface. This remarkable construction goes back to E. Cartan [Ca32] and Segre [Se32] (see also a remark by Webster [We77]), and was recently revisited in the work of Sukhov [Su01],[Su03] in the nondegenerate setting, and in the work of Shafikov and the first two authors in the degenerate setting (see[KS16a],[KS16b],[KL14],[KL16]). We describe this procedure in the case $N = 2$ relevant for our purposes.

Let $M \subset \mathbb{C}^2$ be a smooth real-analytic hypersurface, passing through the origin, and $U = U_z \times U_w$ a sufficiently small neighborhood of the origin. In this case we associate a second order holomorphic ODE to M, which is uniquely determined by the condition that the equation is satisfied by all the graphing functions $h(z,\zeta) = w(z)$ of the Segre family $\{Q_\zeta\}_{\zeta \in U}$ of M in a neighbourhood of the origin.

More precisely, since M is Levi-nondegenerate near the origin, the Segre map $\zeta \longrightarrow Q_\zeta$ is injective and the Segre family has the so-called transversality property: if two distinct Segre varieties intersect at a point $q \in U$, then their intersection at q is transverse. Thus, $\{Q_\zeta\}_{\zeta \in U}$ is a 2-parameter family of holomorphic curves in U with the transversality property, depending holomorphically on $\bar\zeta$. It follows from the holomorphic version of the fundamental ODE theorem (see, e.g., [IY08]) that there exists a unique second order holomorphic ODE $w'' = \Phi(z,w,w')$, satisfied by all the graphing functions of $\{Q_\zeta\}_{\zeta \in U}$.

To be more explicit we consider the complex defining equation $w = R(z,\bar z,\bar w)$, as introduced above. The Segre variety Q_ζ of a point $\zeta = (a,b) \in U$ is now given as the graph

$$w(z) = \rho(z,\bar a,\bar b). \tag{2.5}$$

Differentiating (2.5) once, we obtain

$$w' = \rho_z(z,\bar a,\bar b). \tag{2.6}$$

Considering (2.5) and (2.6) as a holomorphic system of equations with the unknowns $\bar{a}, \bar{b}$, an application of the implicit function theorem yields holomorphic functions A, B such that

$$\bar{a} = A(z, w, w'),\ \bar{b} = B(z, w, w').$$

The implicit function theorem applies here because the Jacobian of the system coincides with the Levi determinant of M for $(z, w) \in M$ ([BER99]). Differentiating (2.6) once more and substituting for $\bar{a}, \bar{b}$ finally yields

$$w'' = \rho_{zz}(z, A(z, w, w'), B(z, w, w')) =: \Phi(z, w, w'). \tag{2.7}$$

Now (2.7) is the desired holomorphic second order ODE $\mathcal{E} = \mathcal{E}(M)$.

More generally, the association of a completely integrable PDE with a CR-manifold is possible for a wide range of CR-submanifolds (see [Su01, Su03]). The correspondence $M \longrightarrow \mathcal{E}(M)$ has the following fundamental properties:

(1) Every local holomorphic equivalence $F : (M, 0) \longrightarrow (M', 0)$ between CR-submanifolds is an equivalence between the corresponding PDE systems $\mathcal{E}(M), \mathcal{E}(M')$ (see subsection 2.4);

(2) The complexification of the infinitesimal automorphism algebra $\mathfrak{hol}^{\omega}(M, 0)$ of M at the origin coincides with the Lie symmetry algebra of the associated PDE system $\mathcal{E}(M)$ (see, e.g., [Ol93] for the details of the concept).

Even though for a real hypersurface $M \subset \mathbb{C}^2$ which is nonminimal at the origin there is no a priori way to associate to M a second order ODE or even a more general PDE system near the origin, in [KS16b] the Shafikov and the first author found an injective correspondence between nonminimal at the origin and spherical outside the complex locus hypersurfaces $M \subset \mathbb{C}^2$ and certain *singular* complex ODEs $\mathcal{E}(M)$ with an isolated singularity at the origin. It is possible to extend this construction to the non-spherical case, which we do in Section 3.

2.4. **Equivalences and symmetries of ODEs.** We start with a description of the jet prolongation approach to the equivalence problem (which is a simple interpretation of a more general approach in the context of *jet bundles*). We refer to the excellent sources [Ol93], [BK89] for more details and collect the necessary prerequisites here. In what follows all variables are assumed to be complex, all mappings biholomorphic, and all ODEs to be defined near their zero solution $y(x) = 0$.

Consider two ODEs, $\mathcal{E}$ given by $y^{(k)} = \Phi(x, y, y', ..., y^{(k-1)})$ and $\tilde{\mathcal{E}}$ given by $y^{(k)} = \tilde{\Phi}(x, y, y', ..., y^{(k-1)})$, where the functions Φ and $\tilde{\Phi}$ are holomorphic in some neighbourhood of the origin in $\mathbb{C}^{k+1}$. We say that a germ of a biholomorphism $H\colon (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0)$ transforms $\mathcal{E}$ into $\tilde{\mathcal{E}}$, if it sends (locally) graphs of solutions of $\mathcal{E}$ into graphs of solutions of $\tilde{\mathcal{E}}$. We define the *k-jet space* $J^k(\mathbb{C}, \mathbb{C})$ to be the $(k + 2)$-dimensional linear space with coordinates $x, y, y_1, ..., y_k$, which correspond to the independent variable x, the dependent variable y and its derivatives up to order k, so that we can naturally consider $\mathcal{E}$ and $\tilde{\mathcal{E}}$ as complex submanifolds of $J^k(\mathbb{C}, \mathbb{C})$.

For any biholomorphism H as above one may consider its *k-jet prolongation* $H^{(k)}$, which is defined on a neighbourhood of the origin in $\mathbb{C}^{k+2}$ as follows. The first two components of the mapping $H^{(k)}$ coincide with those of H. To obtain the remaining components we denote the coordinates in the preimage by (x, y) and in the target domain by (X, Y). Then the derivative $\frac{dY}{dX}$ can be symbolically recalculated, using the chain rule, in terms of x, y, y', so that the third coordinate Y_1 in the target jet space becomes a function of x, y, y_1. In the same manner one obtains the remaining components of the prolongation of the mapping H. Thus, for differential equations of order k, *a mapping H transforms the ODE $\mathcal{E}$ into $\tilde{\mathcal{E}}$ if and only if the prolonged* *mapping $H^{(k)}$ transforms $(\mathcal{E},0)$ into $(\tilde{\mathcal{E}},0)$ as submanifolds in the jet space $J^k(\mathbb{C},\mathbb{C})$.* A similar statement can be formulated for systems of differential equations, as well as for certain singular differential equations, for example, the ones considered in the next subsection.

Some further details and properties of the jet prolongations $H^{(k)}$ are given in Section 4.

2.5. **Tangential sectorial domains and smooth CR-mappings.** Let $M \subset \mathbb{C}^2$ be a real-analytic Levi nonflat hypersurface, which is nonminimal at a point $p \in M$, and $X \ni p$ its complex locus. We choose for M local holomorphic coordinates (2.2) so that $p = 0$, $X = \{w = 0\}$. We next recall the following definition (see [KL16]).

Definition 2.1. A set $D_p \subset \mathbb{C}^2$, $D_p \ni p$ is called a *tangential sectorial domain for M at p* if, in some local holomorphic coordinates (z, w) for M as above, the set D_p looks as

$$\Delta \times \Big(S^+ \cup \{0\} \cup S^-\Big). \tag{2.8}$$

Here $\Delta \subset \mathbb{C}$ is a disc of radius $r > 0$, centered at the origin, and $S^\pm \subset \mathbb{C}$ are sectors

$$S^+ = \big\{|w| < R,\, \alpha^+ < \arg w < \beta^+\big\}, \quad S^- = \big\{|w| < R,\, \alpha^- < \arg w < \beta^-\big\} \tag{2.9}$$

for appropriate $R > 0$ and such that $S^\pm$ contains the direction $\mathbb{R}^\pm$. We also denote by $D_p^\pm$ the domains $\Delta \times S^\pm \subset \mathbb{C}^2$ respectively.

As discussed in [KL16], *for any tangential sectorial domain D_p for M at p, the intersection of M with a sufficiently small neighborhood U_p of p in $\mathbb{C}^2$ is contained in D_p.*

Next, we recall the following classical notion.

Definition 2.2. Let $f(w)$ be a function holomorphic in a sector $S \subset \mathbb{C}$. We say that a formal power series $\hat{f}(w) = \sum_{j\geq 0} c_j z^j$ is the *Poincaré asymptotic expansion of f in S*, if for any $n \geq 0$ we have:

$$\frac{1}{w^n}\left(f(w) - \sum_{j=0}^{n} c_j w^j\right) \to 0 \quad \text{when} \quad w \to 0,\ w \in S.$$

In the latter case, we write: $f(w) \sim \hat{f}(w)$.

For basic properties of the asymptotic expansion we refer to [Wa65]. In particular, we recall that asymptotic expansion in a full punctured neighborhood of a point means the usual holomorphicity of a function.

The notion of Poincaré asymptotic expansion can be naturally extended to function holomorphic in products of sectors and the respective formal power series in several variables. This allows us to formulate the following

Definition 2.3. We say that a C^∞ CR-function f in a neighborhood of p in M is *sectorially extendable,* if for some (and then any sufficiently small) tangential sectorial domain D_p for M at p, there exist functions $f^\pm \in \mathcal{O}(D_p^\pm)$ such that

(i) each $f^\pm$ coincides with f on $D_p^\pm \cap M$, and

(ii) both $f^\pm$ admit the same Poincaré asymptotic representation

$$f^\pm \sim \sum_{k,l\geq 0} a_{kl} z^k w^l$$

in the respective domains $D_p^\pm$.

We can similarly define the sectorial extendability of CR-mappings or infinitesimal CR-automorphisms of real-analytic hypersurfaces. Crucially, it is not difficult to see (as discussed in [KL16]) that *restricting two holomorphic functions $f^{\pm}$, as in Definition 2.3, onto a nonminimal hypersurface M as above defines a C^∞ CR-function on M near 0, sectorially extandable into the initial tangential sectorial domain.* This observation will be the final ingredient for the proof of Theorem 1.

2.6. **Summability of formal power series.** In this section, we shall recall some known facts about multisummability of formal power series and we shall recall a key theorem due to Braaksma that says that any formal solution of a system of nonlinear differential equations at an irregular singularity is multisommable in any direction but a finite number of them. This means there are holomorphic solutions in some sectors with vertex at the singularity and having the formal solution as asymptotic power series. This has a long although recent history and we refer to [Ram80, Ram93, Bal00, HS99, RS93] for more information.

Definition 2.4. Let $s>0$. A formal power series $\hat{f}=\sum_{n\geq 0} f_n z^n$ is said to be a *Gevrey series of order s* if there exist $A,B>0$ suh that $|f_n|\leq AB^n\Gamma(1+sn)$ for all n. The space of such power series is denoted by $\mathbb{C}[[z]]_s$.

In other words, we have $|f_n|\leq \tilde{A}\tilde{B}^n(n!)^s$ for some appropriate constants. Let $I=]a,b[$ be an open interval of $\mathbb{R}$ and let $r>0$. We denote by $\mathcal{S}_r(I)$ the open sector of $\mathbb{C}$:

$$\mathcal{S}_r(I):=\{z\in\mathbb{C}|\quad a<\arg z<b,\quad 0<|z|<r\}.$$

Definition 2.5. A holomorphic function $f\in\mathcal{O}(\mathcal{S}_r(I))$ is said to have an *s-Gevrey asymptotic expansion* at 0 if there exists a formal power series $\hat{f}=\sum_{j\geq 0} f_j z^j$ such that, for all $I'\subset\subset I$, there exist $C>0$ and $0<r'\leq r$ such that for all integer $n>0$

$$\left|f(z)-\sum_{k=0}^{n-1} f_j z^j\right|\leq C^n\Gamma(1+sn)|z|^n,\quad \forall z\in\mathcal{S}_{r'}(I').$$

We shall write $f\sim_s\hat{f}$. The space of these functions will be denoted by $\mathcal{A}_s(I)$.

Note that the above Gevrey asymptotic property strengthens the Poincaré asymptotic property introduced in the previous section. We also remark that asymptotic series $\hat{f}$ of such a function belongs to $\mathbb{C}[[z]]_s$.

Definition 2.6. A formal power series $\hat{f}\in\mathbb{C}[[z]]_{\frac{1}{k}}$ is said to be *k-summable* in the direction d if there exists a sector $\mathcal{S}_r(I)$, bissected by d and of opening $|I|>\frac{\pi}{k}$, and a holomorphic function $f\in\mathcal{O}(\mathcal{S}_r(I))$ such that $f\sim_{\frac{1}{k}}\hat{f}$. We also say that $\hat{f}$ is k-summable on I.

Such a holomorphic function f is unique (this is a consequence of Watson Lemma [Mal95]) and called the *k-sum of $\hat{f}$*. We emphasize that a *k-summable power series is $\frac{1}{k}$-Gevrey.* In order to describe the properties of solutions of differential equations with irregular singularity, we need the more general notion of multi-summability.

Definition 2.7. Let $r\geq 1$ be an integer and let $\mathbf{k}:=(k_1,\ldots,k_r)\in(\mathbb{R})^r$ with $0<k_1<\cdots<k_r$. For any $1\leq j\leq r$, let $I_j:=]a_j,b_j[$ be an open interval of length $|I_j|=b_j-a_j>\frac{\pi}{k_j}$ such that $I_j\subset I_{j-1}$, $2\leq j\leq r$. A formal power series $\hat{f}\in\mathbb{C}[[z]]$ is said to be $\mathbf{k}$-*multisummable on* $\mathbf{I}=(I_1,\ldots,I_r)$ if there exist formal power series $\hat{f}_j$ such that $\hat{f}:=\sum_{j=1}^r\hat{f}_j$ and such that each

$\hat{f}_j$ is k_j-summable on I_j with sum f_j, $1 \leq j \leq r$. We shall also say that $\hat{f}$ *is* $\mathbf{k}$*-multisummable in the multidirection* $\mathbf{d} = (d_1, \ldots, d_r)$ where d_j bissects the sector $\{a_j < \arg z < b_j\}$.

In that case, we say that $\mathbf{f} = (f_1, \ldots, f_r)$ *is the multisum of* $\hat{f}$. Such a multisum is unique according the relative Watson lemma [Mal95][Théorème 2.2.1.1]. From it, one can build the (unique) $\mathbf{k}$-sum of $\hat{f}$ on $\mathbf{I}$, denoted by $\mathbf{f}_{\mathbf{k},\mathbf{I}}$, that satisfies $\mathbf{f}_{\mathbf{k},\mathbf{I}} \sim_{\frac{1}{k_1}} \hat{f}$ on I_1[Bra92][p.524]. Here we have used the definition of W. Balser [Bal92] but there are other equivalent definitions due to Ecalle[Eca, MR91] and Malgrange-Ramis[RM92].

Next, we shall emphasize the folowing important property:

Proposition 2.8. [RM92]*[proposition 3.2,p. 358]*[Mal95]*[Théorème 2.2.3.1] Let* Φ *be a germ of holomorphic function at* 0 *of* $\mathbb{C}^{p+1}$*. Let* $\hat{f}_i$ *be a formal power series such that* $\hat{f}_i(0) = 0$, $i = 1, \ldots, p$*. Assume that* $\hat{f}_i$ *is* $\mathbf{k}$*-multisummable on* $\mathbf{I} = (I_1, \ldots, I_r)$ *with multisum* $\mathbf{f} = (f_1, \ldots, f_r)$*. Then,* $\Phi(z, \hat{f}_1(z), \ldots, f_p(z))$ *is also* $\mathbf{k}$*-multisummable on* $\mathbf{I} = (I_1, \ldots, I_r)$ *with multisum* $\Phi(z, \mathbf{f}) = (\Phi(z, f_{1,1}, \ldots, f_{p,1}), \ldots, \Phi(z, f_{1,r}, \ldots, f_{p,r}))$*.*

In particular, we conclude that the class of multisummable functions forms an algebra and is closed under the division operation, provided the denominator has no constant terms in its expansion.

The reason for introducing these notions is that these are the natural spaces to which solutions of nonlinear differential equations with irregular singularity must belong.

Let $r \in \mathbb{N}$, $k_j \in \mathbb{N}$, $j = 1, \ldots, r$, $0 < k_1 < \ldots < k_r$. We set $\mathbf{k} := (k_1, \ldots, k_r)$. Let $\mathbf{I} = (I_1, \ldots, I_r)$ where $I_j =]\alpha_j, \beta_j[$ is an open interval with $\beta_j - \alpha_j > \pi/k_j$. We also assume that $I_j \subset I_{j-1}, j = 1, \ldots, r$ where $I_0 = \mathbb{R}$. Consider

$$\operatorname{diag}\{x^{k_1} I^{(1)}, \ldots, x^{k_r} I^{(r)}\} x \frac{dy}{dx} = \Lambda y + x g(x, y) \tag{2.10}$$

where $I^{(j)}$ denotes the identity matrix of dimension $n_j \in \mathbb{N}$ and $n = n_1 + \ldots n_r$, $y \in \mathbb{C}^n$, $\Lambda = \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}$, Λ is invertible and g is a $\mathbf{k}$-sum of some $\hat{g}(x, y) \in \mathbb{C}[[x, y]]$ on $\mathbf{I}$ uniformly in a neighborhood of $0 \in \mathbb{C}^n$ (g analytic at $(0, 0)$ in $\mathbb{C} \times \mathbb{C}^n$ is a special case). Let $\hat{y} = \sum_{h=1}^{\infty} c_h x^h$ be a formal solution of (2.10). This means that

$$\operatorname{diag}\{x^{k_1} I^{(1)}, \ldots, x^{k_r} I^{(r)}\} x \frac{d\hat{y}(x)}{dx} = \Lambda y + x \hat{g}(x, \hat{y}(x)).$$

Then the following holds (cf. [RS94, Bra92, BBRS91])

Theorem 4. [Bra12] *The formal solution* $\hat{y}$ *of (2.10) is* $\mathbf{k}$*-multisummable on* $\mathbf{I} = (I_1, \ldots, I_r)$ *if* $\arg \lambda_h \notin]\alpha_j + \pi/(2k_j), \beta_j - \pi/(2k_j)[$ *for all* $h \in [n_1 + \ldots + n_{j-1} + 1, n_1 + \ldots + n_j]$

Corollary 2.9. [Bra92]*[Corollary p.525] Consider an analytic nonlinear differential equation of the form*

$$z^{\nu+1} \frac{dy}{dz} = F(z, y) \tag{2.11}$$

where $z \in \mathbb{C}$, $y \in \mathbb{C}^n$*, and* F *is analytic in a neighborhood of the the origin in* $\mathbb{C} \times \mathbb{C}^n$, $\nu > 0$*. Then, there exist a positive integers* q *and* $0 < k_1 < \ldots < k_r$ *such that a formal power series solution* $\hat{y}$ *of (2.11) is* $(\frac{k_1}{q}, \ldots, \frac{k_r}{q})$*-multisummable.*

As shown in [Bra91][p.60], there exists an analytic transformation and a ramification $x = z^{1/q}$ which transforms (2.11) into (2.10). As a consequence, we can also apply Theorem 4 in the situation when the righthand side F is a $(\frac{k_1}{q}, \ldots, \frac{k_r}{q})$-sum, uniformly in a neighborhood of $0 \in \mathbb{C}^n$.

The point for not stating this directly in the theorem is that both $\mathbf{k}$ and q need to be known and cannot be read off immediately on (2.11).

Remark 2.10. Let $\hat{y}$ be a formal power series solution of (2.11). Let $\mathbf{k}/\mathbf{q} := (\frac{k_1}{q}, \ldots, \frac{k_r}{q})$ as above. Then $\hat{y}$ has a $\mathbf{k}/\mathbf{q}$-sum $y^{\pm}$ defined in a sector containing the direction $\mathbb{R}^{\pm}$. Indeed, having done an appropriate analytic transformation and a ramification $x = z^{1/q}$, we consider (2.10). Let $\epsilon_j > 0$ and let $\tilde{I} := \cup_j \cup_{h\in[n_1+\ldots+n_{j-1}+1, n_1+\ldots+n_j]}]\arg\lambda_h - \epsilon_j, \arg\lambda_h + \epsilon_j[$. It is always possible to choose the ϵ_j's small enough so that the exists a $\tau_+ \notin \tilde{I}$ and so that $|\tau_+| < \frac{\pi}{2k_r} + \frac{1}{2}\min\frac{\epsilon_j}{2}$. Therefore, for all j, $-\tau_+ - \frac{\pi}{2k_j} - \frac{\epsilon_j}{2} < 0 < -\tau_+ + \frac{\pi}{2k_j} + \frac{\epsilon_j}{2}$. This means that $\mathbb{R}^+$ belongs to the sector I_j^+ bissected by τ_+ and of opening $\frac{\pi}{k_j} + \epsilon_j$, for all j. Setting $\tau_- = \tau_+ + \pi$, then $\mathbb{R}^-$ belongs to the sector I_j^- bissected by τ_- and of opening $\frac{\pi}{k_j} + \epsilon_j$, for all j. According to theorem Theorem 4, $\hat{y}$ is $\mathbf{k}$-multisummable on $\mathbf{I}^{\pm}$ and its $\mathbf{k}$-sum $\mathbf{y}_{\mathbf{k},\mathbf{I}^{\pm}}$ is defined on $\mathbb{R}^{\pm}$. To obtain the same result for 2.11, one has to divide τ_+ by q and set $\tau_- = \tau_+ + \pi/q$.

3. Complete system for a generic nonminimal hypersurface

We start with the proof of Theorem 1. We assume both reference points p, p^* to be the origin. As was discussed in the Introduction, in the finite type case the assertion of Theorem 1 follows from [BER00]. In the Levi-flat case the assertion is obvious. Hence, we assume in what follows that *both M, M^* are nonminimal at the reference point 0 but are Levi-nonflat.*

In this section, we prove Theorem 1 for the class of m-nonminimal at the origin hypersurfaces, satisfying the generic assumption that *the minimal part $M \setminus X$ of M is Levi-nondegenerate* (thus the origin is a generic nonminimal points, in the terminology of Section 2). As was explained in Section 2, any such hypersurface can be written in appropriate local holomorphic coordinates by an equation (2.4).

3.1. Mappings of nonminimal hypersurfaces in general position.

Let us consider two hypersurfaces $M, M^* \subset \mathbb{C}^2$, given near the origin by (2.4), and a formal power series map

$$H = (F, G): \quad (M, 0) \mapsto (M^*, 0)$$

between them. We first show that such a map has the following specific form.

Lemma 3.1. *Any formal power series map*

$$(z, w) \mapsto \big(F(z, w), G(z, w)\big)$$

between germs at the origin of two hypersurfaces of the form (2.4) *satisfies:*

$$G = O(w), \quad G_z = O(w^{m+1}). \tag{3.1}$$

Proof. We interpret (2.4) as:

$$w = \bar{w} + \bar{w}^m \cdot z\bar{z} \cdot O(1).$$

Then the basic identity gives:

$$G(z, w) = \bar{G}(\bar{z}, \bar{w}) + \bar{G}^m(\bar{z}, \bar{w}) \cdot \bar{F}(\bar{z}, \bar{w}) \cdot F(z, w) \cdot O(1), \text{ where } w = \bar{w} + \bar{w}^m \cdot z\bar{z} \cdot O(1).$$

Putting in the latter identity $\bar{z} = \bar{w} = 0$, we get $G(z, 0) \equiv 0$. Further, differentiating with respect to z, evaluating at $\bar{z} = 0$ at which one has $w = \bar{w}$, we get:

$$G_z(z, \bar{w}) = \bar{G}(0, \bar{w})^m \cdot \bar{F}(0, \bar{w}) \cdot O(1),$$

which already implies the assertion of the lemma. □

Lemma 3.1 immediately implies that, when considering formal invertible mappings between hypersurfaces of the form (2.4), *we can restrict to transformations of the form:*

$$z \mapsto z + f(z,w), \quad w \mapsto w + wg_0(w) + w^m g(z,w)$$

with

$$f_z(0,0) = 0, \quad g_0(0) = 0, \quad g(z,w) = O(zw) \tag{3.2}$$

(normalizing the coefficients of z, w for F, G respectively is possible by means of a linear scaling applied to the source hypersurface).

3.2. **Associated complete system.** Our first goal is to show the following:

Proposition 3.2. *Associated with a hypersurface* (2.4) *is a second order singular holomorphic ODE* $\mathcal{E}(M)$ *given by*

$$w'' = w^m \Phi\left(z, w, \frac{w'}{w^m}\right), \tag{3.3}$$

where $\Phi(z,w,\zeta)$ *is a holomorphic near the origin in* $\mathbb{C}^3$ *function with* $\Phi = O(\zeta)$*. The latter means that all Segre varieties of* M *(besides the complex locus* $X = \{w = 0\}$ *itself), considered as graphs* $w = w_p(z)$*, satisfy the ODE* (3.3)*.*

Proof. The argument of the proof very closely follows the one given in the proof of an analogues statement in [KS16b], [KL16] for the case of *m-nonminimal hypersurfaces*, and we leave the details of the proof to the reader. □

Based on the connection between mappings of hypersurfaces and that of the associated ODEs discussed in Section 2 and Lemma 3.1, we come to the consideration of ODEs (3.3) and formal power series mappings (3.2) between them. We further recall that the fact that a mapping $(F(z,w), G(z,w))$ transforms an ODE $\mathcal{E}$ into an ODE $\mathcal{E}^*$ is equivalent to the fact that the second jet prolongation $(F^{(2)}, G^{(2)})$ transforms the ODEs $\mathcal{E}, \mathcal{E}^*$ into each other, where the ODEs are considered as submanifolds in $J^2(\mathbb{C},\mathbb{C})$. Applying this to two nonsingular ODEs $\mathcal{E} = \{w'' = \Psi(z,w,w')\}$, $\mathcal{E}^* = \{w'' = \Psi^*(z,w,w')\}$ and employing the classical jet prolongation formulas (e.g., [BK89]), we obtain:

$$\begin{aligned}\Psi(z,w,w') = \frac{1}{J}\Bigg((F_z + w'F_w)^3 \Psi^*\Big(F(z,w), G(z,w), \frac{G_z + w'G_w}{F_z + w'F_w}\Big) + \\ + I_0(z,w) + I_1(z,w)w' + I_2(z,w)(w')^2 + I_3(z,w)(w')^3\Bigg),\end{aligned} \tag{3.4}$$

where $J := F_zG_w - F_wG_z$ is the Jacobian determinant of the transformation and

$$\begin{aligned} I_0 &= G_zF_{zz} - F_zG_{zz} \\ I_1 &= G_wF_{zz} - F_wG_{zz} - 2F_zG_{zw} + 2G_zF_{zw} \\ I_2 &= G_zF_{ww} - F_zG_{ww} - 2F_wG_{zw} + 2G_wF_{zw} \\ I_3 &= G_wF_{ww} - F_wG_{ww}. \end{aligned} \tag{3.5}$$

Setting then $\Psi(z,w,w') := w^m\Phi\left(z,w,\frac{w'}{w^m}\right)$ (and similarly for Φ^*) and switching to the notations in (3.2), we obtain the transformation rule for the class of ODEs (3.3) and mappings (3.2) between them:

$$
\begin{aligned}
w^m\Phi\left(z,w,\frac{w'}{w^m}\right) &= \frac{1}{J}\Big[\big(1+f_z+w'f_w\big)^3(1+g_0(w)+w^{m-1}g\big)^m\cdot \\
&\cdot w^m\Phi^*\Big(z+f,w+wg_0(w)+w^mg,\frac{w^mg_z+w'(1+wg_0'+g_0+mw^{m-1}g+w^mg_w)}{w^m(1+g_0(w)+w^{m-1}g)^m(1+f_z+w'f_w)}\Big)+ \\
&\qquad + I_0(z,w)+I_1(z,w)w'+I_2(z,w)(w')^2+I_3(z,w)(w')^3\Big], \qquad (3.6)
\end{aligned}
$$

where

$$
\begin{aligned}
J &= (1+f_z)(1+g_0+wg_0'+w^mg_w+mw^{m-1}g)-w^mf_wg_z, \\
I_0 &= w^m\big(g_zf_{zz}-(1+f_z)g_{zz}\big), \\
I_1 &= \big(1+wg_0'+g_0+mw^{m-1}g+w^mg_w\big)f_{zz}-w^mf_wg_{zz}- \\
&\quad -2(1+f_z)(mw^{m-1}g_z+w^mg_{zw})+2w^mg_zf_{zw}, \\
I_2 &= w^mg_zf_{ww}-(1+f_z)(wg_0''+2g_0'+m(m-1)w^{m-2}g+2mw^{m-1}g_w+w^mg_{ww})- \\
&\quad -2f_w(mw^{m-1}g_z+w^mg_{zw})+2(1+wg_0'+g_0+mw^{m-1}g+w^mg_w)f_{zw}, \\
I_3 &= (1+wg_0'+g_0+mw^{m-1}g+w^mg_w)f_{ww}- \\
&\quad -f_w(wg_0''+2g_0'+m(m-1)w^{m-2}g+2mw^{m-1}g_w+w^mg_{ww}).
\end{aligned} \qquad (3.7)
$$

Importantly, after putting $w'=\zeta w^m$, (3.6) becomes an identity of formal power series in the *independent* variables z,w,ζ.

We now extract from (3.7) four identities of power series *in* z,w *only*, in the following way. For the first identity, we extract in (3.7) terms with $(w')^0$ and divide the resulting identity by w^m. For the second identity, we extract in (3.7) terms with $(w')^1$. For the third identity, we extract in (3.7) terms with $(w')^2$ and multiply the resulting identity (which has a pole in w of order m) by w^m. For the last identity, we extract in (3.7) terms with $(w')^3$ and multiply the resulting identity (which has a pole in w of order $2m$) by w^{2m}. The four resulting identities of formal power series in z,w can be written as:

$$
\begin{aligned}
&I_0=w^mT_0(z,w,j^1(f,g,g_0)), \quad I_1=T_1(z,w,j^1(f,g,g_0)), \\
&w^mI_2=T_2(z,w,j^1(f,g,g_0)), \quad w^{2m}I_3=T_3(z,w,j^1(f,g,g_0)),
\end{aligned} \qquad (3.8)
$$

where $j^1(f,g.g_0)$ denotes the 1-jet of f,g,g_0 (the collection of derivatives of order ≤ 1), and $T_k(\cdot,z,w)$ are four precise holomorphic at the origin functions, exact form of which is of no interest to us. We though emphasize two important properties of the identities (3.8):

(a) the derivatives f_w,g_w come in each T_k with the factor w^m, and the derivative g_0' comes in each T_k with the factor w;

(b) the derivatives f_w,g_w,f_{zw},g_{zw} all come in all the left hand sides in (3.8) with the factor w^m, the derivatives f_{ww},g_{ww} all come in all the left hand sides in (3.8) with the factor w^{2m}, and the derivatives g_0',g_0'' come in all the left hand sides in (3.8) with the factor w.

It is also not difficult to verify that the identities (3.8) are well defined, i.e. the formal power series under considerations all come into the right hand side in (3.8) with the zero constant term.

Based on the observations (a),(b), we proceed as follows. Let us expand f, g as:

$$f(z,w)=\sum_{j=0}^{\infty} f_j(w)z^j, \quad g(z,w)=\sum_{j=1}^{\infty} g_j(w)z^j \tag{3.9}$$

(we point out that the function $g_0(w)$, as in (3.2), is *not* present in the expansion (3.9)!). In view of (3.2) we have

$$f_1(0)=g_1(0)=0. \tag{3.10}$$

We also introduce the new functions

$$y_1:=f_0, \quad y_2:=g_0, \quad y_3:=f_1, \quad y_4:=g_1, \quad y_5:=w^m f_0', \quad y_6:=wg_0', \quad y_7:=w^m f_1', \quad y_8:=w^m g_1'. \tag{3.11}$$

It is *important* that all the y_j do not have a constant term, as follows from (3.2),(3.10) and the fact that transformation maps the origin to itself. We clearly have

$$w^m y_1'=y_5, \quad wy_2'=y_6, \quad w^m y_3'=y_7, \quad w^m y_4'=y_8. \tag{3.12}$$

We then consider in the last two identities in (3.8) terms with z^0, z^1, respectively. This gives us *four second order singular ODEs for the functions* f_0, f_1, g_0, g_1. In the two identities with z^0, only the second order derivatives f_0'', g_0'' participate (the other derivatives have order ≤ 1). It is not difficult to solve the latter identities for $w^{2m}f_0'', w^{m+1}g_0''$ (by applying the Cramer rule to the a nondegenerate linear system). We obtain, by combining the information in (3.7),(3.12) and the observations (a),(b) above:

$$w^{2m}f_0''=U(y_1,y_2,...,y_8,w), \quad w^{m+1}g_0''=U(y_1,y_2,...,y_8,w), \tag{3.13}$$

where U and V are two holomorphic at the origin functions in all their variables, exact form of which is of no interest to us. Using the y-notations and (3.12), the equations (3.13) give:

$$w^m y_5'=\tilde{U}(y_1,y_2,...,y_8,w), \quad w^m y_6'=\tilde{V}(y_1,y_2,...,y_8,w), \tag{3.14}$$

where, again, $\tilde{U}$ and $\tilde{V}$ are two holomorphic at the origin functions in all their variables, exact form of which is of no interest to us.

To obtain the missing conditions for y_7', y_8', we use the system of two second order ODEs obtained by collecting in the last two identities of (3.8) terms with z^1. Considering this system as a (nondegenerate) linear system in $w^{2m}f_1'', w^{2m}g_1''$ and solving by Cramer rule, we get:

$$w^{2m}f_1''=X(y_1,y_2,...,y_8,w^{2m}f_0'',w), \quad w^{2m}g_1''=Y(y_1,y_2,...,y_8,w^{m+1}g_0'',w), \tag{3.15}$$

where X and Y are two holomorphic at the origin functions in all their variables, exact form of which is of no interest to us. Combining this with (3.13) and using (3.12), we finally obtain

$$w^m y_7'=\tilde{X}(y_1,y_2,...,y_8,w), \quad w^m y_8'=\tilde{Y}(y_1,y_2,...,y_8,w), \tag{3.16}$$

By putting (3.12),(3.14),(3.16) together, we have proved the following

Proposition 3.3. *The formal vector function $Y_0(w):=(y_1(w),...,y_8(w))$ satisfies a meromorphic differential equation*

$$w^m\frac{dY}{dw}=A(w,Y), \tag{3.17}$$

where $A(w,Y)$ is a holomorphic at the origin function.

Applying now the fundamental Theorem 4 (or rather its Corollary 2.9) on the multisummability of formal solutions of nonlinear differential equation at an irregular singularity as well as remark 2.10 (see Section 2), we immediately obtain

Corollary 3.4. *There exist sectors $S^+, S^- \subset \mathbb{C}$, containing the positive and the negative real lines, directions $d^\pm$, functions $f_0^\pm(w), g_0^\pm(w), f_1^\pm(w), g_1^\pm(w)$ holomorphic in the respective sectors, and a multi-order $\mathbf{k} = (k_1, ..., k_l)$ such that the following holds.*

(i) The functions $f_0^\pm(w), g_0^\pm(w), f_1^\pm(w), g_1^\pm(w)$ are the $\mathbf{k}$-multisums of f_0, g_0, f_1, g_1 in the directions $d^\pm$, respectively;

(ii) The holomorphic in respectively $S^\pm$ functions $Y^\pm(w)$, constructed via $f_0^\pm(w), g_0^\pm(w), f_1^\pm(w), g_1^\pm(w)$ by using formulas (3.11), satisfy the ODE (3.17).

The last point is a consequence of uniqueness of multisummable functions. Since Y is $\mathbf{k}$-multisommable on some multisectors, so are functions $w^m f_0', wg_0', w^m f_1', w^m g_1'$. Thus equalities 3.11 hold.

Our next goal is to show that the "barred" power series $\bar{f}_0(w), \bar{g}_0(w), \bar{f}_1(w), \bar{g}_1(w)$ belong to the same summability class as the original series. For doing so, let us consider the associated with (3.17) ODE

$$w^m \frac{dZ}{dw} = \bar{A}(w, Z), \tag{3.18}$$

where $A(w, Y)$ is as in (3.17). We first note that the "barred" power series $\bar{Y}_0(w)$ satisfies the ODE (3.18). Now, let us write $\mathbf{Y} := (Y, Z)$ and

$$\mathbf{A}(w, \mathbf{Y}) := \begin{pmatrix} A(w, Y) & 0 \\ 0 & \bar{A}(w, Z) \end{pmatrix},$$

and then consider the system

$$w^m \frac{d\mathbf{Y}}{dw} = \mathbf{A}(w, \mathbf{Y}). \tag{3.19}$$

Applying Corollary 2.9 of Theorem 4 for the "decoupled" system (3.19), we find sectors $S^+, S^- \subset \mathbb{C}$ containing the positive and the negative real lines (which we without loss of generality assume to be equal to the ones in Corollary 3.4), direction $d^\pm$ (which we without loss of generality assume to be equal to the ones in Corollary 3.4), and functions

$$\overline{f_0^\pm}(w), \overline{g_0^\pm}(w), \overline{f_1^\pm}(w), \overline{g_1^\pm}(w), \tag{3.20}$$

holomorphic in the respective sectors $S^\pm$, which are the $\mathbf{k}$-multisums in the directions $d^\pm$ of $\bar{f}_0(w), \bar{g}_0(w), \bar{f}_1(w), \bar{g}_1(w)$, respectively (we, again, assume without loss of generality that the multi-order $\mathbf{k}$ equals to the one in Corollary 3.4). In addition, the holomorphic in respectively $S^\pm$ function $\overline{Y^\pm}(w)$, constructed via $\overline{f_0^\pm}(w), \overline{g_0^\pm}(w), \overline{f_1^\pm}(w), \overline{g_1^\pm}(w)$ by using formulas (3.11), satisfies the ODE (3.17).

We now continue the argument leading to the proof of Theorem 1. We further consider the first two equations in (3.8). Read together, they can be treated as a system of linear equations in f_{zz}, g_{zz} determinant of which at the origin is non-vanishing. Applying the Cramer rule, we obtain the following system of equations:

$$f_{zz} = P(z, w, j^1(f, g), g_0, wg_0', f_{zw}, g_{zw}), \quad g_{zz} = Q(z, w, j^1(f, g), g_0, wg_0', f_{zw}, g_{zw}), \tag{3.21}$$

where P, Q are appropriate functions holomorphic in their arguments. We now consider the intimately related Cauchy problem

$$f_{zz} = P(z, w, j^1(f, g), \alpha_0, \alpha_1, f_{zw}, g_{zw}), \quad g_{zz} = Q(z, w, j^1(f, g), \alpha_0, \alpha_1, f_{zw}, g_{zw}) \tag{3.22}$$

With the Cauchy data

$$f(0, w) = \beta_0, \quad f_z(0, w) = \beta_1, \quad g(0, w) = 0, \quad g_z(0, w) = \beta_2, \tag{3.23}$$

where α_i, β_j are additional parameters. By the parametric version of the Cauchy-Kowalevski theorem, namely the Ovcyannikov's theorem [Ovs65, Trè68], the latter Cauchy problem has a unique analytic solutions

$$f = \varphi(z, w, \alpha_0, \alpha_1, \beta_0, \beta_1, \beta_2), \quad g = \psi(z, w, \alpha_0, \alpha_1, \beta_0, \beta_1, \beta_2),$$

where φ and ψ depend analytically on all their arguments. Hence, taking into account (3.2),(3.9), we have the identities:

$$\begin{aligned} f(z,w) &= \varphi\big(z, w, g_0(w), wg_0'(w), f_0(w), f_1(w), g_1(w)\big), \\ g(z,w) &= \psi\big(z, w, g_0(w), wg_0'(w), f_0(w), f_1(w), g_1(w)\big) \end{aligned} \tag{3.24}$$

(we emphasize that the substitution of formal power series into φ, ψ is well defined here, since *all* the formal data being substituted has no constant term!).

We are now in the position to prove Theorem 1 in the m-admissible case.

Proof of Theorem 1. Let us introduce the functions

$$\begin{aligned} f^{\pm}(z,w) &= \varphi\big(z, w, g_0^{\pm}(w), w\cdot(g_0^{\pm})'(w), f_0^{\pm}(w), f_1^{\pm}(w), g_1^{\pm}(w)\big), \\ g^{\pm}(z,w) &= \psi\big(z, w, g_0^{\pm}(w), w\cdot(g_0^{\pm})'(w), f_0^{\pm}(w), f_1^{\pm}(w), g_1^{\pm}(w)\big), \end{aligned} \tag{3.25}$$

well defined in the product of a disc Δ in z centered at the origin and the sectors $S^{\pm}$ in w (this product forms a *tangential sectorial domain*, as described in Section 2. $f^{\pm}(z,w), g^{\pm}(z,w)$ are asymptotically represented in their domains by $f(z,w), g(z,w)$ respectively, as follows from (3.24). Based on (3.20), we similarly introduce $\overline{f^{\pm}}(z,w)$, $\overline{g^{\pm}}(z,w)$, asymptotically representing $\bar{f}(z,w), \bar{g}(z,w)$, respectively.

Let us now consider the (complexified) basic identity

$$G(z,w) - \rho^*\big(F(z,w), \overline{F}(\xi,\eta), \overline{G}(\xi,\eta)\big)|_{w=\rho(z,\xi,\eta)} = 0 \tag{3.26}$$

for the map (F,G) between the germs at the origin of the initial hypersurfaces $M = \{w = \rho(z,\bar{z},\bar{w})\}$ and $M^* = \{w = \rho^*(z,\bar{z},\bar{w})\}$. We claim that the sectorial map $(F^{\pm}(z,w), G^{\pm}(z,w))$ constructed via $f^{\pm}, g^{\pm}$ by the formula (3.2) satisfies the basic identity (3.26) as well, i.e.

$$G^{\pm}(z,w) - \rho^*\big(F^{\pm}(z,w), \overline{F^{\pm}}(\xi,\eta), \overline{G^{\pm}}(\xi,\eta)\big)|_{w=\rho(z,\xi,\eta)} = 0, \quad (z,\xi,\eta) \in \Delta\times\Delta\times S^{\pm}. \tag{3.27}$$

To prove the claim, let us analyze the identity (3.27). The left hand side of it, which we denote by

$$\chi(z,\xi,\eta),$$

is holomorphic in $\Delta\times\Delta\times S^{\pm}$, respectively. Accordingly, the identity (3.27) holds if and only if we have:

$$\left.\frac{\partial^{p+q}}{\partial z^p \partial \xi^q}\chi(z,\xi,\eta)\right|_{z=\xi=0} \equiv 0, \quad p,q \geq 0. \tag{3.28}$$

However, it is not difficult to verify (by applying the chain rule) that for each fixed $p,q \geq 0$ the left hand side in (3.28) is an analytic function $R_{p,q}$ in $f_0^{\pm}(\eta), g_0^{\pm}(\eta), f_1^{\pm}(\eta), g_1^{\pm}(\eta), \overline{f_0^{\pm}}(\eta), \overline{g_0^{\pm}}(\eta), \overline{f_1^{\pm}}(\eta), \overline{g_1^{\pm}}(\eta)$ and their derivatives of order $\leq p+q$, and in η. Hence, each left hand side in (3.28) is the $\mathbf{k}$-multisum of the identical analytic expressions $R_{p,q}$ in formal series, where $f_0^{\pm}(\eta), g_0^{\pm}(\eta), f_1^{\pm}(\eta), g_1^{\pm}(\eta)$ are replaced by the asymptotic expansions f_0, g_0, f_1, g_1, respectively, and $\overline{f_0^{\pm}}(\eta), \overline{g_0^{\pm}}(\eta), \overline{f_1^{\pm}}(\eta), \overline{g_1^{\pm}}(\eta)$ by their asymptotic expansions $\bar{f}_0(\eta), \bar{g}_0(\eta), \bar{f}_1(\eta), \bar{g}_1(\eta)$, respectively. In view of the (valid!) formal basic identity (3.26), the latter formal series in η vanish identically for any $p,q \geq 0$. The uniqueness property within the class of **k**-multisummable series in the directions $d^{\pm}$ implies now that all the left hand sides in (3.28) *all vanish identically.*

As was explained in Section 2, the property (3.27) for a sectorial map defined in a tangential sectorial domain implies that the restriction of the map onto the source manifold is a C^∞ CR-map onto the target. Thus, the claim under discussion implies the assertion of the theorem. $\square$

4. The exceptional case

In this section, we prove Theorem 1 in full generality. For that, we have to consider the case when, for an m-nonminimal at the origin hypersurface $M \subset \mathbb{C}^2$, the minimal part $M \setminus X$ contains Levi degenerate points. In this case, M can *not* be associated an ODE (3.3). We overcome this difficulty by introducing *associated ODEs of high order.*

The proof of Theorem 1 in the general case has several ingredients, each of which we put in a separate subsection below.

4.1. **k-summability of initial terms.** In what follows, for hypersurfaces under consideration we consider the defining equation (2.2). We will also need the complex defining equation

$$w = \Theta(z,\bar z,\bar w) \quad \Theta(z,\bar z,\bar w) = \bar w + \sum_{j,k\geq 1} \Theta_{jk}(\bar w) z^k \bar z^l, \quad \Theta \not\equiv 0. \tag{4.1}$$

Furthermore, we require the additional condition

$$\Theta_{11}(\bar w) \not\equiv 0 \tag{4.2}$$

in (4.1) (the geometric meaning of it will be discussed below). The fact that the minimal part $M \setminus X$ contains Levi degenerate points reads as

$$\operatorname{ord}_0 \Theta_{11}(\bar w) > m. \tag{4.3}$$

We start by considering for a formal power series map (F,G) between germs at the origin of hypersurfaces (4.1) the expansion:

$$F = \sum_{j\geq 0} F_j(w) z^j, \quad G = \sum_{j\geq 0} G_j(w) z^j. \tag{4.4}$$

Arguing similarly to the proof of Lemma 3.1, it is not difficult to prove

Lemma 4.1. *The components of the formal map (F,G) satisfy:*

$$F_z(0,0) = F_1(0) \neq 0, \quad G_w(0,0) = G_0'(0) \neq 0, \quad G(z,w) = O(w), \quad G_z(z,w) = O(w^m). \tag{4.5}$$

In this way, by performing a scaling, we may assume

$$F_z(0,0) = F_1(0) = 1, \quad G_w(0,0) = G_0'(0) = 1.$$

Our goal in this subsection is to prove the following

Proposition 4.2. *There exist sectors $S^+, S^- \subset \mathbb{C}$, containing the positive and the negative real lines respectively, directions $d^{\pm} \subset S^{\pm}$, a multi-order $\mathbf{k} = (k_1, ..., k_l)$, and functions $F_j^{\pm}(w), G_j^{\pm}(w)$ holomorphic in the respective sectors, such that for each $j \geq 0$, the functions $F_j^{\pm}(w), G_j^{\pm}(w)$ are the $\mathbf{k}$-multisums of $F_j, G_j 1$ in the directions $d^{\pm}$, respectively.*

Proposition 4.2 is proved in several steps.

Step I. We first observe that the assertion of Proposition 4.2 is invariant under *biholomorphic* transformations of the target. Indeed, a holomorphic coordinate change

$$z \mapsto U(z,w), \quad w \mapsto V(z,w)$$

in the target changes the components of the map as follows:

$$\widetilde{F} = U(F(z,w), G(z,w)), \quad \widetilde{G} = V((F(z,w), G(z,w)). \tag{4.6}$$

The new coefficient functions $\widetilde{F}_j, \widetilde{G}_j$ can be computed by differentiating (4.6) in z sufficiently many times and evaluating at $z = 0$. Now the desired invariance property follows from the properties of multisummable functions (see Section 2) and the chain rule.

Step II. In this step, we make use of the following efficient blow-up procedure introduced in [LM07] by Mir and the second author.

Lemma 4.3 (Blow-up Lemma, see [LM07]). *Let $M \subset \mathbb{C}^2$ be a real-analytic hypersurface, which is Levi-degenerate at the origin and Levi-nonflat. Assume that M is given in coordinates* (4.1) *and that the distinguished curve*

$$\Gamma = \{(z,w) \in M : \ z = 0\} \subset M \tag{4.7}$$

does not contain Levi-degenerate points of M other than the origin. Then there exists a blow-down map

$$B(\xi,\eta) : \quad (\mathbb{C}^2, 0) \longrightarrow (\mathbb{C}^2, 0), \quad B(\xi,\eta) = (\xi\eta^s, \eta), \quad s \in \mathbb{Z}, \quad s \geq 2, \tag{4.8}$$

and a real-analytic nonminimal at the origin hypersurface $M_B \subset \mathbb{C}^2_{(\xi,\eta)}$ with the complex locus $X = \{\eta = 0\}$ such that:

(i) $B(M_B) \subset M, \quad B(X) = \{0\}$;

(ii) $M_B \setminus X$ is Levi-nondegenerate, and M_B is given by an equation of the kind (2.4).

We note at this point that the condition for Γ in Lemma 4.3 is precisely equivalent to (4.2).

We have to now revisit the proof of the Blow-up Lemma. Recall that this proof goes as follows. For an m-nonminimal hypersurface, transformations bringing to coordinates of the kind (4.1) are associated with curves $\gamma \subset M$ passing through 0 and transverse to the complex tangent at 0. Such a curve γ is being transformed into the distingusihed (4.7) in the new coordinates (2.2).

We then choose γ in such a way that $\gamma \cap \Sigma = \{0\}$ for the Levi degeneracy set $\Sigma \subset M$, and bring to coordinates (4.1). This means that for the resulting hypersurface (4.1) we have $\Theta_{11} \not\equiv 0$. For each $k \geq 2$, let us denote

$$m(k) := \min_{p+q=k} \operatorname{ord}_0 \Theta_{pq}.$$

We have $m(j) \geq m$ for all $j \geq 2$. After that, an integer s in (4.8) is determined as any integer satisfying all the inequalities

$$2s + m(2) \leq ks + m(k), \quad k \geq 3. \tag{4.9}$$

In fact, one can require the unique (stronger) inequality

$$m(2) < s, \tag{4.10}$$

and thus avoid considering $m(k)$, $k \geq 3$.

We now proceed as follows. We may assume that both M and M^* are given by coordinates (2.2) with $\Theta_{11} \not\equiv 0$. We then fix an integer s, which satisfies (4.10) for both M and M^*. Next, we consider the formal curve $\gamma \subset M$ - the pre-image of (4.7) under the given formal map $H = (F, G)$. Let us choose an *analytic* curve $\tilde{\gamma} \subset M$ tangent to γ to order $s + 1$, and a *biholomorphic* map H_1

transforming $\tilde{\gamma}$ into (4.7) and M into a hypersurface $\widetilde{M}$ of the kind (2.2). Put $H_2 := H \circ H_1^{-1}$, so that $H = H_2 \circ H_1$. Finally, put

$$\tilde{\Gamma} := H(\tilde{\gamma}).$$

Note that, since γ and $\tilde{\gamma}$ are tangent to order $s+1$, the same is true for (4.7) and $\tilde{\Gamma}$.

We then can decompose H^{-1} as a product

$$H^{-1} = H_1^{-1} \circ H_2^{-1}, \tag{4.11}$$

where H_1^{-1} is a *biholomorphic* map transforming (4.7) into $\tilde{\gamma}$ and $\widetilde{M}$ into M, and H_2^{-1} is a *formal* invertible map transforming $\tilde{\Gamma}$ into (4.7) and M^* into the real-analytic hypersurface $\widetilde{M}$. Importantly, in view of the tangency condition, the formal map H_2^{-1} satisfies

$$\operatorname{ord}_0 F_0(w) \geq s+1, \tag{4.12}$$

where F_0 is as in (4.4). Moreover, the blow up integer s can be kept the same as before for the hypersurface $\widetilde{M}$ as well. Indeed, a transformation satisfying (4.12) clearly preserves the corresponding integer $m^*(2)$ in (4.10)(as we chose $s > m^*(2)$), so that the inequalities (4.10) still hold true for the same s and the hypersurface $\widetilde{M}$.

Finally, we recall that, in view of the considerations of Step I, the assertion of Proposition 4.2 applied for H_2^{-1} is equivalent to that for H^{-1}.

We summarize the considerations of Step II as follows: in view of the decomposition (4.11) and the subsequent properties of H_2^{-1},

it is sufficient to prove Proposition 4.2 for maps (F,G) satisfying, in addition, the inequality (4.12).

Step III. In this step, we are finally able to reduce Proposition 4.2 to the results already proved in the generic case. For that, we use the above blow up procedure.

In accordance with the outcome of the previous step, we consider a map $(F,G) : (M,0) \mapsto (M^*,0)$ satisfying, in addition, (4.12). Here the integer s in (4.12) is an admissible integer for the blow down map (4.8) both in the source and in the target. After performing the blow ups (with the integer s in (4.8)), we obtain real-analytic hypersurfaces M_B, M_B^*, respectively.

Re-calculating the map (F,G) in the "blown up" coordinates (ξ,η) gives:

$$G_B(\xi,\eta) = G(\xi\eta^s,\eta), \quad F_B(\xi,\eta) = \frac{F_0(\eta)}{\eta^s} + F_1(\eta)\xi + \cdots, \tag{4.13}$$

where dots stand for a power series in ξ,η of the kind $O(\xi^2)$. In view of (4.12), $F_B(\xi,\eta)$, $G_B(\xi,\eta)$ are *well defined power series.* It is immediate then that the formal map

$$H_B(\xi,\eta) := \big(F_B(\xi,\eta),\, G_B(\xi,\eta)\big)$$

transforms $(M_B,0)$ into $(M_B^*,0)$. Furthermore, in view of (4.5), the formal map $H_B(\xi,\eta)$ is *invertible*, so that the results of Section 3 are applicable to it. Expanding now

$$F_B(\xi,\eta) = \sum_{j\geq 0} F_j^B(\eta)\xi^j, \quad G_B(\xi,\eta) = \sum_{j\geq 0} G_j^B(\eta)\xi^j,$$

and applying to H_B the assertion of Corollary 3.4 and the formulas (3.24), we immediately obtain for the components F_j^B, G_j^B the desired $\mathbf{k}$-summability property (identical to the one stated in Proposition 4.2). At the same time, the relations (4.13) show that

$$G_j^B(\eta) = \eta^{sj} G_j(\eta). \tag{4.14}$$

We immediately obtain from (4.14) the assertion of Proposition 4.2 for the components G_j (with the same sectors, multi-directions and multi-order $\mathbf{k}$ as for F_B, G_B). Finally, since we have

$$F(\xi\eta^s,\eta)=\big(G(\xi\eta^2,\eta)\big)^s\cdot F_B(\xi,\eta),$$

the chain rule and the multisummability property for F_j^B, G_j imply the assertion of Proposition 4.2 for the components F_j. This finally proves Proposition 4.2.

□

4.2. **Associated ODEs of high order.** In this section we consider the case when the source and the target m-nonminimal hypersurfaces satisfy the additional k-*nondegeneracy* condition. The latter means that for some $k\geq 1$ we have

$$\operatorname{ord}_0\Theta_{k1}=m \tag{4.15}$$

for the defining function (4.1). As a well known fact (e.g. [Me95]) the property of being m-nonminimal k-nondegenerate is invariant under (formal) invertible transformations. In view of (4.3), we may assume that $k\geq 2$ in our setting.

The main goal of this section is to show that an m-nonminimal k-nondegenerate hypersurface M is associated a *system* $\mathcal{E}(M)$ *of* k *singular ODEs of orders* $\leq k+1$. By the latter we mean (as in the generic case) that all the Segre varieties Q_p of M for $p\notin X$ considered as graphs $w=w_p(z)$ satisfy the system of ODEs $\mathcal{E}(M)$.

For producing the associated ODEs, we consider the Segre family of an m-nonminimal hypersurface (4.1) satisfying the additional k-nondegeneracy condition, and produce for it an elimination procedure, in the spirit of that discussed in Section 2. This Segre family looks as:

$$w=b+O(ab^mz) \tag{4.16}$$

(we use the notation $p=(\bar a,\bar b)$). Differentiating (4.16) k times in z and using (4.15), we obtain:

$$w^{(k)}=ab^m(\alpha+o(1)),\quad \alpha\neq 0 \tag{4.17}$$

(here α is a fixed constant). Dividing (4.17) by the m-th power of (4.16) gives:

$$\frac{w^{(k)}}{w^m}=\alpha a+o(a). \tag{4.18}$$

Solving the system (4.18),(4.16) for a,b by the implicit function theorem yields

$$a=A\left(z,w,\frac{w^{(k)}}{w^m}\right),\quad b=B\left(z,w,\frac{w^{(k)}}{w^m}\right) \tag{4.19}$$

for two holomorphic near the origin in $\mathbb{C}^3$ functions $A(z,w,\zeta),B(z,w,\zeta)$ with $A=O(\zeta)$ and $B=O(w)$. Differentiating then (4.16) j times for each $j=1,...,k-1,k+1$ and substituting (4.19) into the results finally gives us:

$$w'=\Phi_1\left(z,w,\frac{w^{(k)}}{w^m}\right),\cdots,w^{(k-1)}=\Phi_{k-1}\left(z,w,\frac{w^{(k)}}{w^m}\right),\ \ w^{(k+1)}=\Phi\left(z,w,\frac{w^{(k)}}{w^m}\right). \tag{4.20}$$

Here $\Phi_1(z,w,\zeta),...,\Phi_{k-1}(z,w,\zeta),\Phi(z,w,\zeta)$ are all holomorphic near the origin in $\mathbb{C}^3$ functions of the kind

$$O(w^m\zeta) \tag{4.21}$$

(as follows from the elimination procedure).

Definition 4.4. The system of ODEs (4.20) is called *associated with* M and is denoted by $\mathcal{E}(M)$.

It is immediate, in the same way as in the nondegenerate case, that *all the Segre varieties Q_p of M for $p \notin X$ considered as graphs $w = w_p(z)$ satisfy the system of ODEs $\mathcal{E}(M)$.*

We would need in the sequel one nondegeneracy propery for the ODE system (4.20). For obtaining it, let us recall that defining equations (4.1) of hypersurfaces under consideration satisfy the reality condition:

$$w \equiv \Theta(z, \bar{z}, \bar{\Theta}(\bar{z}, z, w)) \ \forall z, \bar{z}, w \tag{4.22}$$

(see, e.g., [BER99]). Gathering in (4.22) terms with $z^k \bar{z}^1$ and using (4.1), we obtain

$$0 = \Theta_{k1}(w) + \bar{\Theta}_{1k}(w).$$

Hence we have, in view of (4.15):

$$\mathrm{ord}_0\, \Theta_{1k} = m. \tag{4.23}$$

It immediately follows then from the above elimination procedure that

Property $(*)$**.** The term with $z^0 w^m \zeta^k$ in the expansion of the function Φ_1 in (4.20) is non-zero; without loss of generality, we assume its coefficient in what follows to be equal to $\pm i$, even though its exact value is of no special interest to us.

As the final outcome of this subsection, we have the following:

under the k-nondegeneracy assumption, the formal map (F, G) under consideration is a map of the kind (4.5) *transforming the system $\mathcal{E}(M)$ into $\mathcal{E}(M^*)$; the systems $\mathcal{E}(M)$ and $\mathcal{E}(M^*)$ have the form* (4.20) *and satisfy, in addition,* (4.21) *and the Property* $(*)$ *above.*

4.3. **Proof of the main theorem under the k-nondegeneracy assumption.** In what follows, we have to take into consideration the space $J^{k+1}(\mathbb{C}, \mathbb{C})$ of $(k+1)$-jets of holomorphic maps from $\mathbb{C}$ into itself. We use the notations

$$(z, w, w_1, ..., w_{k+1})$$

for the coordinates in the jet space (here w_j corresponds to the derivative $w^{(j)}(z)$). A system (4.20) shall be regarded then as a submanifold in $J^{k+1}(\mathbb{C}, \mathbb{C})$ of dimension 3 (with the local coordinates z, w, w_k):

$$w_1 = \Phi_1\left(z, w, \frac{w_k}{w^m}\right), \cdots, w_{k-1} = \Phi_{k-1}\left(z, w, \frac{w_k}{w^m}\right), \ \ w_{k+1} = \Phi\left(z, w, \frac{w_k}{w^m}\right). \tag{4.24}$$

Next, we consider the $(k+1)$-jet prolongation

$$H^{(k+1)}(z, w, w_1, ..., w_{k+1}) = \left(F(z, w), G(z, w), G^{(1)}(z, w, w_1), G^{(2)}(z, w, w_1, w_2), ..., G^{(k+1)}(z, w, w_1, ..., w_{k+1})\right)$$

of the map (F, G). Introducing the total derivation operator

$$D := \partial_z + w_1 \partial_w + \sum_{j \geq 1} w_{j+1} \partial_{w_j}, \tag{4.25}$$

we can inductively compute the components of the prolonged map (see [BK89][(3.96d) of section 2.3.1]) as

$$G^{(j)} = \frac{DG^{(j-1)}}{DF}, \ j \geq 1, \quad \text{where} \quad G^{(0)} := G. \tag{4.26}$$

Note that, in fact,

$$DF = F_z + w_1 G_w.$$

As follows from (4.26), each $G^{(j)}(z,w,w_1,...,w_j)$ is an expression, *rational* in the first jet variable w_1 and *polynomial* in the remaining jet variables $w_2,...,w_j$; its coefficients are *universal polynomials* in the j-jet of (F,G). For certain precise values of j (e.g. $j=1,2,3$), the j-jet prolongation formulas can be written explicitly. For example, we have:

$$\begin{aligned} G^{(1)}(z,w,w_1) &= \frac{G_z+w_1G_w}{F_z+w_1F_w}, \\ G^{(2)}(z,w,w_1,w_2) &= \frac{1}{(F_z+w_1F_w)^3}\Big[(F_z+w_1F_w)(G_{zz}+2w_1G_{zw}+(w_1)^2G_{ww}+w_2G_w)- \\ &\quad -(G_z+w_1G_w)(F_{zz}+2w_1F_{zw}+(w_1)^2F_{ww}+w_2F_w)\Big]. \end{aligned} \tag{4.27}$$

For some higher orders see, e.g., [BK89]. However, for a general j, only certain summation formulas exist, which can not always be worked out. That is why we will use only certain properties of the prolonged maps, which are useful for our consideration. For example, we can claim that, for maps of the kind (4.5), the denominator of it is non-vanishing at $z=w=w_1=...=w_j=0$. This can be easily proved by induction, by using (4.26) and the fact that $F_z(0,0)=1$.

According to the outcome of the previous section and the discussion in Section 2, the prolonged map $H^{(k+1)}$ transforms the submanifolds $\mathcal{E}(M),\mathcal{E}(M^*)\subset J^{k+1}(\mathbb{C},\mathbb{C})$ into each other. That is, we have the following basic identity:

$$\begin{aligned} G^{(1)}(z,w,w_1) &= \Phi_1^*\left(F(z,w),G(z,w),\frac{G^{(k)}(z,w,W_k)}{G^m(z,w)}\right), \\ &\cdots \\ G^{(k-1)}(z,w,w_1,\dots,w_{k-1}) &= \Phi_{k-1}^*\left(F(z,w),G(z,w),\frac{G^{(k)}(z,w,W_k)}{G^m(z,w)}\right), \\ G^{(k+1)}(z,w,w_1,\dots,w_{k+1}) &= \Phi^*\left(F(z,w),G(z,w),\frac{G^{(k)}(z,w,W_k)}{G^m(z,w)}\right), \end{aligned} \tag{4.28}$$

subject to the restriction

$$w_1=\Phi_1\left(z,w,\frac{w_k}{w^m}\right),\cdots,w_{k-1}=\Phi_{k-1}\left(z,w,\frac{w_k}{w^m}\right),\ \ w_{k+1}=\Phi\left(z,w,\frac{w_k}{w^m}\right) \tag{4.29}$$

(here we used the star notation for the target ODE system, and set $W_k:=(w_1,...,w_k)$).We claim that, by setting

$$\zeta:=\frac{w_k}{w^m}, \tag{4.30}$$

we can understand (4.28) as an identity of formal power series in the *independent* variables z,w,ζ.

Indeed, we first note that the substitution $w_k=w^m\zeta$ makes all expressions in (4.29) power series in z,w,ζ (divisible by ζ, in view of (4.21)). Further, we consider the singular expression $\frac{G^{(k)}(z,w,w_1,..,w_k)}{G^m(z,w)}$ in (4.28) as a ratio of two formal power series $P(z,w,w_1,...,w_k),Q(z,w,w_1,...,w_k)$, each of which is polynomial in $w_1,...,w_k$. The denominator Q can be factorized as $w^m\cdot\widetilde{Q}(z,w)$ with $\widetilde{Q}(0,0)\neq 0$ (as follows from (3.2)). Next, the "constant" term of the polynomial P obtained by setting $w_j=0$ for all j, can be inductively computed using the scheme

$$c_1=\frac{G_z}{F_z},\quad c_j=\frac{\partial_z(c_{j-1})}{F_z},\quad 2\le j\le k+1 \tag{4.31}$$

(as follows from (4.26)), and it follows then from (4.5) that the desired constant term $c_k(z,w)$ is divisible by w^m. All the other terms in P are (i) either divisible by w_k, hence the substitution $w_k = w^m\zeta$ makes them divisible by w^m, or (ii) divisible by some w_j, $j = 1, ..., k-1$, and hence the substitution $w_j = \Phi_j(z, w, w_1, ..., w_j)$ makes them divisible by w^m (in view of (4.21)). We conclude that P subject to restriction (4.29) is divisible by w^m (after the substitution $w_k = w^m\zeta$), and this proves the claim.

We have now to work out the well defined basic identity (4.28),(4.29). Let us expand

$$
\begin{aligned}
&F = z + S(z,w) + \sum_{j=0}^{k} f_j(w)z^j + f(z,w),\\
&G = T(w) + w^m R(z,w) + wg_0(w) + w^m \sum_{j=1}^{k} g_j(w)z^j + w^m g(z,w),\\
&S_z(0,0) = 0, \quad f(z,w) = O(z^{k+1}), \quad g(z,w) = O(z^{k+1}),
\end{aligned}
\tag{4.32}
$$

where f_j, g_j, f, g are formal power seires, f_j, g_j all vanish to order $k+1$, and $T(w), S(z,w), R(z,w)$ are certain *fixed polynomials* in their variables, exact form of which is of no interest to us (the desired representation of g is possible in view of (4.5)). Let us in what follows treat f_j, g_j and their derivatives as "additional parameters". For this purpose, we denote

$$
\alpha_{ij} := f_i^{(j)}(w), \quad \beta_{ij} := g_i^{(j)}(w), \quad \alpha = \{\alpha_{ij}\}, \quad \beta = \{\beta_{ij}\}, \quad 0 \le i \le k, \quad 0 \le j \le k+1.
$$

We then consider the last equation in (4.28) subject to (4.29) as an identity in z, w, ζ and collect within it all terms with ζ^0. Then:

(i) in the left hand side, we obtain the expression $c_{k+1}(z,w)$ from (4.31); it is easy to see that this expression can be written as

$$
\frac{1}{(F_z)^{k+1}}\big(\partial_z^{k+1}G.F_z - \partial_z^{k+1}F.G_z + \cdots\big),
$$

where dots stand for a polynomial in $F_z, F_{zz}, ..., F_{z^k}, G_z, g_{zz}, ..., G_{z^k}$; substituting (4.32), we obtain

$$
w^m\big(\partial_z^{k+1}g.(1+A) + \partial_z^{k+1}f.B + C\big),
$$

where A, B, C are *holomorphic* expressions in $j^k f, j^k g, z, w, \alpha, \beta$, and A, B vanish at the origin. (In fact, A, B, C have more specific form, but we do not need these further details);

(ii) for the right hand side, we argue as above and conclude that, for the singular argument $\frac{G^{(k)}(z,w,w_1,..,w_k)}{G^m(z,w)}$, evaluating $\zeta = 0$ and substituting (4.32) makes the numerator divisible by w^m. Taking further (4.21) into account, we conclude that the right hand side in the identity under consideration as well has the form

$$
w^m\widetilde{C},
$$

where $\widetilde{C}$ is an expression, *holomorphic* in $j^k f, j^k g, z, w, \alpha, \beta$.

We summarize that, gathering in the last identity in (4.28) terms with ζ^0 gives:

$$
\partial_z^{k+1}g.(1+A) + \partial_z^{k+1}f.B = \widehat{C}, \tag{4.33}
$$

where $A, B, \widehat{C}$ are holomorphic expressions as above, and A, B vanish at the origin.

It remain for us to obtain one more identity of the kind (4.33), solvable already in $f_{z^{k+1}}$. For doing so, we consider in the last identity in (4.28) (subject to restriction (4.29)) terms with ζ^k.

Claim. The result of collecting terms with ζ^k in (4.28) (subject to restriction (4.29)) can be written in the form

$$-\partial_z^{k+1}f.\big(\pm i+L_0\big)+\sum_{j=1}^{k+1}L_j\cdot\partial_z^{k+1-j}\partial_w^j f+\sum_{j=0}^{k+1}M_j\cdot\partial_z^{k+1-j}\partial_w^j g+N=\widetilde{N}, \tag{4.34}$$

where the expressions $L_j, M_j, N, \widetilde{N}$ are described identically to the expressions $A, B, \widehat{C}$ in (4.33) and, moreover, L_0 vanishes at the origin.

To prove the claim, we have to analyze the jet prolonged component $G^{(k+1)}$ with more details. Recall that $G^{(k+1)}$ is a rational in w_1 and polynomial in $w_2,...,w_{k+1}$ expression, coefficients of which are certain universal polynomials in $j^{k+1}(F,G)$. Its denominator is nonvanishing for $z=w=w_1=...=w_{k+1}$, as discussed above. Hence, we may expand

$$G^{(k+1)}=\sum_{l_1,\dots,l_{k+1}\geq 0}E_{l_1,\dots,l_{k+1}}(w_1)^{l_1}\cdots(w_{k+1})^{l_{k+1}}, \tag{4.35}$$

where $E_{l_1,\dots,l_{k+1}}$ are all certain universal polynomials in $j^{k+1}(F,G)$ and the ratio $\frac{1}{F_z}$ (the latter fact can be seen from (4.26), induction in k and the chain rule). Recall that the constant term $E_{0,\dots,0}$ can be computed via (4.31). For all the other terms, we have to distinguish $E_{l_1,\dots,l_{k+1}}$ depending on the *highest order derivatives* $\partial_z^p\partial_w^q F, \partial_z^p\partial_w^q G$, $p+q=k+1$. In this regard, we have

Lemma 4.5. *The only coefficients $E_{l_1,\dots,l_{k+1}}$ in* (4.35) *depending on the highest order derivatives $\partial_z^p\partial_w^q F, \partial_z^p\partial_w^q G$, $p+q=k+1$ are $E_{l,0,\dots,0}$, $l\geq 0$. Moreover, $E_{1,0,\dots,0}$ has the form identical to the left hand side of* (4.34)*, where the expressions L_j, M_j, N are certain universal polynomials in $j^k(F,G)$ and $\frac{1}{F_z}$, and, moreover, L_0 vanishes when $j^k(F,G)$ is evaluated at $z=w=0$.*

Proof. For $k=0$, the assertion follows from the formula (4.27) and the chain rule. For $k>0$, we apply the iterative formula (4.26), induction in k and the chain rule. Then the assertion of the lemma follows by a straightforward inspection. ☐

We immediately conclude that, when collecting terms with ζ^k in the left hand side of the last identity in (4.28), highest order derivatives may arise only from terms with $(w_1)^l$, $l\geq 1$. Next, we note that the term $E_{1,0,\dots,0}\cdot w_1$, subject to constraint (4.29), contributes

$$w^m(\pm i+o(1))\cdot E_{1,0,\dots,0}$$

(as follows from the Property $(*)$ of Φ_1). Substituting the expansions (4.32) for F, G, we obtain an expression of the kind (4.34) multiplied by w^m. Further, the constant term $E_{0,\dots,0}$ does not contribute to ζ^k (as it doesn't depend on the w_j's). All terms with $(w_1)^l$, $l\geq 2$ may contribute to ζ^k, however, in view of the factorization property (4.21) their contribution gives at least the factor $O(w^{2m})$ in front of the highest order derivatives. All other terms do *not* contribute to ζ^k with the highest order derivatives, as follows from Lemma 4.5. They, however, necessarily give the factor w^m, in view of (4.21),(4.30).

We finally conclude that the left hand side of the identity under discussion has the form identical to the left hand side in (4.34) multiplied by w^m.

To study the right hand side of the last identity in (4.28) subject to (4.29), we recall that (i) the denominator of the singular argument $\frac{G^{(k)}(z,w,w_1,..,w_k)}{G^m(z,w)}$ has the form $w^m\cdot\widetilde{G}(z,w)$, $\widetilde{G}(0,0)\neq 0$; (ii) the constant term in the numerator of the same expression is divisible by w^m, after substituting (4.32) (as discussed above); (iii) the substitutions (4.30),(4.29) together with the factorization (4.21) makes the rest of the numerator divisible by w^m; (iv) the factorization (4.21) applied to

Φ^* makes the right hand side under consideration in addition divisible by w^m (after substituting (4.32)).

In this way, we conclude that the right hand side of the identity under discussion has the form identical to the right hand side in (4.34) multiplied by w^m. Dividing the latter identity by w^m finally proves the claim.

□

We are now in the position to prove Theorem 1 in the k-nondegenerate case.

Proof of Theorem 1 under the k-nondegeneracy assumption. Let us consider the identities (4.33),(4.34) as a linear system in $\partial_z^{k+1} f, \partial_z^{k+1} g$. Solving it by the Cramer rule, we obtain the following system of PDEs with the additional parameters α, β:

$$\begin{aligned} \partial_z^{k+1} f &= U\Big(z, w, j^k(f,g), \{\partial_z^{k+1-j}\partial_w^j f\}_{j=1}^{k+1}, \{\partial_z^{k+1-j}\partial_w^j g\}_{j=1}^{k+1}, \alpha, \beta\Big), \\ \partial_z^{k+1} g &= V\Big(z, w, j^k(f,g), \{\partial_z^{k+1-j}\partial_w^j f\}_{j=1}^{k+1}, \{\partial_z^{k+1-j}\partial_w^j g\}_{j=1}^{k+1}, \alpha, \beta\Big), \end{aligned} \tag{4.36}$$

Solving this system by the Ovcynnikov theorem (see Section 3) as a Cauchy problem with the initial data

$$\partial_z^j f(0,w) = \partial_z^j g(0,w) = 0, \quad 0 \le j \le k$$

and the additional parameters α, β, we obtain:

$$f(z,w) = \varphi\big(z, w, \alpha, \beta\big), \quad g(z,w) = \psi\big(z, w, \alpha, \beta\big) \tag{4.37}$$

for two functions φ, ψ, holomorphic in all their arguments. We recall now that, by definition, α and β stand for formal power series *without constant term*, so that substituting back $f_i^{(j)}(w)$ for α_{ij} and $g_i^{(j)}(w)$ for β_{ij} is well defined.

Let us note finally that, combining Proposition 4.2 and the expansion (4.32), we may apply the assertion of Proposition 4.2 to the functions $f_0, ..., f_k, g_0, ..., g_k$. Substituting the arising sectorial functions $f_j^{\pm}, g_j^{\pm}$ into (4.37), we obtain sectorial holomorphic transformations $(F^{\pm}, G^{\pm})$. Then, arguing identically to the proof of Theorem 1 in the generic case (end of Section 3), we obtain the assertion of Theorem 1 in the k-nondegenerate case. □

4.4. **Pure order of a nonminimal hypersurface.** It might still happen that an m-nonminimal hypersurface (4.1) does *not* satisfy the k-nondegeneracy assumption. To deal with this case, we do (in appropriate coordinates) a blow up in the space of parameters of the Segre family. Related to this procedure is an important invariant of real hypersurface which we will call the *pure order*. From the point of view of our method, the pure order replaces, in a certain sense, the nonminimality order.

Definition 4.6. Let $M \subset \mathbb{C}^2$ be a real-analytic Levi-nonflat hypersurface given by (4.1). The *pure order* of M at 0 is the integer p such that

$$p + 1 = \min_{k,l \ge 1} \{l + \mathrm{ord}_0\, \Theta_{kl}(\bar w)\}. \tag{4.38}$$

In other words, $p+1$ is the the minimal possible $l+s$ such that for some $k>0$ the term with $z^k \bar z^l \bar w^s$ in the expansion of Θ does not vanish.

Note that:

(i) for a Levi-nonflat hypersurface p is well defined and nonnegative;

(ii) for a Levi-nondegenerate hypersurface we have $p = 0$;

(iii) for an m-nonminimal hypersurface we have $p \ge m$;

(iv) for an m-nonminimal hypersurface with $M \setminus X$ Levi-nondegenerate (the generic case from Section 3) we have $p = m$.

We start with showing that the integer p is a (formal) invariant of a real-analytic hypersurface.

Proposition 4.7. *The pure order of a Levi-nonflat hypersurface is invariant under (formal) invertible transformations of hypersurfaces* (4.1)*.*

Proof. We note that the pure type introduced above actually comes from an *invariant pair* as introduced in [ELZ09]; the invariance of those is proved in that paper. □

We now apply the notion of the pure type to prove the following factorization property for CR-maps.

Proposition 4.8. *Let $M, M^* \subset \mathbb{C}^2$ be two real-analytic nonminimal at the origin hypersurfaces, and $H = (F, G) : (M, 0) \mapsto (M^*, 0)$ a formal map. Then*

$$F_z(0,0) \neq 0, \quad G_w(0,0) = G_0'(0) \neq 0, \quad G(z,w) = O(w), \quad G_z(z,w) = O(w^{p+1}), \tag{4.39}$$

where p is the pure order of M, M^* at 0.

Proof. The proof of all the assertions except the last one goes identically to the proof of (3.2). For the property $G_z(z,w) = O(w^{p+1})$, we consider the basic identity

$$G(z,w) = \Theta^*(F(z,w), \bar F(\bar z, \bar w), \bar G(\bar z, \bar w))|_{w=\Theta(z,\bar z,\bar w)}. \tag{4.40}$$

Putting $\bar z = 0$, we get $w = \bar w$, and further differentiating in z gives:

$$G_z(z,\bar w) = \frac{\partial}{\partial z}\Big[\Theta^*(F(z,\bar w), \bar F(0,\bar w), \bar G(0,\bar w))\Big]. \tag{4.41}$$

We note now that every non-zero term in the expansion of Θ^* in $z/\bar z, \bar w$ has total degree at least $p+1$ in $\bar z, \bar w$ (by the definition of p). At the same time, since (F, G) preserves the origin, we have $g(0,\bar w) = O(\bar w)$, $F(0\bar w) = O(\bar w)$, so that the whole expression in the square brackets in (4.41) becomes divisible by $\bar w^{p+1}$. This property persists after differentiating in z, and this proves the proposition. □

Next, we prove

Proposition 4.9. *Let $M \subset \mathbb{C}^2$ be a real-analytic Levi-nonflat hypersurface, and p is its pure type at* 0*. Then there exist local holomorphic coordinates* (4.1) *for M at the origin with* (4.2)*, such that for certain $k \geq 1$ we have:*

$$\mathit{ordr}_0\, \Theta_{k1}(\bar w) = p. \tag{4.42}$$

Proof. Let us choose any coordinates (4.1) for M at 0 with (4.2). As was discussed above, change of coordinates (4.1) corresponds to chossing a curve $\gamma \subset M$ being transformed to the canonical curve (4.7). Let us choose $\gamma \subset M$ of the kind

$$z = \alpha u, \quad w = u + iq(u), \quad u \in \mathbb{R}$$

for an appropriate real-valued $q(u)$ and a generic $\alpha \in \mathbb{C}$. Then there exists a biholomorphic transformation of the form,

$$z \mapsto z - \alpha w, \quad w \mapsto g(z,w), \quad g(0,0) = 0, \tag{4.43}$$

mapping M into another hypersurface M^* of the form (4.1) and γ into the curve (4.7) (e.g., [CM74][LM07]). If we now fix in the expansion (4.1) of M the non-zero term $z^k \bar z^j \bar w^l$, $j + l = p+1$ with the minima $k \geq 1$, then it is easy to verify from the basic identity that the substitution

(4.43) creates, for a ganeric α, a non-zero term $z^k\bar z\bar w^p$ in the expansion (4.1) for M^*. In view of the invariance of the pure order this means the validity of (4.42) for M^*. Moreover, for a generic α in (4.43) the condition (4.2) persists as well, and this proves the proposition. □

4.5. **Proof of the main theorem.** Having Theorem 1 proved in the k-nondegenerate case (subsection 4.3) and having the relations (4.39),(4.42), we are now in the position to prove Theorem 1 in its full generality.

Proof of Theorem 1. According to the outcome of subsection 4.3, it remains to prove the theorem in the case when M is m-nonminimal at 0 but is not k-nondegenerate for any $k\geq 1$. Let us choose for M, M^* local holomorphic coordinates according to Proposition 4.9. Then we have the identity (4.42), for both the source and the target. Let us then consider the Segre family $\mathcal{S}=\{Q_p\}_{p=(\bar a,\bar b)}$ of M, considered as a 2-parameter holomorphic family in a,b. Then, let us perform the following blow up in the space of parameters:

$$a=\tilde a\tilde b,\quad b=\tilde b. \tag{4.44}$$

Let us denote the new parameterized family by $\widetilde{\mathcal{S}}$, and keep denoting for simplicity the new parameters by a,b. Then, considering an element of the family $\mathcal{S}$ as a graph $w=w(z)$ and expanding in z,a,b, we see that a terms $\lambda z^k a^j b^l$ gets transformed (after the blow up (4.44)) into $\lambda z^k a^j b^{j+l}$. We obtain from here the crucial corollary that *all terms in the expansion of* $w(z,a,b)$ *except the term very first term* $z^0a^0b^1$ *are divisible by* b^{p+1}. Furthermore, the non-zero (in view of (4.42)) term $\lambda z^k ab^p$, $k\geq 1$ gets transformed into $z^k ab^{p+1}$. We conclude that the transformed family $\widetilde{\mathcal{S}}$ has the form identical to (4.16) with the nondegeneracy property (4.17), with the only difference that m is replaced by $p+1$. Hence, arguing identically to subsection 4.2, we conclude that the family $\widetilde{\mathcal{S}}$ (and hence the family $\mathcal{S}$!) satisfy a system of ODEs, identical to (4.20) with the only difference that, again, m is replaced by $p+1$. The same statement applies for the target M^*, and we conclude that the given formal map (F,G) between $(M,0)$ and $(M^*,0)$ satisfies an identity similar to (4.28) with m replaced by $p+1$.

We finally recall that (F,G) satisfy the factorization (4.39), which is identical to (4.5) with, again, m replaced by $p+1$. This allows to repeat the proof in the k-nondegenerate case word-by-word (we recall that the crucial Proposition 4.2 is valid without any further assumptions and thus is applicable to the map (F,G)), which finally proves the theorem.

□

Proof of Theorem 3. The assertion of the theorem immediately follows from the crucial Corollary 3.4 and Proposition 4.2, and the representations (3.24),(4.37). □

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