Convergence of formal maps

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Analysis and geometry in several complex variables II
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Outline

Setting

- $M \subset \mathbb{C}^N_Z$, $M \subset \mathbb{C}^{N'}_{Z'}$ germs of real-analytic CR submanifolds (e.g. hypersurfaces), through 0
- $M = \{\varrho = 0\}$, $M' = \{\varrho' = 0\}$
- $H \in \mathbb{C}[\![Z]\!]^{N'}$ formal map with $H(0) = 0$
- $H \colon M \to M' :\Leftrightarrow \varrho'(H(Z), \overline{H(Z)}) = a(Z, \bar{Z})\varrho(Z, \bar{Z})$

If $H \in \mathbb{C}\{Z\}^{N'}$, then $H \colon M \to M'$ iff $H(M) \subset M'$.

Theorem (L.-Mir 2016)

If M is minimal at 0, M' is strictly pseudoconvex, and $H \in \mathbb{C}[\![Z]\!]^{N'}$ is a formal map with $H \colon M \to M'$, then H is convergent.

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About those assumptions...

Minimality

There exist examples of nonminimal M which allow divergent formal automorphisms (Kossovskiy-Shafikov). Minimality is, in our setting, equivalent to finite commutator type.

Strict pseudoconvexity

M' needs to satisfy some kind of curvature condition. Strict pseudoconvexity is a simple condition and the question has been long open.

Holomorphic nondegeneracy

A holomorphically degenerate manifold M' is one which can generically be split into a product $M' = \mathbb{C} \times \hat{M}'$. Any nontrivial interplay with the fiber directions can possibly lead to divergent maps.

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Why should I care?

There is no a priori reason why a formal map should converge. But:

Theorem (Chern and Moser 76)

If $M, M' \subset \mathbb{C}^N$ are strictly pseudoconvex, and $H\colon M \to M'$ is a formal map, then H converges.

In the equidimensional case, there is a theorem under optimal assumptions.

Theorem (Bauendi, Mir, and Rothschild 02)

Let $M, M' \subset \mathbb{C}^N$ be real-analytic generic CR submanifolds that are holomorphically nondegenerate and of finite type. Then any formal biholomorphism $H\colon M \to M'$ is convergent.

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Extending Chern-Moser

The following question can be traced back to work of Forstneric and Huang in the 80s:

Chern-Moser in positive codimension

Does any formal holomorphic map sending a real-analytic strongly pseudoconvex hypersurface $M \subset \mathbb{C}^N$ into another such hypersurface $M' \subset \mathbb{C}^{N'}$ necessarily converge?

It has been more recently asked again by e.g. Rothschild (2003). Our result gives a positive answer to this question.

Issues to overcome

The strategy

For *invertible* maps: Prolongation of

$$\varrho'(H, \bar{H}) = 0,$$

(i.e. application of CR vector fields) yields a “reflection identity”

$$H(Z) = R(Z, \bar{Z}, \overline{j_Z^k H}),$$

which can be used iteratively along “Segre maps”; then apply minimality criterion of Baouendi-Ebenfelt-Rothschild.

Issues to overcome

Problem 1: Singularities

The strategy typically fails to work, but one can instead often find a system

$$S(Z, \bar{Z}, \overline{j_Z^k H}, H(Z)) = 0, \quad \frac{\partial S}{\partial Z'}(Z.\bar{Z}, \overline{j_Z^k}H, H(Z)) \neq 0.$$

Such identities can be shown, after some additional nontrivial work, to be the right substitutres for the reflection identity.

Problem 2: Prolongation not good enough

Additional information about the location of the "characteristics" with respect to the "image characteristics" is needed.

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Earlier and related work

- L. (2001) : strongly pseudoconvex hypersurfaces + additional stringent conditions on the maps.
- Mir (2002) : Corollary in the case $N' = N + 1$ (codimensional one case).
- Meylan, Mir, Zaitsev (2003) : main result +additional assumption that M' is real-algebraic (instead of real-analytic).
- Ebenfelt, L. (2004) : Finite determination of embeddings (again rather stringent conditions)
- Berhanu, Ming (2014) : Smoothness of finitely smooth mappings between strictly pseudoconvex hypersurfaces on a dense open set.

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Recall the Set-up

Let M, M' be real-analytic generic CR submanifolds in $\mathbb{C}^N$ and $\mathbb{C}^{N'}$, through the origin, and $H\colon M \to M'$, $H(0) = 0$, be a formal holomorphic map.

- $\varrho' = (\varrho'_1, \dots, \varrho'_{d'})$ defining function for M'
- $\bar{L}_1, \dots, \bar{L}_n$ local basis of real-analytic CR vector fields for M
- $\mathbb{C}[\![M]\!]$ be the formal coordinate ring of M
- $\mathbb{C}(\!(M)\!)$ quotient field of $\mathbb{C}[\![M]\!]$

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Covariant derivatives

For $k \in \mathbb{N}$, we define

$$\mathcal{E}_k(H) := \mathrm{Span}_{\mathbb{C}(\!|M|\!)}\{E_j^\alpha : \alpha \in \mathbb{N}^n,\ |\alpha| \le k,\ 1 \le j \le d'\} \subset \mathbb{C}(\!|M|\!)^{N'},$$

where

$$E_j^\alpha := \left(\bar{L}^\alpha \varrho'_{j,Z'_1}(H,\overline{H})\big|_M, \ldots, \bar{L}^\alpha \varrho'_{j,Z'_N}(H,\overline{H})\big|_M\right) \in \mathbb{C}[\![M]\!]^{N'}.$$

- $\mu_k^H := \dim_{\mathbb{C}(\!|M|\!)} \mathcal{E}_k(H)$.
- $\mathcal{E}_k(H)$ is independent of all of the choices.
- $d = \mu_0^H < \mu_1^H < \cdots < \mu_{k_0}^H = \mu_{k_0+1}^H = \cdots$
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Generic degeneracy

Definition

Let H, M, M' be as above.

(a) We define the generic degeneracy of H as $\kappa^H := N' - \mu^H$.

(b) We say that H is a holomorphically nondegenerate formal holomorphic map if $\kappa^H = 0$.

- $0 \leq \kappa^H \leq n'$, where $n' = N - d$ is the CR dimension of M'.

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Further remarks

- M is holomorphically nondegenerate in the sense of Stanton if and only if the identity mapping is holomorphically nondegenerate in the sense of the above definition.
- The more stringent conditions alluded to above, appearing in earlier work, can be expressed by saying that

$$\dim \operatorname{Span}_{\mathbb{C}}\{E_j^{\alpha}\big|_0 : \alpha \in \mathbb{N}^n,\ 1 \le j \le d'\} = \mu^H.$$

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Definition

Let H, M, M' be as above. Let $V = (V_1, \ldots, V_{N'}) \in \mathbb{C}(\!|M|\!)^{N'}$. We say that V is a *formal meromorphic infinitesimal deformation of H* if V is tangent to M' along $H(M)$ i.e. if

$$\sum_{r=1}^{N'} (V_r\, \varrho'_{j,Z'_r}(H, \overline{H}))\big|_M = 0 \ \text{ in } \ \mathbb{C}(\!|M|\!), \quad j \in \{1, \ldots, d'\}.$$

- If $M = M'$ and $H = \mathrm{id}$ then a formal meromorphic infinitesimal deformation of H corresponds to a formal meromorphic vector field tangent to M.

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Degeneracy vs deformations

Proposition

Let H, M, M' be as above. Then the following conditions are equivalent :

(i) *H is a holomorphically degenerate map of generic degeneracy κ;*

(ii) *The space of formal meromorphic infinitesimal deformations of H is a vector space of dimension κ over $\mathbb{C}(\!|M|\!)$.*

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Degeneracy and deformations through the years

- For (constantly degenerate) smooth CR maps between smooth CR manifolds, this is due to Berhanu-Xiao (2015)
- For (constantly degenerate) formal maps between real-analytic CR manifolds, this appears in (L. 2001)
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The hammer...

Proposition

Let $M \subset \mathbb{C}^N$ be a real-analytic generic submanifold through the origin and $\Theta = (\Theta_1, \ldots, \Theta_{N'})$ be a convergent power series mapping with components in $\mathbb{C}\{Z, \bar{Z}, \lambda, Z'\}$ where $Z \in \mathbb{C}^N$, $Z' \in \mathbb{C}^{N'}$, $\lambda \in \mathbb{C}^r$, $N', N, r \geq 1$. Let $H \colon (\mathbb{C}^N, 0) \to \mathbb{C}^{N'}$, $G \colon (\mathbb{C}^N, 0) \to \mathbb{C}^r$ be formal holomorphic power series mappings, vanishing at 0, satisfying

$$\Theta(Z, \bar{Z}, \overline{G(Z)}, H(Z))|_M = 0, \quad \text{and}$$

$$\det \frac{\partial \Theta}{\partial Z'} \left(Z, \bar{Z}, \overline{G(Z)}, H(Z) \right) \Big|_M \neq 0.$$

If M is of finite type at 0, then H is convergent.

...and why it hits.

The convergence proposition is new and has a number of cool features.

- It's enough that the system of equations is valid when restricted to a certain finite type generic submanifold instead of being valid in the ambient complex space.
- The power series map G is essentially a free formal parameter and can be divergent; in any case, the formal map H turns out to be convergent.

This result does not reduce to the implicit function theorem in "nondegenerate" situations. Precursor results have been obtained earlier by Mir.

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Some remarks on the proof

The proof of the proposition consists of :

- A result on the partial convergence properties of formal power series mappings solutions of certain analytic equations containing formal parameters (propagation of convergence along certain subspaces, in the spirit of previous work from Mir, and Meylan, Mir and Zaitsev).
- Application of the partial convergence result in conjunction with the iterated Segre mapping technique introduced by Baouendi-Ebenfelt-Rothschild for finite type CR manifolds.

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Holomorphically nondegenerate maps

Theorem (Nail 1)

Assume that M is of finite type and that H is a holomorphically nondegenerate map. Then H is convergent.

Proof: Apply the "hammer" to the system

$$0 = \bar{L}^{\alpha_k} \varrho'_{r_k}(H(Z), \overline{H(Z)}) = \Theta_k(Z, \bar{Z}, \overline{G(Z)}, H(Z)), \quad k = 1, \ldots, N',$$

for a suitable choice of the α_k, where $G(Z) = D_Z^{k_0} H(Z)$.

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Maps with “nondegenerate deformations”

Theorem (Nail 2)

Assume that M is of finite type, M' is a hypersurface (for simplicity), and that H is a holomorphically degenerate map of generic degeneracy $\kappa > 0$. Assume that for every κ-tuple $(V^1, \dots, V^\kappa)$ of $\mathbb{C}(\!|M|\!)$-linearly independent formal meromorphic infinitesimal deformations of H, $V^j = (V^j_1, \dots, V^j_{N'}) \in (\mathbb{C}(\!|M|\!))^{N'}$, the Gram matrix

$$\begin{pmatrix} \mathcal{L}^{\varrho'}_{H(Z)}(V^1, \bar{V}^1) & \dots & \mathcal{L}^{\varrho'}_{H(Z)}(V^1, \bar{V}^\kappa) \\ \vdots & & \vdots \\ \mathcal{L}^{\varrho'}_{H(Z)}(V^\kappa, \bar{V}^1) & \dots & \mathcal{L}^{\varrho'}_{H(Z)}(V^\kappa, \bar{V}^\kappa) \end{pmatrix}$$

is nonsingular. Then H is convergent.

The proof is inspired by the work of Berhanu-Ming, combining our convergence proposition with the tool of the meromorphic infinitesimal deformations. One considers the system

$$0 = \bar{L}^{\alpha_k} \varrho'_{r_k}(H(Z), \overline{H(Z)}) = \Theta_k(Z, \bar{Z}, \overline{G(Z)}, H(Z)), \quad k = 1, \ldots, N' - \kappa,$$

complemented with the “missing equations”

$$0 = \sum_{\ell=1}^{N'} \bar{V}^j_k \varrho'_{\bar{Z}_k}(H(Z), \overline{H(Z)}) = \Theta_{N'-\kappa+j}, \quad j = 1, \ldots, \kappa$$

(where we clear the denominators).
The assumption on the Levi form allows us to apply the convergence proposition.

The main result...

follows because in the strictly pseudoconvex case, if H is degenerate, the matrix

$$\begin{pmatrix} \mathcal{L}^{\varrho'}_{H(Z)}(V^1, \bar{V}^1) & \cdots & \mathcal{L}^{\varrho'}_{H(Z)}(V^1, \bar{V}^\kappa) \\ \vdots & & \vdots \\ \mathcal{L}^{\varrho'}_{H(Z)}(V^\kappa, \bar{V}^1) & \cdots & \mathcal{L}^{\varrho'}_{H(Z)}(V^\kappa, \bar{V}^\kappa) \end{pmatrix}$$

is of full rank (because $\mathcal{L}$ is positive definite).

Thank you for your attention!