DEFORMATIONS OF CR MANIFOLDS, PARAMETRIZATIONS OF AUTOMORPHISMS, AND APPLICATIONS

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Abstract. We prove a parametrization theorem for maps of deformations of minimal, holomorphically nondegenerate real-analytic CR manifolds. This is used to deduce results on biholomorphic equivalence; we show that one can, for any germ of a minimal, holomorphically nondegenerate real-analytic CR manifold (M,p) construct a function which completely characterizes the CR manifolds biholomorphically equivalent to (M,p). As an application, we show that for any $p \in M$, the equivalence locus $E_p = \{q \in M\colon (M,q)$ biholomorphically equivalent to $(M,p)\}$ is a locally closed real-analytic submanifold of M, and give a criterion for the global CR automorphism group to be a (finite-dimensional) Lie group.

1. Biholomorphic equivalence and equivalence loci

Let M and M' be connected, real-analytic CR manifolds, $p \in M$, $q \in M'$. We will write $(M,p) \sim (M',q)$ and say that the germs (M,p) and (M',q) are biholomorphically equivalent if there exists a neighbourhood U of p in M and a real-analytic CR diffeomorphism $h\colon U \to M'$ $h(p) = q$. In this paper, we shall present a number of results which give answers to the question of how to decide whether $(M,p) \sim (M',q)$ for real-analytic CR manifolds which are holomorphically nondegenerate (in the sense of Stanton [16]) and minimal; and we shall discuss some applications of these results.

In order to illustrate our later results, let us start with what is essentially an (important) application. The *equivalence locus* E_p of a point $p \in M$ is defined by

$$E_p = \{q \in M\colon (M,p) \sim (M,q)\}. \tag{1}$$

One can ask a number of questions about E_p; our main result for the local structure of this set is the following:

Theorem 1. *If M is a connected real-analytic CR manifold which is minimal and holomorphically nondegenerate, then for every $p \in M$, E_p is a locally closed real-analytic submanifold of M.*

Let us recall that M is *minimal* if for every connected CR submanifold $N \subset M$ of the same CR dimension as M we necessarily have $N = M$. The second condition which we assume in Theorem 1, *holomorphic nondegeneracy*, can be phrased in a number of different ways. It is equivalent to the space of germs of infinitesimal CR diffeomorphisms $\mathfrak{hol}(M,p)$ to be totally real for some (or equivalently all) $p \in M$; it is also equivalent to the fact that for no $p \in M$, $(M,p) \sim (\hat{M} \times \mathbb{C}, 0)$ for some real analytic CR manifold $\hat{M}$; finally, for a minimal real-analytic CR manifold M, it is equivalent to $\dim \operatorname{Aut}(M,p) < \infty$ for (one or all) $p \in M$ by [9].

Theorem 1 answers a question raised at the "Emerging applications of complexity for CR mappings"[1] workshop at the American Institute of Mathematics in 2010 in the (important) setting of real-analytic, minimal, holomorphically nondegenerate CR manifolds. It gives interesting insights into the structure of CR manifolds: in a sense, it exhibits homogeneous CR manifolds as the building blocks of more general CR manifolds. The notion of homogeneity employed here is that a CR manifold M is homogeneous if for any

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[1]see `http://www.aimath.org/pastworkshops/crmappings.html`

$p, q \in M$ there exists a neighbourhood $U(p)$ and a CR-diffeomorphism $h\colon U(p) \to M$ with $h(p) = q$. We will discuss one application of this fact later, when we discuss the group of global CR automorphisms.

More generally, we show that given a real-analytic deformation of a germ of a real-analytic CR manifold (M, p), the locus of deformation parameters giving rise to a biholomorphically equivalent germ forms a semianalytic set in the space of deformation parameters. A real-analytic deformation $(M_\varepsilon, p)_{\varepsilon \in X}$ of such a germ (M_{ε_0}, p) by real-analytic parameters $\varepsilon \in X$ (which we assume to be a real-analytic manifold for simplicity, an assumption which the reader will see can be relaxed considerably), can be realized in the following way: First, (M, p) can be thought of as a generic, real-analytic submanifold of $(\mathbb{C}^N, p)$, where $N = n + d$, n the CR dimension of M, and $2n + d$ the real dimension of M. The deformation is then given by a germ of a real-analytic CR submanifold of $\mathbb{C}^N \times X$ at $\{p\} \times X$ having the same CR dimension as M_{ε_0} with $(\pi_1 \pi_2^{-1}(\varepsilon_0), p) = (M_{\varepsilon_0}, p)$, where π_j is the projection onto the first resp. second component; we write $\pi_1 \pi_2^{-1}(\varepsilon) = M_\varepsilon$.

Theorem 2. *Let (M, p) be a germ of a real-analytic CR manifold which is minimal and holomorphically nondegenerate, and assume that $(M_\varepsilon, p)_{\varepsilon \in X}$ is a real-analytic deformation of (M, p) as above. Then the space*

$$E_M = \{\varepsilon \in X \colon (M, p) \sim (M_\varepsilon, p)\}$$

is a semianalytic subset of X.

The stronger conclusion that E_M is a submanifold fails in the setting of a general deformation, since X, unlike E_p above, need not possess any homogeneity properties. For an example, we consider the deformation $\operatorname{Im} w = |z|^2 + f(\varepsilon)|z|^8$ of the Heisenberg hypersurface, where f is any germ of a real-analytic function at ε_0. Then $(E_M, \varepsilon_0) = \{\varepsilon \colon f(\varepsilon) = 0\}$, since the Chern-Moser normal form [7] implies that a hypersurface $\operatorname{Im} w = |z|^2 + c|z|^8$, where $c \in \mathbb{R} \setminus \{0\}$, is not biholomorphically equivalent to $\operatorname{Im} w = |z|^2$. We will discuss a number of examples later in section 7. Theorem 1 is an immediate consequence of Theorem 2 since in Theorem 1, E_p is also homogeneous. Indeed, any semianalytic subset is a real-analytic submanifold in a neighborhood of some of its points: since, by definition, E_p is acted upon transitively by local holomorphic diffeomorphisms, it is a closed real-analytic submanifold around any of its point.

A natural question which occurs at this point is whether the nice structure of the equivalence loci described here is a real-analytic phenomenon. We shall show in section 7 that there are counterexamples in class C^k, $k < \infty$; we do not know at the present time of any counterexample of class C^∞.

On the other hand, it would be interesting to study the set E_M (and equivalence locus E_p) in the presence of additional structure, for instance if M and the deformation M_ε are real-algebraic. We conjecture that, in this situation, E_M should be a semi-algebraic subset rather than just semi-analytic: the methods used in the paper, however, are of an intrinsically analytic nature and do not allow at the moment to draw this stronger conclusion.

The proof of Theorem 2 is based on a parametrization theorem for mappings of deformations of real-analytic, minimal holomorphically nondegenerate CR manifolds. In order to state this theorem, it is helpful to have another, more extrinsic, point of view for deformations of CR manifolds. Recall that every germ of a real-analytic CR manifold (M, p) can be identified with a germ of a real-analytic generic submanifold of some $\mathbb{C}^N$, where $N = n + d$, $\dim_{\mathbb{R}} M = 2n + d$, and $\dim_{CR} M = n$. In this setting, a deformation is just a germ ρ of a real-analytic map $(\mathbb{C}^N \times X, \{p\} \times X) \to \mathbb{R}^d$ with

$$\rho(p, \varepsilon) = 0, \quad \rho_{Z_1} \wedge \cdots \wedge \rho_{Z_N}(p, \varepsilon) \neq 0, \quad \varepsilon \in X. \tag{2}$$

With this notation, (M_ε, p) is given by the defining function $Z \mapsto \rho(Z, \varepsilon)$. We will write $G_p^k(\mathbb{C}^N)$ for the space of k-jets of germs of biholomorphisms of $(\mathbb{C}^N, p)$. For any two germs of real-analytic CR manifolds (M, p), (M', p'), we write $\operatorname{Bihol}((M, p), (M', p'))$ for the space of germs of real-analytic CR diffeomorphisms between (M, p) and (M', p'); if (M, p) and (M', p') are generic, real-analytic submanifolds of $\mathbb{C}^N$, we have the natural inclusion $\operatorname{Bihol}((M, p), (M', p')) \subset \operatorname{Bihol}((\mathbb{C}^N, p), (\mathbb{C}^N, p'))$ into the space of germs of biholomorphisms of $\mathbb{C}^N$ at p which map p to p'.

Theorem 3. *Let (M, p) be a germ of a generic real-analytic submanifold of $\mathbb{C}^N$, which is minimal and holomorphically nondegenerate, and assume that and $(M_\varepsilon, p)_{\varepsilon \in X}$ is a real-analytic deformation of $(M, p) = (M_{\varepsilon_0}, p)$. Then there exists an integer k, a finite set L and for each $J \in L$, a germ of a real-analytic function $e_J \colon G_p^k(\mathbb{C}^N) \times X \to \mathbb{R}$, which is a real polynomial in its first variable, at $G_p^k(M) \times \{\varepsilon_0\}$ and a germ of a* *real-analytic map* $\Psi_J\colon \mathbb{C}^N \times G_p^k(\mathbb{C}^N) \times X$ *at* $\{p\} \times \{e_J \neq 0\}$, *holomorphic in its first variable, with the following properties:*

i) *For any* $H \in \operatorname{Bihol}((M,p),(M_\varepsilon,p))$ *there exists* $J \in L$ *such that* $e_J(j_p^k H, \varepsilon) \neq 0$*;*

ii) *If* $H \in \operatorname{Bihol}((M,p),(M_\varepsilon,p))$ *and* $e_J(j_p^k H,\varepsilon) \neq 0$, *then* $\Psi_J(Z, j_p^k H, \varepsilon) = H(Z)$ *as germs at* p*;*

iii) Ψ_J *can be written as*

$$\Psi_J(Z,\Lambda,\varepsilon) = \sum_{\alpha\in\mathbb{N}^n} \frac{p_\alpha(\Lambda,\varepsilon)}{e_J(\Lambda,\varepsilon)^{d_\alpha}} (Z-p)^\alpha,$$

for some $p_\alpha(\Lambda,\varepsilon)$ *which is a real polynomial in* Λ *and real-analytic in* ε, *and some integers* d_α.

As a sidenote on terminology, if we speak about a *real* polynomial in a complex variable s, we mean that it is a polynomial of the underlying real coordinates of a complex variable, i.e. an element of $\mathbb{C}[s,\bar{s}]$; a *complex* polynomial in s is an element of $\mathbb{C}[s]$ as usual.

The result from which Theorem 3 follows is more general: in it, we use the defining function of the deformed manifold as a parameter. We recall that in general, the space of k-jets of holomorphic maps $(\mathbb{C}_x^p,0) \to (\mathbb{C}^q,0)$, which we denote by $\operatorname{Hol}((\mathbb{C}_x^p,0),(\mathbb{C}^q,0))$, is defined by

$$J^k((\mathbb{C}_x^p,0),(\mathbb{C}^q,0)) = {}^{\mathfrak{m}\mathbb{C}\{x\}^q}\!\big/_{\mathfrak{m}^{k+1}},$$

where $\mathfrak{m}$ is the maximal ideal in $\mathbb{C}\{x\}$. Also recall that

$$J^k((\mathbb{C}_x^p,0),(\mathbb{C}^q,0)) = {}^{\hat{\mathfrak{m}}\mathbb{C}[[x]]^q}\!\big/_{\hat{\mathfrak{m}}^{k+1}},$$

where $\hat{\mathfrak{m}}$ is the maximal ideal in $\mathbb{C}[[x]]$, and that the canonical projection, which we denote by

$$j_0^k\colon \hat{\mathfrak{m}}\mathbb{C}[[x]]^q \to J^k((\mathbb{C}_x^p,0),(\mathbb{C}^q,0)),$$

restricts to the canonical projection of $\mathfrak{m}\mathbb{C}\{x\}^q$ onto $J^k((\mathbb{C}_x^p,0),(\mathbb{C}^q,0))$; we shall consequently use the same notation for both. The space $\operatorname{Hol}((\mathbb{C}_x^p,0),(\mathbb{C}^q,0))$ gets endowed with the natural inductive limit topology of uniform convergence on a compact neighbourhood. Typically, we shall denote a variable in jet space by $\Lambda \in J^k((\mathbb{C}_x^p,0),(\mathbb{C}^q,0))$, and the reader can identify Λ with the collection

$$\Lambda = (\Lambda_\alpha\colon \alpha\in\mathbb{N}^p, 1 \le |\alpha| \le k),$$

with each Λ_α a variable in $\mathbb{C}^q$, such that

$$j_0^k H = \left(\frac{\partial^{|\alpha|} H}{\partial x^\alpha}(0)\colon 1 \le |\alpha| \le k \right).$$

As a last point, the jet group mentioned above is realized as

$$G_0^k(\mathbb{C}^N) = j_0^k(\operatorname{Bihol}((\mathbb{C}^N,0),(\mathbb{C}^N,0))) \subset J^k((\mathbb{C}^N,0),(\mathbb{C}^N,0)).$$

When we say that we use the defining function of the deformed manifold as a parameter, we will use its *complex form*: If $(M,0) \subset (\mathbb{C}^N,0)$ is a germ of a generic real-analytic manifold of real codimension d, then there exist coordinates $(z,w) \in \mathbb{C}^n \times \mathbb{C}^d$ and a germ of a holomorphic map $Q(z,\chi,\tau)\colon \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^d \to \mathbb{C}^d$ such that M is defined by the equation $w = Q(z,\bar{z},\bar{w})$ for (z,w) close to 0.

Theorem 4. *Let* $(M',0)$ *be a germ of a generic minimal real-analytic submanifold of* $\mathbb{C}^N$ *which is holomorphically nondegenerate and minimal at* 0*, of real codimension* d*, and write* $n = N-d$*. Then there exist an integer* ℓ*, a finite set* L*, for any* $J \in L$ *a real polynomials* e_J *on* $J^\ell((\mathbb{C}^N,0),(\mathbb{C}^N,0)) \times J^\ell((\mathbb{C}^{2n+d},0),(\mathbb{C}^d,0))$*, and a real-analytic map* Ψ_J *defined on the open subset*

$$U_J = \{(\Lambda, Q(z,\chi,\tau))\colon e_J(\Lambda, j_0^\ell Q) \neq 0\} \subset J^\ell((\mathbb{C}^N,0),(\mathbb{C}^N,0)) \times \mathbb{C}\{z,\chi,\tau\}^d,$$

where $(z,\chi,\tau) \in \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^d$*, with values in* $\operatorname{Bihol}((\mathbb{C}^N,0),(\mathbb{C}^N,0))$*, which have the following properties:*

i) *if* $w = Q(z,\bar{z},\bar{w})$ *defines a germ of a real-analytic submanifold* M_Q *at* 0 *and* $H \in \operatorname{Bihol}((M_Q,0),(M',0))$*, then there exists a* $J \in L$ *such that* $e_J(j_0^\ell H, j_0^\ell Q) \neq 0$*;*

ii) *if* $H \in \operatorname{Bihol}((M_Q,0),(M',0))$ *and* $e_J(j_0^\ell H, j_0^\ell Q) \neq 0$*, then*

$$H = \Psi_J(j_0^\ell H, Q);$$

iii) *each Ψ_J can be written in the following form:*

$$\Psi_J(\Lambda, Q)(Z) = \sum_\alpha \frac{p_\alpha(\Lambda, j_0^{c_\alpha} Q)}{e_J(\Lambda, j_0^\ell Q)^{2d_\alpha}} Z^\alpha,$$

where p_α are real polynomials, and c_α, d_α are integers.

Remark 1. In the statement of Theorem 3, we use the following notion of real analyticity: We say that a real-analytic map A defined on an open subset U of a complex vector space E is defined as a holomorphic map $\tilde{A}$ on $U \times \bar{U} \subset E \times \bar{E}$ which agrees with A along the diagonal, i.e. $A(e) = \tilde{A}(e, \bar{e})$; a holomorphic map is a map which is Gateaux-holomorphic and continuous. In fact, our maps fulfill an even stronger version of holomorphicity, i.e. the coefficients of the Ψ_J fulfill "convergence estimates" of the form discussed in e.g. [12].

The last theorem has a rather interesting consequence for the question of deciding whether two minimal, holomorphically nondegenerate CR manifolds are biholomorphically equivalent. This question, which in general goes under the *biholomorphic equivalence problem*, goes back to Poincaré [15]; Theorem 4 allows us to find all CR manifolds which are biholomorphically equivalent to a fixed real-analytic CR manifold, which is minimal and holomorphically nondegenerate; to be more precise, we have the following theorem.

Theorem 5. *Let $(M', 0)$ be a germ of a real-analytic generic submanifold of $(\mathbb{C}^N, 0)$, of CR dimension n and real codimension d. Then there exists an integer ℓ, a finite set L, and for each $J \in L$ a real polynomials e_J defined on $J^\ell((\mathbb{C}^N, 0), (\mathbb{C}^N, 0)) \times J^\ell((\mathbb{C}^{2n+d}, 0), (\mathbb{C}^d, 0))$ and a real-analytic map ψ_J defined on the open subset*

$$U_J = \{(\Lambda, Q) \colon e_J(\Lambda, j_0^\ell Q) \neq 0\} \subset J^\ell((\mathbb{C}^N, 0), (\mathbb{C}^N, 0)) \times \mathbb{C}\{z, \chi, \tau\}^d,$$

where z, χ are variables in $\mathbb{C}^n$ and τ a variable in $\mathbb{C}^d$ with values in $\mathbb{C}\{z, \chi, \tau\}$, which satisfy the following properties:

i) *If $w = Q(z, \bar{z}, \bar{w})$ defines the germ of a real-analytic generic submanifold $(M_Q, 0)$, then M_Q is biholomorphically equivalent to $(M', 0)$ if and only if there exists $\Lambda_0 \in G_0^\ell(\mathbb{C}^N)$ and a $J \in L$ such that $e_J(\Lambda_0, j_0^\ell Q) \neq 0$ and $\psi_J(\Lambda_0, Q) = 0$.*

ii) *Writing $Y = (z, \chi, \tau)$, each ψ_J is of the form*

$$\psi_J(\Lambda, Q)(Y) = \sum_\alpha \frac{p_{J,\alpha}(\Lambda, j_0^{c_\alpha} Q)}{e_J(\Lambda, j_0^\ell Q)^{d_\alpha}} Y^\alpha,$$

where c_α, d_α are integers.

The e_J and $p_{J,\alpha}$ are real polyonmials which can each be computed in finitely many steps from finite order data of $(M', 0)$.

Let us explain how Theorem 4 implies Theorem 5. We choose a germ of a real analytic defining function $\rho(Z, \bar{Z})$ for $(M', 0)$, and fix J. We write the function

$$\tilde{\rho}(Z, \zeta, \Lambda, Q) = \rho\left(\Psi_J(\Lambda, Q)(Z), \overline{\Psi_J(\Lambda, Q)}(\zeta)\right)$$

and note that there exists a biholomorphism H from $(M_Q, 0)$ to $(M', 0)$ with $e_J(j_0^\ell H, Q) \neq 0$ if and only if $\tilde{\rho}(Z, \zeta, j_0^\ell H, Q) = 0$ on $(M_Q, 0)$. We therefore set $\psi_j(\Lambda, Q)(z, \chi, \tau) = \tilde{\rho}(z, Q(z, \chi, \tau), \chi, \tau, \Lambda, Q)$ to obtain a map ψ_j with the required properties. A short computation involving iii) of Theorem 4 shows that the ψ_j is of the form required in ii).

Theorem 4 solves the biholomorphic equivalence problem in the following sense: In order to decide whether $\mathrm{Bihol}((M, p), (M', p'))$ is empty or not, we first choose normal coordinates for M' and assume that $p' = 0$ and then normal coordinates (z, w) for (M, p) such that $p = 0$ and M is given by $w = Q(z, \chi, \tau)$ near 0. With the polynomials e_J and $p_{J,\alpha}$ from Theorem 5, we thus just need to decide whether the real-algebraic sets

$$W_J = \left\{\Lambda \in J^\ell((\mathbb{C}^N, 0), (\mathbb{C}^N, 0)) \colon p_{J,\alpha}(\Lambda, j_0^{c_\alpha} Q) = 0 \text{ for all } \alpha\right\},$$
$$V_J = \left\{\Lambda \in J^\ell((\mathbb{C}^N, 0), (\mathbb{C}^N, 0)) \colon e_J(\Lambda, j_0^\ell Q) = 0\right\},$$

satisfy $W_J \subset V_J$ for all J or not; in the first case, there does not exist a map, in the other case, for any $\Lambda \in W_J \setminus V_J$, $\psi_J(\Lambda, Q) \in \mathrm{Bihol}((M_Q, 0), (M', 0))$.

We will now return to the question of the equivalence locus. A particular point is that one would–of course–like a way to actually compute E_p. Our next theorem shows that this is, in principle, a linear problem (if we want to compute the connected component of E_p containing p, at least). Recall that $\mathfrak{hol}(M,p)$ denotes the space of germs of infinitesimal CR automorphisms, which if (M,p) is realized as a generic real-analytic submanifold of $\mathbb{C}^N$ is the space

$$\mathfrak{hol}(M,p) = \left\{ X = \sum_j a_j(Z) \frac{\partial}{\partial Z_j} : a_j \in \mathcal{O}_p, \quad \operatorname{Re} X \text{ tangent to } M \right\}$$

of holomorphic vector fields whose real part is tangent to M near p (we note that in what follows, we shall abuse notation slightly and write $\mathfrak{hol}(M,p)$ for this space and the "intrinsic" space interchangeably; it will be clear from the context which point of view to take).

Theorem 6. *Assume that M is a real-analytic CR manifold which is minimal and holomorphically nondegenerate. For any $p \in M$, we have*

$$T_q E_p = \mathfrak{hol}(M,q)(q)$$

for all $q \in E_p$. In particular, $p \mapsto \dim_{\mathbb{R}} E_p$ is a lower semicontinuous function. More generally, there exists a neighbourhood $U \subset E_p$ of p in E_p, a neighbourhood V of p in M, and a map $\psi(w,q)$ defined in $V \times U$, such that $w \mapsto \psi(w,q)$ is a real-analytic CR diffeomorphism on V and $\psi(p,q) = q$.

Theorem 6 can be used to produce a general jet parametrization result which allows points to move in the equivalence loci; this result is stated and proved in section 8 as Theorem 15.

Our last application concerns another question, namely the global automorphism group of a real-analytic CR manifold M. This is the subgroup $\operatorname{Aut}^\omega_{CR}(M) \subset \operatorname{Diff}^\omega(M)$ of real-analytic CR diffeomorphisms $h\colon M \to M$, where we consider $\operatorname{Diff}^\omega(M)$ with the real-analytic compact-open topology. This group has been studied quite extensively, see e.g. [3] and [13]. With the tools developed in this paper, we can prove

Theorem 7. *Let M be a connected, real-analytic CR manifold which is minimal and holomorphically nondegenerate. Assume that there exists a compact subset $K \subset M$ with the property that for every $p \in M$, the connected component V_p of E_p containing p intersects K nontrivially. Then $\operatorname{Aut}^\omega_{CR}(M)$ is a finite dimensional Lie group in the real-analytic compact-open topology. Furthermore, there exists a $k \in \mathbb{N}$ such that the C^k compact-open topology on $\operatorname{Aut}^\omega_{CR}(M)$ and the real-analytic compact-open topology on it agree.*

In particular, the automorphism group of every compact, real-analytic CR manifold which is holomorphically nondegenerate and minimal is a finite-dimensional Lie group, a fact which in the case of manifolds embedded in Stein spaces had been proved by [13].

The plan of the paper is as follows: In section 2 we discuss the mapping identities which we need to use for our parametrization. In section 3, parametrizations "along the Segre varieties" are deduced in the spirit of Theorem 4, i.e. leaving the defining function of the source manifold as a parameter. The proof of Theorem 4 is given in section 4. Theorem 2 needs some preparations which are given in section 5; it is based on some results from real-algebraic geometry and finite order equivalences. These also allow us to prove a weaker statement of Theorem 6 in the setting of a general deformation, which actually implies Theorem 6 in the homogeneous setting. In section 8 we shall discuss how to deduce Theorem 7 from the earlier results, using well-known arguments from the literature.

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2. Deformation-stable mapping identities

Our goal in this section is to derive the mapping identities which we will use in order to construct a parametrization of automorphisms of deformations; none of the techniques or results in this section are new, but we need to prepare the equations in a particular way suitable for our treatment. We consider germs $(M,0)$ and $(M',0)$ of generic, real-analytic submanifolds of $\mathbb{C}^N$, of the same real codimension, which are given in normal coordinates by $w = Q(z,\bar z,\bar w)$ and $w = Q'(z,\bar z,\bar w)$, respectively. Assume that $H = (f,g)$ is a germ of a biholomorphism near $0 \in \mathbb{C}^N$. Then we have the basic identity

$$g(z, Q(z,\chi,\tau)) = Q'(f(z, Q(z,\chi,\tau)), \bar H(\chi,\tau)), \tag{3}$$

which is valid in a neighbourhood of $0 \in \mathbb{C}^{2N}$; as usual, $\bar{H}$ denotes the mapping obtained from H by taking the complex conjugates of the power series coefficients. Differentiating with respect to z, one obtains

$$g_w(z, Q)Q_z = Q'_z(f(z, Q), \bar{H}(\chi, \tau))(f_z + f_w Q_z), \tag{4}$$

where we have suppressed the independent variables in Q to make the equation more compact, and use matrix notation, i.e.

$$g_w = \begin{pmatrix} g_{1,w_1} & \dots & g_{1,w_d} \\ \vdots & & \vdots \\ g_{d,w_1} & \dots & g_{d,w_d} \end{pmatrix}, \quad Q_z = \begin{pmatrix} Q^1_{z_1} & \dots & Q^1_{z_n} \\ \vdots & & \vdots \\ Q^d_{z_1} & \dots & Q^n_{z^n} \end{pmatrix}, \quad Q'_z = \begin{pmatrix} {Q'}^1_{z_1} & \dots & {Q'}^1_{z_n} \\ \vdots & & \vdots \\ {Q'}^d_{z_1} & \dots & {Q'}^n_{z^n} \end{pmatrix},$$

$$f_z = \begin{pmatrix} f_{1,z_1} & \dots & f_{1,z_n} \\ \vdots & & \vdots \\ f_{n,z_1} & \dots & f_{n,z_n} \end{pmatrix}, \quad f_w = \begin{pmatrix} f_{1,w_1} & \dots & f_{1,w_d} \\ \vdots & & \vdots \\ f_{n,w_1} & \dots & f_{n,w_d} \end{pmatrix}.$$

Since the $n \times n$-matrix $f_z + f_w Q_z$ when evaluated at 0 is just $f_z(0)$, which is invertible, the inverse $(f_z + f_w Q_z)^{-1}$ is again defined in a neighbourhood of 0, and we can apply it to (4) to obtain

$$Q'_z(f(z, Q), \bar{H}(\chi, \tau)) = g_w(z, Q)Q_z(f_z + f_w Q_z)^{-1}.$$

By Cramer's rule, the entries of the matrix on the right hand side are rational functions in f_z, f_w, Q_z, g_w, with the denominator being given by $\det(f_z + f_w Q_z)$; i.e. for every j, there exists a polynomial P_{e_j} such that

$$Q'_{z_j}(f(z, Q(z, \chi, \tau), \bar{H}(\chi, \tau)) = \frac{P_{e_j}(f_z(z, Q), f_w(z, Q), g_w(z, Q), Q_z(z, \chi, \tau))}{\det(f_z(z, Q) + f_w(z, Q)Q_z)}.$$

Continuing this process, we obtain for every α a polynomial P_α with

$$Q'_{z^\alpha}\left(f(z, Q(z, \chi, \tau), \bar{H}(\chi, \tau)\right) = \frac{P_\alpha\left(j^{|\alpha|}_{(z,Q(z,\chi,\tau))} H, Q_{z^\beta}(z, \chi, \tau) \colon |\beta| \le |\alpha|\right)}{(\det(f_z(z, Q) + f_w(z, Q)Q_z))^{2|\alpha|-1}}. \tag{5}$$

It is convenient to express these formulas in terms of the map

$$\pi^k_{\mathcal{M}} \colon \mathcal{M} \to H^k_d(\mathbb{C}^N), \quad \pi_{\mathcal{M}}(Z, \zeta) = j^k_Z S_{\bar{\zeta}},$$

and the analogous map $\pi_{\mathcal{M}'}$; here $\mathcal{M}$ and $\mathcal{M}'$ denote the complexifications of M and M', respectively, $H^k_d(\mathbb{C}^N)$ denotes the bundle of k-jets of germs of complex submanifolds of $\mathbb{C}^N$ of codimension d, and $S_{\bar{\zeta}} = \{Z \colon (Z, \zeta) \in \mathcal{M}\}$ is the Segre variety associated to ζ. In terms of our normal coordinates $Z = (z, w)$, we introduce conjugate variables $\zeta = (\chi, \tau)$, so that $\mathcal{M}$ is given by $w = Q(z, \zeta)$, and $\mathcal{M}'$ by $w' = Q'(z', \zeta')$. We can thus use coordinates (z, χ, τ) for $\mathcal{M}$ (and also $\mathcal{M}'$). In terms of these coordinates,

$$\pi^k_{\mathcal{M}'}(z', \zeta') = (Q'_{z'^\alpha}(z', \zeta') \colon |\alpha| \le k). \tag{6}$$

We can now formulate

Lemma 1. *For every $k \in \mathbb{N}$ there exists a polynomial P_k such that for any germs of real-analytic generic submanifolds $(M, 0)$, $(M', 0)$ given in normal coordinates $Z = (z, w) \in \mathbb{C}^n \times \mathbb{C}^d = \mathbb{C}^N$ by $w = Q(z, \chi, \tau)$ and $w' = Q'(z', \chi', \tau')$, respectively, if $H = (f, g) \colon (M, 0) \to (M', 0)$ is a germ of a biholomorphism, then*

$$\pi^k_{\mathcal{M}'}(H(Z), \bar{H}(\zeta)) = \frac{P_k\left(j^k_Z H, (Q_{z^\alpha}(z, \zeta))_{|\alpha| \le k}\right)}{(\det(f_z(z, Q) + f_w(z, Q)Q_z))^{2k-1}}. \tag{7}$$

3. Inverting the reflection map

In this section, we will collect some definitions and facts from [9], reformulating some of them following [11] and making them suitable for our purpose here. First of all, we will use the following notion of *type* of a power series in variables (x, t) (where we think about x as "tangential" and t as "transversal" later):

$$\operatorname{tp} \varphi(x, t) = \operatorname{tp} \sum_{\alpha, \beta} \varphi_{\alpha,\beta} x^\alpha t^\beta = \min\left\{(|\alpha|, |\beta|) \in \mathbb{N}^2 \colon \varphi_{\alpha,\beta} \neq 0\right\}. \tag{8}$$

The minimum here is taken with respect to the lexicographic ordering on $\mathbb{N}^2$ defined by

$$(m,n) \le (k,l) \text{ if and only if } \begin{cases} n < l \text{ or} \\ n = l \text{ and } m \le k. \end{cases}$$

We also define the *determinantal type* $\operatorname{dtp} \Phi(x,t)$ of a matrix-valued power series map Φ as the minimum of the type of the determinants of its minors of maximal size, and the determinantal type of a map $\Psi(x,t)$ as the determinantal type of its Jacobian. We allow here one of the sizes of the matrix to be infinite (with the understanding that the corresponding minimum is actually already realized by a submatrix of finite size).

In order to define the Segre maps associated to a generic real-analytic submanifold given in normal coordinates $Z = (z,w)$ by $w = Q(z,\chi,\tau)$, we use the following notation for coordinates: $(x^{[1;k]};t) \in \mathbb{C}^{nk} \times \mathbb{C}^d$, where $x^{[j;k]} = (x^j, \dots, x^k)$. We then inductively define $S^j \colon \mathbb{C}^{nj} \times \mathbb{C}^d \to \mathbb{C}^N$:

$$S^0(t) = (0,t); \quad S^1(x^1;t) = (x^1;t), \quad S^{j+1}(x^{[1;j+1]};t) = \left(x^1, Q\left(x^1, \bar{S}^j(x^{[2;j+1]};t)\right)\right).$$

If we evaluate at $t = 0$, we will denote the corresponding maps by $S_0^j(x^{[1;j]}) = S^j(x^{[1;j]};0)$. An important property of the Segre maps thus defined is that the map

$$\mathcal{S}^q(x^{[1;q]};t) = \left(S^{q-1}\left(x^{[2,q]};t\right), \bar{S}^q\left(x^{[1;q]};t\right)\right)$$

is valued in $\mathcal{M}$. We can now recall the sequence of invariant pairs (n_1^q, n_2^q) introduced in [9] to measure the vanishing of $\pi_{\mathcal{M}}$. We thus consider the differential $d\pi_{\mathcal{M}}^k$ as a matrix-valued power series map and define

$$(n_1^q, n_2^q) = \min_{k\in\mathbb{N}} \operatorname{dtp}(d\pi_{\mathcal{M}}^k \circ \mathcal{S}^q);$$

the minimum is again taken with respect to the lexicographic ordering defined above, and is finite for every q (i.e. not equal to (∞,∞)) if and only if M is holomorphically nondegenerate (see [9]). (n_1^q, n_2^q) measures how $\operatorname{dtp} \pi_{\mathcal{M}}$ vanishes "along the Segre maps" and form a lexicographically decreasing sequence in q. We can now summarize the results of §3 of [9] as follows:

Lemma 2. *The numbers (n_1^q, n_2^q) are stable in the following sense: If $(M,0)$ and $(M',0)$ are generic real-analytic submanifolds of $\mathbb{C}^N$ and $H \colon (M,0) \to (M',0)$ is a germ of a biholomorphic map, then with the biholomorphism $\mathcal{H} = (H(Z), \bar{H}(\zeta))$ between $\mathcal{M}$ and $\mathcal{M}'$ we have*

$$\operatorname{dtp} d\pi_{\mathcal{M}'} \circ (\mathcal{H} \circ \mathcal{S}^q) = (n_1^q, n_2^q).$$

Finally, we recall the definition of the transversal jet space $J_t^k((\mathbb{C}_x^p, \mathbb{C}_t^q, 0), \mathbb{C}^r)$, which is the set of equivalence classes of germs of holomorphic maps $h \colon (\mathbb{C}_x^p \times \mathbb{C}_t^q, 0) \to \mathbb{C}^r$ with respect to the equivalence relation of agreeing up to order k in t, that is,

$$J_t^k((\mathbb{C}_x^p, \mathbb{C}_t^q, 0), \mathbb{C}^r) = {\mathfrak{m}\mathbb{C}\{x,t\}^r}\big/{(t_1, \dots, t_q)^k},$$

and we have the natural map $h \mapsto j_{t,0}^k h \in J_t^k((\mathbb{C}_x^p, \mathbb{C}_t^q, 0), \mathbb{C}^r)$, which after choosing coordinates is given by

$$j_{t,0}^k h = (h_{t^\alpha}(x,0) \colon 1 \le |\alpha| \le k).$$

Thus we can essentially identify $J_t^k((\mathbb{C}_x^p, \mathbb{C}_t^q, 0), \mathbb{C}^r)$ with a space of germs of power series in x.

We now recall the following result, which is essentially a restatement of Theorem 7 and Theorem 8 of [9]:

Theorem 8. *Let $P \colon (\mathbb{C}^r, 0) \to (\mathbb{C}^s, 0)$ be a holomorphic map of generically full rank s, and $(n_1, n_2) \in \mathbb{N}^2$. We write $k(\ell) = \max(2n_2 - 1, n_2 + \ell)$. Then there exists an integer k_0, a finite number of Zariski-open subsets $V^1, \dots, V^d \subset J^{k_0}((\mathbb{C}^{p+q}, 0), (\mathbb{C}^r, 0))$ covering $J^{k_0}((\mathbb{C}^{p+q}, 0), (\mathbb{C}^r, 0))$ and for every $\ell \in \mathbb{N}$ holomorphic mappings $\Phi_\ell^j \colon V^j \times J_{t,0}^{k(\ell)}((\mathbb{C}^p \times \mathbb{C}^q, 0), \mathbb{C}^s) \to J_t^\ell((\mathbb{C}^p \times \mathbb{C}^q, 0), \mathbb{C}^r)$, with the property that*

$$j_{t,0}^\ell h = \Phi_\ell^j\left(j_0^{k_0} h, j_{t,0}^{k(\ell)}(P \circ h)\right) \text{ whenever } \operatorname{dtp} P_y \circ h = (n_1, n_2) \text{ and } j_0^{k_0} h \in V_j. \tag{9}$$

Moreover, Φ_ℓ can be chosen to be of the following form:

$$\Phi_\ell^j(\Lambda, \tilde{\Lambda}(x))(x) = \sum_{\alpha\in\mathbb{N}^p} \frac{p_\alpha(\Lambda, \tilde{\Lambda}_\beta \colon |\beta| \le a + b|\alpha|)}{e_j(\Lambda)^{d_{\alpha,\ell}}} x^\alpha, \tag{10}$$

where p_α and e_j are polynomials, $V^j = \{e_j = 0\}^c$, and a, b, and $d_{\alpha,\ell}$ are integers, and we write $\tilde{\Lambda}(x) = \sum_\beta \tilde{\Lambda}_\beta x^\beta$.

Before we can deduce our first parametrization result, we also need to recall the following lemma on derivatives.

Lemma 3. *For every $q \in \mathbb{N}$, $k \in \mathbb{N}$, there exists a real-analytic map $D^q_k\colon J^k((\mathbb{C}^{qn+d},0),(\mathbb{C}^m,0)) \times \mathbb{C}\{z,\chi,\tau\} \to \mathrm{Hol}((\mathbb{C}^{qn+d},0), J^k((\mathbb{C}^N,0),(\mathbb{C}^m,0)))$, which is a complex polynomial in its first argument, such that for any power series map $h(z,w)$, we have that*

$$j^k_Z h\Big|_{Z=S^q\left(x^{[1;q]};t\right)} = D^k_q\left(j^k_{(x^{[1;q]};t)}(h\circ S^q), Q\right), \tag{11}$$

where D^k_q is of the form

$$D^k_q(\Lambda, Q)(x^{[1;q]},t) = \sum_{\alpha,\beta} p_{\alpha,\beta}(\Lambda, j_0^{|\alpha|+|\beta|}Q)(x^{[1;q]})^\alpha t^\beta$$

for some polynomials $p_{\alpha,\beta}$ (which are complex polynomials in their first and real polynomials in their second variable).

For the notion of analyticity used here, we refer the reader to Remark 1.

Proof. In the proof of this Lemma, we write $S_j(x^{[1;j]};t) = (x^1, U^j(x^{[1;j]};t))$ according to the decomposition $\mathbb{C}^N = \mathbb{C}^n \times \mathbb{C}^d$ and employ again the matrix notation (see section 2). We are going to use the facts that $U^q_{x^1}(0;0) = 0_{n\times d}$ and $U^q_t(0;0) = I_{d\times d}$. We start by verifying the claim for $k=1$: setting $f = h\circ S^q$, we have (here the dimension of the matrices involved is left implicit and we write (x,t) for $(x^{[1;q]};t)$)

$$f_{x^1}(x,t) = h_z(S^q(x,t)) + h_w(S^q(x,t))\cdot U^q_{x^1}(x,t),$$
$$f_t(x,t) = h_w(S^q(x,t))\cdot U^q_t(x,t).$$

When $(x,t) = (0,0)$ the system reduces to $f_{x^1}(0,0) = h_z(0,0)$, $f_t(0,0) = h_w(0,0)$, so that it can be solved for h_z and h_w. By Cramer's rule, the solution is a linear function of $f_{x^1}(x,t)$, $f_t(x,t)$ with coefficients which are rational functions of $U^q_{x^1}(x,t)$, $U^q_t(x,t)$; the conclusion for $k=1$ then follows from the definition of U^q.

Assume, now, that $k>1$ and that the conclusion is valid for $k-1$; we decide here to abuse notation in a pretty straightforward manner, and leave it to the reader to make the computation formally correct:

$$h_{z^{k-1}}(S^q(x,t)) = \sum_{\alpha,\beta} q^{k-1,0}_{\alpha,\beta}(j^{k-1}_{(x,t)}f, j_0^{|\alpha|+|\beta|}Q)x^\alpha t^\beta,$$
$$h_{z^{k-2}w}(S^q(x,t)) = \sum_{\alpha,\beta} q^{k-2,1}_{\alpha,\beta}(j^{k-1}_{(x,t)}f, j_0^{|\alpha|+|\beta|}Q)x^\alpha t^\beta,$$
$$\cdots$$
$$h_{w^{k-1}}(S^q(x,t)) = \sum_{\alpha,\beta} q^{0,k-1}_{\alpha,\beta}(j^{k-1}_{(x,t)}f, j_0^{|\alpha|+|\beta|}Q)x^\alpha t^\beta,$$

where each $q^{i,j}_{\alpha,\beta}(\Lambda,Q)$ is a complex polynomial in Λ and a real polynomial in (the appropriate jet of) Q. Differentiating this system with respect to x^1 and (only the last equation) with respect to t, we obtain

$$h_{z^k}(S^q(x,t)) + h_{z^{k-1}w}(S^q(x,t))\cdot U^q_{x^1}(x,t) = \sum_{\alpha,\beta} r^{k,0}_{\alpha,\beta}(j^k_{(x,t)}f, j_0^{|\alpha|+|\beta|}Q)x^\alpha t^\beta,$$
$$h_{z^{k-1}w}(S^q(x,t)) + h_{z^{k-2}w^2}(S^q(x,t))\cdot U^q_{x^1}(x,t) = \sum_{\alpha,\beta} r^{k-1,1}_{\alpha,\beta}(j^k_{(x,t)}f, j_0^{|\alpha|+|\beta|}Q)x^\alpha t^\beta,$$
$$\cdots$$
$$h_{zw^{k-1}}(S^q(x,t)) + h_{w^k}(S^q(x,t))\cdot U^q_{x^1}(x,t) = \sum_{\alpha,\beta} r^{1,k-1}_{\alpha,\beta}(j^k_{(x,t)}f, j_0^{|\alpha|+|\beta|}Q)x^\alpha t^\beta,$$
$$h_{w^k}(S^q(x,t))\cdot U^q_t(x,t) = \sum_{\alpha,\beta} r^{0,k}_{\alpha,\beta}(j^k_{(x,t)}f, j_0^{|\alpha|+|\beta|}Q)x^\alpha t^\beta,$$

where each $r^{i,j}_{\alpha,\beta}(\Lambda, Q)$ is again a complex polynomial in Λ and a real polynomial in Q. Computing in $(x,t) = (0,0)$, one sees that the system can be solved for $h_{z^i w^j}$ as a linear function of the right hand sides, with coefficients which are rational functions of $U^q_{x^1}(x,t)$, $U^q_t(x,t)$; in the same way as before, this implies the claim.

□

Remark 2. In the following, we will often have to substitute $j_0^k\varphi$ into functions which depend on $j_0^\ell\varphi$ for $\ell < k$; in order to lighten the notation, we suppress the application of j_0^ℓ in that case; i.e. if $\psi(\Lambda)$ is a function which depends on a k–jet Λ, we define $\psi(\tilde\Lambda)$ for an ℓ-jet $\tilde\Lambda$ by $\psi(\tilde\Lambda) = \psi(j_0^\ell \tilde\Lambda)$.

We can now state and prove the parametrization theorem for deformations, first along the Segre varieties.

Theorem 9. *Let $(M', 0)$ be a germ of a real-analytic generic submanifold of $\mathbb{C}^N$ which is holomorphically nondegenerate. Then for every $(q,\ell) \in \mathbb{N}^2$ there exist an integer $s(q,\ell)$, a finite set $L = L(q,\ell)$ and for each $J \in L$ a Zariski-open subset $V^J \subset J_0^{s(q,\ell)}((\mathbb{C}^N,0),(\mathbb{C}^N,0)) \times J_0^{s(q,\ell)}((\mathbb{C}^{2n+d},0),(\mathbb{C}^d,0))$, and a real-analytic map $\Psi^J_{q,\ell}$ defined on $V^J \times \mathbb{C}\{z,\chi,\tau\}$ with values in $J^\ell_{t,0}((\mathbb{C}^{nq+d},0),(\mathbb{C}^N,0))$ such that*

$$j^\ell_{t,0}\left(H \circ S^q\right) = \Psi^J_{q,\ell}\left(j_0^{s(q,\ell)} H, j_0^{s(q,\ell)} Q, Q\right),$$

if $w = Q(z,\bar z,\bar w)$ defines a real-analytic generic submanifold M_Q in normal coordinates (z,w), and H is a biholomorphism taking the germ $(M_Q,0)$ into $(M',0)$ satisfying $(j_0^{s(q,\ell)}H, j_0^{s(q,\ell)}Q) \in V^J$; for any such H and Q, there exists at least one J with this property. Furthermore, $\Psi^J_{q,\ell}$ can be chosen such that

$$\Psi^J_{q,\ell}(\Lambda,Q)(x) = \sum_\alpha \frac{q_\alpha(\Lambda, j_0^{c_\alpha}Q)}{|e_J(\Lambda)|^{2d_\alpha}} x^\alpha, \tag{12}$$

with real polynomials q_α, complex polynomials e_J for which $V^J = \{e_J \neq 0\}$, and integers c_α, d_α.

Proof. The proof is by induction on q. We start with $q = 1$ and choose k large enough such that $\operatorname{dtp} d\pi^k_{\mathcal{M}'} = (n_1^1, n_2^1)$, and use Lemma 1 to see that

$$\begin{aligned}
\pi^k_{\mathcal{M}'} \circ \mathcal{H} \circ \mathcal{S}^1 &= \frac{P_k\left(j^k_{(0,t)}H, (Q_{z^\beta}(0,x^1,t))_{|\beta|\le k}\right)}{\det(f_z(0,t) + f_w(0,t)Q_z(0,x^1,t))^{2k-1}} \\
&= \sum_{\alpha,\beta} \frac{P_{\alpha,\beta}\left(j_0^{2k-1+|\beta|}H, j_0^{k+|\alpha|+|\beta|}Q\right)}{\det f_z(0)^{2k-1+|\alpha|+|\beta|}} (x^1)^\alpha t^\beta \\
&=: \mathcal{P}^1_k(H,Q)(x,t).
\end{aligned} \tag{13}$$

The last expression $\mathcal{P}^1_k(H,Q)$ then defines an analytic map in our sense (as a composition map). For any $s \in \mathbb{N}$, we write

$$j^s_{t,0}\mathcal{P}^1_k(H,Q) = \mathcal{Q}^1_{k,s}(j_0^{k+s}H, Q).$$

By Lemma 2, $\operatorname{dtp} d\pi^k_{\mathcal{M}'} \circ \mathcal{H} \circ \mathcal{S}^1 = (n_1^1, n_2^1)$. We can now apply Theorem 8 with $P = \pi^k_{\mathcal{M}'}$ and $(n_1,n_2) = (n_1^1,n_2^1)$ and obtain an integer k_1 and polynomials e_j such that with $\varphi = \mathcal{H} \circ \mathcal{S}^1$ we have

$$j^\ell_{t,0}\varphi(x^1,t) = \left(\tilde\Phi^j_\ell\left(j_0^{k_1}\varphi, j_{t,0}^{k(\ell)}\pi_{\mathcal{M}'} \circ \varphi\right), \hat\Phi^j_\ell\left(j_0^{k_1}\varphi, j_{t,0}^{k(\ell)}\pi_{\mathcal{M}'} \circ \varphi\right)\right),$$

whenever $\operatorname{dtp} d\pi_{\mathcal{M}'} \circ \varphi = (n_1^1,n_2^1)$ and $\tilde e_j(j_0^{k_1}\varphi) \neq 0$, where we split Φ into components $(\tilde\Phi, \hat\Phi)$ corresponding to the coordinates (Z,ζ) for $\mathcal{M}'$, with the analogous notation for φ. We also note that by the definition of $\pi_{\mathcal{M}'}$ and the construction of the $\tilde e_j$ and Φ we can actually write

$$j^\ell_{t,0}\varphi(x^1,t) = \left(\tilde\Phi^j_\ell\left(j_0^{k_1}\hat\varphi, j_{t,0}^{k(\ell)}\pi_{\mathcal{M}'} \circ \varphi\right), \hat\Phi^j_\ell\left(j_0^{k_1}\hat\varphi, j_{t,0}^{k(\ell)}\pi_{\mathcal{M}'} \circ \varphi\right)\right),$$

whenever $\tilde e_j(j_0^{k_1}\hat\varphi) \neq 0$. In particular, if $w = Q(z,\chi,\tau)$ defines the germ of a complexification of a real-analytic generic submanifold in normal coordinates, H takes $w = Q(z,\bar z,\bar w)$ into $(M',0)$

$$j^\ell_{t,0}\left(\mathcal{H}\circ\mathcal{S}^1\right) = \left(\tilde\Phi^j_\ell\left(j_0^{k_1}\left(\bar H \circ \bar S^1\right), j_{t,0}^{k(\ell)}\pi_{\mathcal{M}'} \circ \left(\mathcal{H}\circ\mathcal{S}^1\right)\right), \hat\Phi^j_\ell\left(j_0^{k_1}\left(\bar H \circ \bar S^1\right), j_{t,0}^{k(\ell)}\pi_{\mathcal{M}'} \circ \left(\mathcal{H}\circ\mathcal{S}^1\right)\right)\right), \tag{14}$$

the second component of which is just

$$j_{t,0}^{\ell}\bar{H}\circ\bar{S}^1=\hat{\Phi}_{\ell}^{j}\left(j_0^{k_1}\left(\bar{H}\circ\bar{S}^1\right),j_{t,0}^{k(\ell)}\pi_{\mathcal{M}'}\circ\left(\mathcal{H}\circ\mathcal{S}^1\right)\right).$$

We note that

$$j_0^{k_1}\left(\bar{H}\circ\bar{S}^1\right)=R\left(j_0^{k_1}\bar{H},j_0^{k_1}\bar{Q}\right);$$

here R is some polynomial. Furthermore, one of the $\tilde{e}_j\circ R$ is nonzero if $H\colon(M_Q,0)\to(M',0)$ is a biholomorphism. We can now define $L^1=\{1,\dots,d\}$ and $e_j(\Lambda)=(\det\lambda_0)\tilde{e}_J(R(\Lambda))$, where λ_0 is the part of Λ corresponding to $\bar{f}_z(0)$.

Now by (10), we have with Λ denoting the jet variable in $J_0^{k_1}((\mathbb{C}^N,0),(\mathbb{C}^N,0))\times J_0^{k_1}((\mathbb{C}^{2n+d},0),(\mathbb{C}^d,0))$ that

$$\hat{\Phi}_{\ell}^{j}\left(\Lambda,\tilde{\Lambda}(x)\right)=\sum_{\alpha\in\mathbb{N}^p}\frac{p_\alpha(\Lambda,\tilde{\Lambda}_\beta\colon|\beta|\le a+b|\alpha|)}{\tilde{e}_j(\Lambda)^{\tilde{d}_\alpha}}x^\alpha,$$

so that if we choose c_α large enough and define

$$\tilde{q}_\alpha(\Lambda,\kappa)=p_\alpha\left(R(\Lambda),\left(\left(\frac{\beta!P_{\gamma,\beta}(\bar{\Lambda},\bar{\kappa})}{(\det\bar{\lambda})^{|\alpha|+|\beta|}}\right)_{\beta\le k(\ell)}\right)_{|\gamma|\le a+b|\alpha|}\right).$$

There exists a real polynomial q_α and an integer d_α such that

$$\frac{\tilde{q}_\alpha(\Lambda,\kappa)}{\tilde{e}_j(R(\Lambda))^{\tilde{d}_\alpha}}=\frac{q_\alpha(\Lambda,\kappa)}{|e_J(\Lambda)|^{d_\alpha}},\tag{15}$$

and (13) and (14) together with the last computations then show that under the assumptions of the Theorem, we have that

$$j_{t,0}^{\ell}\bar{H}\circ\bar{S}^1=\Psi_{1,k}^{J}(j_0^sH,j_0^sQ,Q)=\hat{\Phi}_{\ell}^{j}(R(j_0^{k_1}\bar{H},j_0^{k_1}\bar{Q}),\mathcal{Q}_{k,k(\ell)}^{1}(j_0^{k+k(\ell)}H,Q)),$$

for which we set $s(1,\ell):=s=\max(k_1,k+k(\ell))$; clearly, the Ψ thus defined is of the form (12). This proves the conclusion of the theorem for $q=1$.

We now prove the induction step, and assume that the conclusion of the theorem holds for $q-1$. In order to do so, we again choose k large enough and use Lemma 1 to see that

$$\pi_{\mathcal{M}'}^{k}\circ\mathcal{H}\circ\mathcal{S}^q(x^{[1;q]};t)=\frac{P_k\left(j^k_{S^{q-1}(x^{[2;q]},t)}H,(Q_{z^\beta}(x^2,\bar{S}^q(x^{[1;q]};t))_{|\beta|\le k}\right)}{\det\left(f_z(S^{q-1}(x^{[2;q]},t))+f_w(S^{q-1}(x^{[2;q]},t))Q_z(x^2,\bar{S}^q(x^{[1;q]};t))\right)^{2k-1}};\tag{16}$$

in this equation, we substitute for $j^k_{S^{q-1}(x^{[2;q]},t)}H$, using Lemma 3 and the induction hypothesis applied for the map corresponding to $J\in L$ with $e_J(j_0^{s(q-1,|\beta|+k)}H,j_0^{k+|\alpha|+|\beta|}Q)\neq0$, to see that

$$\pi_{\mathcal{M}'}^{k}\circ\mathcal{H}\circ\mathcal{S}^q(x^{[1;q]};t)=\sum_{\alpha,\beta}\frac{P_{\alpha,\beta}^{q-1}(j_0^{s(q-1,|\beta|+k)}H,j_0^{k+|\alpha|+|\beta|}Q)}{\det f_z(0)^{2k-1+|\alpha|+|\beta|}(e_J(j_0^kH,j_0^kQ))^{g_\alpha}}x^\alpha t^\beta=\mathcal{P}_{k,J}^{q}(H,Q)(x,t).\tag{17}$$

Here K and g_α are some integers. Analogously to the case $q=1$, we will write

$$j_{t,0}^{s}\mathcal{P}_{k,J}^{q}(H,Q)=\mathcal{Q}_{k,s,J}^{q}(j_0^{k+s(q-1,s+k)}H,Q)$$

We again apply Theorem 8, this time with $(n_1,n_2)=(n_1^q,n_2^q)$, and with $P=\pi_{\mathcal{M}'}^k$; we thus obtain similarly as before in the case $q=1$ an integer k_q, finitely many complex polynomials e_j, $j=1,\dots,s_q$, and for every such j a map $\hat{\Phi}_{\ell}^{q,j}$ such that

$$j_{t,0}^{\ell}\bar{H}\circ\bar{S}^q=\hat{\Phi}_{\ell}^{q,j}\left(j_0^{k_q}\left(\bar{H}\circ\bar{S}^q\right),j_{t,0}^{k(\ell)}\pi_{\mathcal{M}'}\circ\left(\mathcal{H}\circ\mathcal{S}^{q-1}\right)\right).$$

We are thus lead to define $L^q=\{1,\dots,s\}\times L^{q-1}$, and $e_{j,J}=e_je_J$, and see that if we put $s(q,\ell)=\max(k_q,k(\ell)+s(q-1,k+k(\ell)))$, then if $e_{j,J}(j_0^{s(q,\ell)}H,j_0^{s(q,\ell)}Q)\neq0$, we have that

$$j_{t,0}^{\ell}\bar{H}\circ\bar{S}^q=\Psi_{q,\ell}^{(j,J)}\left(j_0^{s(q,\ell)}H,j_0^{s(q,\ell)}Q,Q\right)=\hat{\Phi}_{\ell}^{q,j}\left(R^q\left(j_0^{k_q}\bar{H},j_0^{k_q}Q\right),\mathcal{Q}_{k,k(\ell)}^{q}(j_0^{s(q,k)}H,Q)\right),$$

where we have written $j_0^{k_q}\bar{H}\circ\bar{S}^q = R^q(j_0^{k_q}\bar{H}, j_0^{k_q}Q)$. We conclude that if we define

$$\Psi^{(j,J)}_{q,\ell}(\Lambda, Q) = \hat{\Phi}^j_\ell\left(R^q(\Lambda), \mathcal{Q}^q_{k,k(\ell),J}(\Lambda, Q)\right),$$

the requirements of the theorem are fulfilled; a computation along the lines of the first part of the proof shows that again Ψ is of the form (12) as desired. □

4. Inverting the Segre map

In the last section, we have obtained a general perturbation version of a parametrization of biholomorphisms along the Segre varieties. If we furthermore assume that $(M', 0)$ is *minimal*, we shall now obtain the proof of Theorem 4.

First, we need to recall a theorem from [9] which we are going to use; in order to formulate it, we need some definitions. For a formal map $A(z)\colon (\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$ we denote by $\nu(A)$ the minimum order of vanishing of minors of the Jacobian of A. We denote by $D = \{(\delta_1, \dots, \delta_n) \in \{1, \dots, m\}^n \colon \delta_1 < \cdots < \delta_n\}$, and for $\delta \in D$ and a map $A \in \mathrm{Hol}$, we define

$$\delta(A) = \begin{vmatrix} \frac{\partial A_1}{\partial z_{\delta_1}} & \cdots & \frac{\partial A_1}{\partial z_{\delta_n}} \\ \vdots & & \vdots \\ \frac{\partial A_n}{\partial z_{\delta_1}} & \cdots & \frac{\partial A_n}{\partial z_{\delta_n}} \end{vmatrix}. \tag{18}$$

We can thus define

$$\nu(A) = \min_{\delta\in D} \operatorname{ord} \delta(A). \tag{19}$$

We will simplify notation and write $\mathrm{Hol}(m, n)$ ($\hat{\mathrm{Hol}}(m, n)$) for the space of (formal) holomorphic maps $(\mathbb{C}^m, 0) \to (\mathbb{C}^n, 0)$. If the dimensions are clear, we simply write Hol or $\hat{\mathrm{Hol}}$. We can thus consider $\nu\colon \hat{\mathrm{Hol}} \to \mathbb{N}$; note that the maps of generic full rank are exactly the ones for which $\nu < \infty$.

Theorem 10. *For every $s \geq 0$ there exists a finite family of polynomials $\psi_1, \dots, \psi_{\ell(s)}$ on $J_0^{s+1}((\mathbb{C}^m, 0), (\mathbb{C}^n, 0))$ and corresponding holomorphic functions*

$$\Phi_k(A, f)\colon \hat{U}_k \times \mathbb{C}[[z]] \to \mathbb{C}[[w]], \quad 1 \leq k \leq \ell(s), \tag{20}$$

where $U_k = \left\{A \in \mathrm{Hol} : \psi_k(j_0^{s+1}A) \neq 0\right\}$, $\hat{U}_k = \left\{A \in \hat{\mathrm{Hol}} : \psi_k(j_0^{s+1}A) \neq 0\right\}$, and Φ_k is linear in its second variable, such that with $A^*g = g \circ A$ we have that

$$\Phi_k(A, A^*g) = g, \quad \text{for every } g \text{ and for } A \in \hat{U}_k \text{ with } \nu(A) = s, \tag{21}$$

and

$$\bigcup_k \hat{U}_k \supset \{A \in \hat{\mathrm{Hol}}\colon \nu(A) = s\}, \quad \bigcup_k U_k \supset \{A \in \mathrm{Hol}\colon \nu(A) = s\}$$

Furthermore, if $A \in U_k$, the operator $\Phi_k(A, \cdot)$ restricts to a linear operator $\mathbb{C}\{w\} \to \mathbb{C}\{z\}$, and the map $\Phi_k\colon U_k \times \mathbb{C}\{z\} \to \mathbb{C}\{w\}$ is holomorphic, where Hol, *$\mathbb{C}\{w\}$, and $\mathbb{C}\{z\}$ are all equipped with their natural inductive limit topologies.*

Moreover, $\Phi_k(A, f)$ can be written as

$$\Phi_k(A, f)(w) = \sum_\alpha \frac{P_{\alpha,k}(j_0^{\ell(\alpha)}A, j_0^{\ell(\alpha)}f)}{\psi_k(j_0^{s+1}A)^{2|\alpha|-1}} w^\alpha. \tag{22}$$

The crucial observation is that one has the necessary invariance property of the Segre maps to apply this theorem to the Segre maps of a biholomorphic perturbation of any fixed CR manifold.

Lemma 4. *Let $S^q(x^{[1;q]}; t) : \mathbb{C}^{qn+d} \to \mathbb{C}^{n+d}$ be the Segre map of order q associated to $(M, 0)$ given in normal coordinates (z, w) by $w = Q(z, \overline{z}, \overline{w})$, and let $\nu(S_0^q) < +\infty$ be defined as above. Assume that $M' = \{w' = Q'(z', \overline{z}', \overline{w}')\}$ is biholomorphic to M, and denote by S'^q_0 the Segre maps based on Q'. Then $\nu(S'^q_0) = \nu(S_0^q)$.*

Proof. Let $q \in \mathbb{N}$, and let $Y : \mathbb{C}^{qn+d} \to \mathbb{C}^{qn+d}$, $Y = (y^1, \dots, y^q, u)$, $y^j \in \mathbb{C}^n$, $u \in \mathbb{C}^d$, be defined as

$$y^{2j+1} = F(S^{q-2j}(x^{[2j+1;q]};t)), \quad y^{2j} = \overline{F}(S^{q-2j+1}(x^{[2j;q]};t)),$$

$$u = G(x^q;t)\ (q\ odd),\ u = \overline{G}(x^q;t)\ (q\ even),$$

where $H = (F,G)$ is a germ of a biholomorphism taking $(M,0)$ to $(M',0)$. We first claim that Y is a germ of a biholomorphism at 0. To compute its differential, we use the notation

$$d_{x^j} y^m = \left(\frac{\partial y^m_r}{\partial x^j_s} \right)_{r,s=1,\dots,n},$$

and similar ones for $d_{x^j}u$, $d_t y^m$ and $d_t u$. Since Y is defined in an upper triangular way, we have (where we set $X = (x^1, \dots, x^q, t)$)

$$dY(X) = \begin{pmatrix} d_{x^1}y^1(X) & d_{x^2}y^1(X) & \cdots & d_{x^q}y^1(X) & d_t y^1(X) \\ 0 & d_{x^2}y^2(X) & \cdots & d_{x^q}y^2(X) & d_t y^2(X) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & d_{x^q}y^q(X) & d_t y^q(X) \\ 0 & 0 & \cdots & d_{x^q}u(X) & d_t u(X) \end{pmatrix}.$$

By definition, $d_{x^q}u(0) = d_{x^q}G(0,0) = 0$ and $|d_t u(0)| = |d_t G(0,0)| \neq 0$, thus in order to check that $dY(0)$ is invertible it is sufficient to verify that the determinants $|d_{x^k}y^k(0)|$, $k = 1, \dots, q$, don't vanish. Assume that $k = 2j+1$ (the other case is analogous); then

$$y^{2j+1}(X) = F(S^{q-2j}(x^{[2j+1;q]};t)) = F(x^{2j+1}, Q(x^{2j+1}, \overline{S}^{q-2j-1}(x^{[2j+2,q];t})))$$

from which follows (here $(z,w) \in \mathbb{C}^n \times \mathbb{C}^d$ are the normal coordinates)

$$d_{x^{2j+1}} y^{2j+1}(0) = d_z F(0) + d_w F(0) \cdot d_z Q(0).$$

Since $d_z Q(0) = 0$ (in fact $d_{(z,\chi)}Q(0) = 0$) and $|d_z F(0,0)| \neq 0$, the claim is verified.

The Lemma is now a direct consequence of Lemma 3 in [9] and of the claim after restricting to $t = 0$ (since $G(x^q, 0) = 0$). □

Proof of Theorem 4. By the Baouendi-Ebenfelt-Rothschild minimality criterion [1], there exists a q for which S'^q_0 is of generically full rank, i.e. $\nu(S'^q_0) < \infty$. By Theorem 9 there exist an integer $s(q,0)$ and a finite family of complex polynomials e_J, $J \in L^q$ defined on $J^{s(q,0)}((\mathbb{C}^N,0),(\mathbb{C}^N,0)) \times J^{s(q,0)}((\mathbb{C}^{2n+d},0),(\mathbb{C}^d,0))$, and corresponding maps $\Psi^J_{q,0}$ such that if H and Q are as in the Theorem, then

$$H \circ S^q_0 = \Psi^J_{q,0}(j_0^{s(q,0)}H, j_0^{s(q,0)}Q, Q).$$

We now apply Theorem 10 with $s = \nu(S'^q_0)$ and obtain a finite set of polynomials ψ_k and maps Φ_k, $k = 1, \dots, \ell(s)$ such that if $\psi_k(j_0^{s+1}S^q_0) = \tilde{\psi}_k(j_0^{s+1}S^q_0) \neq 0$ and $\nu(S^q_0) = s$, then

$$\Phi_k(S^q_0, H \circ S^q_0) = H.$$

We set $\ell = \max(s+1, s(q,0))$ and consider the finite set of polynomials given by $e_{J,j} = |e_J|^2 \tilde{\psi}_j$, i.e. we set $L = L^q \times \{1, \dots, \ell(s)\}$. If the assumptions of the theorem are satisfied for H and Q, then for one pair (J,j) we have that $e_{J,j}(j_0^\ell H, j_0^\ell Q)$ is nonzero by Lemma 4 and Theorem 9. For any pair $(J,j) \in L$ we proceed to define

$$\Psi_{(J,j)}(\Lambda, Q) = \Phi_j \left(S^q_0, \Psi^J_{q,0}(\Lambda, j_0^{s(q,0)}Q, Q) \right).$$

Then, if $H \in \mathrm{Bihol}((M_Q,0),(M',0))$ with $e_{(J,j)}(j_0^\ell H, j_0^\ell Q) \neq 0$, we have that

$$\Psi_{(J,j)}(j_0^\ell H, Q) = \Phi_j \left(\Psi^J_{q,0}(S^q_0, j_0^\ell H, j_0^{s(q,0)}Q, Q) \right) = \Phi_j \left(S^q_0, H \circ S^q_0 \right) = H.$$

The proof of the theorem is concluded by an easy computation utilizing (12) and (22) in order to show iii). □

5. Finite order equivalence and semi-algebraicity

5.1. **Finite order equivalences.** Let (M,p) and (M',p') be two germs of real submanifolds of $\mathbb{C}^N$ of real codimension d, locally given by defining functions ρ, ρ' respectively, and let $k \in \mathbb{N}$, i.e. $\rho\colon (\mathbb{C}^N,p) \to (\mathbb{R}^d,0)$ is a germ of a real-analytic function with $d\rho(p)$ of rank d. We note that alternatively, we can parametrize (M,p) by a germ of a real-analytic function $\psi(t)$ with $t \in (\mathbb{R}^{2n+d},0)$ with $d\psi(0)$ of rank $2n+d$. Then (M,p) and (M',p') are said to be *equivalent of order k* or *k-equivalent* if there exists a local biholomorphism $H \in \mathcal{B}((\mathbb{C}^N,p),(\mathbb{C}^N,p'))$ and a $(d\times d)$-matrix-valued map A, such that $A(p,\bar{p})$ is non-singular and

$$\rho'(H(Z),\overline{H(Z)}) = A(Z,\overline{Z})\rho(Z,\overline{Z}) + o(|Z|^k),$$

or equivalently, if

$$\rho'\left(H(\psi(t)),\overline{H(\psi(t))}\right) = o(|t|^k). \tag{23}$$

Of course, only the k-jet $j_p^k H \in J^k((\mathbb{C}^N,p),(\mathbb{C}^N,p'))$ is involved in the definition, and the property only depends on the k-jets of (M,p) and (M',p') in the following sense:

Lemma 5. *Let $(M,0)$ be a real-analytic submanifold of $\mathbb{C}^N$ of codimension d. Then there exists a sequence of real polynomials $(\varphi_\ell)_\ell$, only depending on (M,p),*

$$\varphi_\ell\colon J_0^\ell((\mathbb{C}^N,0),(\mathbb{R}^d,0)) \times J_0^\ell((\mathbb{C}^N,0),(\mathbb{C}^N,0)) \to \mathbb{R}^{n_\ell},$$

with the following property: If ρ' is a defining function of a real-analytic submanifold $(M',0)$, also of codimension d, then H defines an ℓ-equivalence of $(M,0)$ and $(M',0)$ if and only if

$$\varphi_\ell(j_0^\ell\rho', j_0^\ell H) = 0.$$

For the proof, let $\psi\colon (\mathbb{R}^{2n+d},0) \to (\mathbb{C}^N,0)$ be a real-analytic parametrization of $(M,0)$, and define

$$\varphi_\ell\left(j_0^\ell\rho', j_0^\ell H\right) = \left(\left.\frac{\partial^{|\alpha|}}{\partial t^\alpha}\right|_{t=0} \rho'\left(H(\psi(t)),\overline{H(\psi(t))}\right)\right)_{|\alpha|\le\ell},$$

which is of the form claimed above by the chain rule and satisfies the properties of the Lemma by the definition given above in (23).

In what follows, we will apply this system to a deformation $\rho(Z,\overline{Z},\epsilon)$ (where $\epsilon \in X$ and X is a germ of real-analytic manifold, see section 1) of the germ ρ. For convenience, we will use the notation

$$\mathfrak{r}_k(\epsilon) = j_0^k\rho(\cdot,\epsilon); \tag{24}$$

$\mathfrak{r}_k : X \to J^k((\mathbb{C}^N,0),(\mathbb{R}^d,0))$ is thus a real-analytic mapping. In the case when the deformation is given by the collection of germs (M,q) for q in a neighborhood of p in M, we will consider $\mathfrak{r}_k$ as a germ of a real-analytic mapping defined near p.

From now on we assume that the real-analytic CR manifold M is minimal and holomorphically nondegenerate, as in the previous section. We will recall some known results regarding the relationship among finite order, formal and biholomorphic equivalence that we are going to employ later. First of all, from [18, Theorem 5.1] we have the following fact: for every $\kappa > 1$ there exists $k > 1$ such that for every k-equivalence $H : (M,0) \to (M,0)$ there exists a formal equivalence $\widetilde{H}$ which coincides with H up to order κ. (Zaitsev's result applies to equivalences of formal real-analytic sets, without minimality or nondegeneracy assumptions, given that they are equivalent to any finite order, which is trivially fulfilled in our setting). Combining this statement with [2], we have that $\widetilde{H}$ is actually a local biholomorphism. Moreover, [18, Theorem 2.1] and [2] show that if $(M,0)$ and $(M',0)$ are k-equivalent for all $k \in \mathbb{N}$ then they are also biholomorphically equivalent. We also refer the reader to [4], where the result was proved in the finitely nondegenerate case.

Remark 3. If $(M',0)$ is biholomorphic to $(M,0)$, then for all $\kappa > 1$ and for the *same* $k(\kappa)$ as above we also have that for every k-equivalence $H : (M,0) \to (M',0)$ there exists a formal (hence, in our setting, convergent) equivalence $\widetilde{H}$ which coincides with H up to order κ. In fact, let $\phi : M \to M'$ be a biholomorphism, and $H' : M \to M'$ a k-equivalence; then $H = \phi^{-1} \circ H'$ is a k-equivalence $M \to M$. Let $\widetilde{H}$ be an automorphism of M which coincides with H up to order κ; then $\widetilde{H}' = \phi \circ \widetilde{H}$ agrees with $\phi \circ H = H'$ up to order κ.

As observed above, the k-equivalence condition gives rise to a real algebraic subset $\{\varphi_k = 0\}$ defined in Lemma 5; what we are really interested in, however, is the projection of this set on its first component. For this purpose, we shall now recall some well-known notions and facts about real algebraic geometry.

5.2. **Basics from real-algebraic geometry.** A set $A \subset \mathbb{R}^n$ is said to be *semi-algebraic* if it is a finite union of intersections of sets defined by real polynomial equations and inequalities:

$$A = \bigcup_{i=1}^{k} \bigcap_{j=1}^{h(i)} A_{ij}$$

where A_{ij} is either of the form $\{P_{ij} = 0\}$ or $\{P_{ij} > 0\}$ for a real polynomial $P_{ij} \in \mathbb{R}[x_1, \ldots, x_n]$. The importance of semi-algebraic sets is highlighted by the following fundamental result:

Theorem 11. (Tarski-Seidenberg) *Let $A \subset \mathbb{R}^n$ be a semi-algebraic set, and let π be the projection $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$. Then $\pi(A)$ is semi-algebraic.*

More generally, if $\mathcal{R}$ is any ring of real functions over a set E, a subset $A \subset E$ is called *definable* if it can be expressed as before, with A_{ij} being either of the form $\{f_{ij} = 0\}$ or $\{f_{ij} > 0\}$ for some $f_{ij} \in \mathcal{R}$. We will need the following consequence of Łojasiewicz's generalization of the Tarski-Seidenberg result (see e.g. [6]):

Theorem 12. (Łojasiewicz) *Let X be an analytic manifold, let $A \subset X \times \mathbb{R}^k$ be definable over the ring $C^\omega(X)[x_1, \ldots, x_k]$ and let $\pi : X \times \mathbb{R}^k \to X$ be the projection on the first factor. Then $\pi(A)$ is semi-analytic, i.e., definable over $C^\omega(X)$.*

We have now gathered all the tools we need in order to prove Theorem 2.

Proof of Theorem 2. Let $(M_\epsilon, 0)_{\epsilon \in X}$ be a deformation of $(M, 0)$. Given a subset $E \subset X$, we will denote by $C^\omega(E)$ the set of real-analytic functions defined in a neighborhood of E in X; moreover, given a point $\epsilon_0 \in X$ we denote by $C^\omega_{\epsilon_0}$ the set of germs of real-analytic functions at ϵ_0. In what follows, for every k we regard the map $\mathfrak{r}_k$ defined in section 5 as a real-analytic map $X \to J^k((\mathbb{C}^{2n+d}, 0), (\mathbb{C}^d, 0))$; i.e., we take a choice of real-analytic functions $Q(z, \chi, \tau, \epsilon)$ such that for $\epsilon \in M$, $Q(z, \bar z, \bar w, \epsilon)$ defines $(M_\epsilon, 0)$ in normal coordinates (z, w), and write

$$\mathfrak{r}_k(\epsilon) = j_0^k Q(\cdot, \cdot, \cdot, \epsilon);$$

we also write $\mathfrak{r}(\epsilon) = Q(\cdot, \cdot, \cdot, \epsilon) \in \mathbb{C}\{z, \chi, \tau\}$. Let e_j and $p_{j,\alpha}$ be defined as in Theorem 5, and fix any j_0; moreover, fix any relatively compact, open semianalytic set $B \subset X$. Since $C^\omega(\overline{B})$ is Noetherian (see [8], and also [10, 14]), the same is true for $C^\omega(\overline{B})[\Lambda]$: it follows that there exists $K \in \mathbb{N}$ such that

$$\{p_{j_0,\alpha}(\Lambda, \mathfrak{r}_{c_\alpha}(\epsilon)) = 0\} = \{p_{j_0,\alpha}(\Lambda, \mathfrak{r}_{c_\alpha}(\epsilon)) = 0\}_{|\alpha| \le K}$$

for $(\Lambda, \epsilon) \in J^\ell((\mathbb{C}^N, 0), (\mathbb{C}^N, 0)) \times \overline{B}$.

Consequently, if we define for every j the set $A_j \subset J^\ell((\mathbb{C}^N, 0), (\mathbb{C}^N, 0)) \times B$ as

$$A_j = \{(\Lambda, \epsilon) \in J^\ell((\mathbb{C}^N, 0), (\mathbb{C}^N, 0)) \times B : e_j(\Lambda, \mathfrak{r}_\ell(\epsilon)) \neq 0, \psi_j(\Lambda, \mathfrak{r}(\epsilon)) = 0\}$$

and set $A = A_1 \cup \ldots \cup A_k$, then A is definable over the real polynomials in $J^\ell((\mathbb{C}^N, 0), (\mathbb{C}^N, 0))$ with coefficients in $C^\omega(B)$. Theorem 5 then implies that (where we denote by π the projection $J^\ell((\mathbb{C}^N, 0), (\mathbb{C}^N, 0)) \times X \to X$) $\pi(A)$ corresponds to the the points $\epsilon \in B$ such that $(M_\epsilon, 0)$ is locally biholomorphic to $(M, 0)$, i.e. $\pi(A) = E_M \cap B$. Thus Łojasiewicz's theorem implies that $\pi(A)$, and thus E_M, is a semianalytic subset of B. $\square$

6. Lifting

In order to proceed with the proof of the lifting statements, we still need to collect two facts. We first recall that, if $A_1 \subset \mathbb{R}^n$ and $A_2 \subset \mathbb{R}^m$ are semi-algebraic set, a map $f : A_1 \to A_2$ is said to be a *semi-algebraic map* if the graph of f is a semi-algebraic subset of $\mathbb{R}^{n+m}$.

A (semi algebraic) *cell decomposition* of a semi-algebraic set A is a finite collection of subsets $\{C^q_j\}$ such that each C^q_j is semi-algebraically homeomorphic to $B^q = \{x \in \mathbb{R}^q : |x| < 1\}$ (C^q_j is then called a *cell of dimension q*) and satisfying the following properties:

(1) $A = \bigcup_{j,q} C^q_j$;

(2) the closure $\overline{C}_j^q$ of a q-cell is the union of C_j^q and cells of strictly smaller dimension.

In other words, the sets C_j^q form what is called a *stratification*. An important result in the theory of semi-algebraic sets is that a cell decomposition always exists (see [5]).

Remark 4. Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ be semi-algebraic sets and let $f : X \to Y$ be a semi-algebraic homeomorphism. Moreover, let $K \subset X$ be any subset. Then there exists $p \in K$, a neighborhood U of p in $\mathbb{R}^n$ and a real analytic submanifold $N \subset U$ such that $K \cap U \subset N$ and $f|_N$ is a real-analytic map. To verify this fact, let $\Gamma(f) \subset \mathbb{R}^{n+m}$ be the graph of f, and let π_1, π_2 be the projections of $\Gamma(f)$ onto X and Y respectively. We note that $\Gamma(f)$ admits a stratification by real analytic cells (see [5, Theorem 2.6.12]: the decomposition can be in fact chosen to be a triangulation); let C be a cell of maximal dimension intersecting $K' = \pi_1^{-1}(K)$, and choose $p' \in C \cap K'$, $p = \pi_1(p')$. Then there exists a neighborhood U' of p' such that $U' \cap K' \subset C$: if not, we can choose a sequence of points p'_k converging to p' such that $p'_k \in D \cap K'$ for some cell $D \neq C$, which by the choice of C is of necessarily lower dimension; hence $p \in \bar{D} \cap C = \emptyset$ since the cells form a stratification (for a similar argument, see also the proof of Lemma 6 below). Take $N = \pi_1(C \cap U')$ (N is a real analytic submanifold since $\pi_1|_C$ is a diffeomorphism); then K is locally contained in N and $f = \pi_2 \circ \pi_1^{-1}$ is real analytic.

We will also need the following local triviality result, due to Hardt (see e.g. [5]), concerning semi-algebraic continuous maps.

Theorem 13. (Hardt) *Let $f : X \to Y$ be a continuous, semi-algebraic map. Then there exist a finite semi-algebraic stratification $\{Y_1, \dots, Y_k\}$ of Y, a collection $\{F_1, \dots, F_k\}$ of semi-algebraic sets, and semi-algebraic homeomorphisms $g_j : f^{-1}(Y_j) \to Y_j \times F_j$ such that*

$$f|_{f^{-1}(Y_j)} = p \circ g_j$$

where p is the projection $p : Y_j \times F_j \to Y_j$.

Let $(M_\epsilon, 0)_{\epsilon \in X}$ be a real-analytic deformation of the germ $(M, 0)$, as defined in section 1, and E_M be the locus $\{\epsilon \in X : (M_\epsilon, 0) \sim (M, 0)\}$. By definition, for all $\epsilon \in E_M$ there exists a local biholomorphism $(M, 0) \to (M_\epsilon, 0)$; however it is not clear, in principle, that these biholomorphisms can be chosen to depend nicely on ϵ. The next lemma shows that this can in fact be done at least boundedly in the neighborhood of each point of a dense subset of E_M.

Lemma 6. *Let $p \in E_M$, and fix a neighborhood U of p. Then there exists $q_0 \in E_M \cap U$ such that for every biholomorphism $H^{q_0} : (M, 0) \to (M_{q_0}, 0)$, any $\kappa \in \mathbb{N}$, and for every neighbourhood W of $j_0^\kappa H^{q_0}$ there exists a neighbourhood V of q_0 with the following property: for all $q \in V \cap E_M$ there is a biholomorphism $H^q : (M, 0) \to (M_q, 0)$ such that the $j_0^\kappa \phi_q \in W$.*

Proof. Choose $k > 0$ (associated to κ) as in Remark 3, and let φ_k be the set of polynomials defined in Lemma 5. Then $A_k = \{\varphi_k = 0\}$ is a real algebraic subset of $J_0^k((\mathbb{C}^N, p'), (\mathbb{R}^d, 0)) \times J_0^k((\mathbb{C}^N, 0), (\mathbb{C}^N, 0))$; let π be the projection to $J_0^k((\mathbb{C}^N, p'), (\mathbb{R}^d, 0))$. Let $\mathfrak{r}_k(q)$ be the map defined in (24); then $(M_q, 0)$ is k-equivalent to $(M, 0)$ if and only if $\mathfrak{r}_k(q)$ belongs to $\pi(A_k)$; in particular the points of E_M satisfy this property. Let $C_1, \dots, C_m$ be a partition of $\pi(A_k)$ into semi-algebraic sets in such a way that $\pi|_{\pi^{-1}(C_j)}$ is trivial for $1 \le j \le m$ (see Hardt's Theorem above and [5]), and let C_j^i be a cell decomposition respecting $\{C_1, \dots, C_m\}$ and forming a stratification in the sense specified above, i.e. $\overline{C}_j^i$ is the union of C_j^i and cells of strictly smaller dimension (cfr. [5]). Let $d = \max\{d' : \exists q \in E_M \cap U \text{ s.t. } \mathfrak{r}_k(q) \in C_j^i \text{ and } \dim C_j^i = d'\}$, and choose $q_0 \in E_M \cap U$ realizing the maximum, $\mathfrak{r}_k(q_0) \in C_{j_0}^{i_0}$. We claim that, for $q' \in E_M$ lying in a small enough neighborhood of q_0, we still have $\mathfrak{r}_k(q') \in C_{j_0}^{i_0}$. Otherwise, there would exist a sequence $q'_n \to q_0$ (hence also $\mathfrak{r}_k(q'_n) \to \mathfrak{r}_k(q_0)$) such that $\mathfrak{r}_k(q'_n) \in C_{j_1}^{i_1}$ for some fixed indexes i_1, j_1, with $\dim C_{j_1}^{i_1} \le d$. It would follow that $C_{j_0}^{i_0} \cap \overline{C}_{j_1}^{i_1} \neq \emptyset$, which contradicts the fact that the sets C_j^i form a stratification. So $\mathfrak{r}_k(q') \in C_{j_0}^{i_0}$ for all the $q' \in V' \cap E_M$ for some small enough neighborhood V' of q_0; in particular $\mathfrak{r}_k(q') \in C_{j_0}$. By the triviality over $\pi^{-1}(C_{j_0})$ we have that, fixed a biholomorphism $H^{q_0} : (M, 0) \to (M_{q_0}, 0)$, there exists a neighbourhood V of q_0 such that for all $q' \in V$ there exists a k-equivalence $\widetilde{H}^{q'} : (M, 0) \to (M_{q'}, 0)$ whose k-jet belongs to $j_0^k (j_0^\kappa)^{-1} W$ where W is the given neighbourhood of $j_0^\kappa H^{q_0}$. By [18, Theorem 5.1] and [2] it now follows that there also exists a biholomorphism $H^{q'} : (M, 0) \to (M_{q'}, 0)$ whose κ-jet belongs to W. $\square$

If the deformation $(M_q, 0)$ is given by moving the basepoint q of a fixed germ (M, p), then the conclusion of Lemma 6 can be strenghtened:

Corollary 7. *Let $p \in M$. Then for every $\kappa \in \mathbb{N}$ and for every neighbourhood V of $j_p^\kappa \mathrm{id}$ there exists a neighborhood U of p such that for all $q \in E_p \cap U$ there is a biholomorphism $\phi_q : (M,p) \to (M,q)$ with $j_p^\kappa \phi_q \in V$.*

Proof. Let q_0 be as in Lemma 6, and fix any biholomorphism $\phi_{q_0} : (M,p) \to (M,q_0)$. Then the claim is proved by composing the family ϕ_q of Lemma 6 with the inverse of ϕ_{q_0}. $\square$

We now want to show that, if E_M contains a real-analytic submanifold, the lifting property of Lemma 6 gives real-analytic sections:

Lemma 8. *Assume that E_M contains a (real analytic) submanifold R of X; let $p \in R$ and fix an open neighborhood U of p. Then for every $\kappa \in \mathbb{N}$ there exist $q_1 \in R \cap U$, a neighbourhood V of q_1, and a real analytic map $\mathcal{L}_\kappa : V \cap R \to G_0^\kappa(\mathbb{C}^N)$ and (for all $q \in V \cap R$) a biholomorphism $\phi_q : (M,0) \to (M_q,0)$ whose κ-jet is $\mathcal{L}_\kappa(q)$.*

Proof. Along the same lines as Lemma 6, we work with the k which is obtained from κ as in Remark 3. The conclusion will follow from the following claim: there exist $x_0 \in A_k$ (with $\pi(x_0) \in \mathfrak{r}_k(R \cap U)$), $q_1 \in R \cap U$ with $\mathfrak{r}_k(q_1) = \pi(x_0)$, a neighborhood V' of $\mathfrak{r}_k(q_1)$, a neighbourhood V of q_1, and a submanifold L of a neighborhood of x_0 in A_k such that the map $\pi : L \to V' \cap \pi(A_k)$ is $1-1$ and the map $\pi^{-1} \circ \mathfrak{r}_k : V \cap R \to L$ is real analytic. Let $C_{j_0}^{i_0}$ be the d-cell chosen in the proof of Lemma 6; since π is trivial over C_{j_0}, there exists a semi-algebraic homeomorphism $f_{j_0} : A_k \cap \pi^{-1}(C_{j_0}) \to C_{j_0} \times F_{j_0}$ (for a certain semi-algebraic set F_{j_0}) such that $\pi|_{\pi^{-1}(C_{j_0})} = p \circ f_{j_0}$ (where p is the projection $C_{j_0} \times F_{j_0} \to C_{j_0}$). We apply Remark 4 to $X = A_k \cap \pi^{-1}(C_{j_0})$, $Y = F_{j_0} \times C_{j_0}$ and $K = \pi^{-1}(\mathfrak{r}_k(R) \cap C_{j_0}) \cap A_k$: we can then choose $x_0 \in A_k$ in such a way that $\pi(x_0) \in \mathfrak{r}_k(R)$ and f_{j_0} is real-analytic on a submanifold N of a neighborhood of x_0 in A_k, locally containing $\pi^{-1}(\mathfrak{r}_k(R)) \cap A_k$. The claim is then obtained by taking L to be the intersection of $f_{j_0}^{-1}(C_{j_0}^{i_0} \times \{f_{j_0}(x_0)\})$ with a small neighborhood of x_0 in N. $\square$

Again, if $(M_q, 0)$ is given by the germs (M,q) induced at $q \in M$ by a fixed germ (M,p), the analogous of Corollary 7 holds with the same proof to give us real-analytic sections:

Corollary 9. *Let $p \in M$. Then for every $\kappa \in \mathbb{N}$ and for every neighbourhood V of $j_p^\kappa \mathrm{id}$ there exists a neighborhood U of p, a real-analytic map $\mathcal{L} \colon U \to V \subset J_p^\kappa((M,p),M)$, and biholomorphisms $\phi_q \colon (M,p) \to (M,q)$ defined for $q \in E_p \cap U$ such that for all $q \in E_p \cap U$ we have $\mathcal{L}(q) = j_p^\kappa \phi_q$.*

Theorem 14. *Let $(M_\varepsilon, 0)_{\varepsilon \in X}$ be a deformation of $(M,0)$, which we assume to be holomorphically nondegenerate and minimal. Then there exists a dense subset $D \subset E_M$ with the following properties:*

i) *Each component of D is real-analytic submanifold of X;*
ii) *For every $q \in D$ there exists a neighbourhood U of q in D and a real-analytic map $\varphi(p,q)$ defined in a neighbourhood of $\{0\} \times U$ in $M \times D$ such that for each $q \in U$ the map $p \mapsto \varphi(p,q) \colon (M,0) \to (M_q,0)$ is a CR equivalence.*

Proof. We first note that since E_M is a semianalytic subset of X by Theorem 2, we can find a dense open subset D' of E_M each of whose components are real-analytic submanifolds of X. By Lemma 8, we can find a dense subset $D \subset D'$ each of whose points fulfills the property of Lemma 8 with κ being the number ℓ required in Theorem 3. From this Theorem, we conclude that $\phi_q = \Psi(\cdot, (\mathcal{L}(q))^{-1}, q)^{-1}$, where the inverse is taken as a map in the first component, which is of the form claimed in ii). $\square$

Proof of Theorem 6. The second part of the Theorem is now either an immediate consequence of Theorem 14 and the homogeneity of E_p, or one could repeat the proof of that theorem with Lemma 8 replaced by Corollary 9 and using the fact that E_p is a real-analytic submanifold. The first part of the Theorem is implied by the second part, since we obtain an infinitesimal CR automorphism of M at p by taking any curve $\gamma(t)$ in E_p with $\gamma(0) = p$ and setting

$$X = \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_{t=0} \psi(\cdot, \gamma(t));$$

since $\psi(p,q) = q$, $X(p) = \gamma'(0)$. $\square$

7. Examples

7.1. **Real-analytic examples.** By Theorem 1, if M is a holomorphically nondegenerate, minimal real-analytic manifold, and $p \in M$, the structure of the locus E_p is – at least on a neighborhood of p – very simple. We present here several examples to illustrate how, from a global point of view, the situation can be more complicated. Also of interest are questions regarding the relationship between E_p and the loci nearby, as well as the possibility to extend the automorphisms of E_p to automorphisms of a neighborhood of E_p in M.

In some cases, where no simple invariant (for example involving degeneracy or type) is available, the actual computation of the locus can be a complicated task. In such a situation, one can employ the characterization in Theorem 6 , which, while only allowing to find the connected component of E_p through p, in principle reduces the calculation to a linear problem.

Example 5. For a generic (M, p) the locus is trivial, i.e. $E_p = \{p\}$. On the opposite side of the spectrum, we have the homogeneous manifolds where $E_p = M$. Many of the results we have proved in this paper for the equivalence locus (and actually, some more) are already known in the setting of homogeneous CR manifolds from work of Zaitsev [17].

An interesting aspect of our results here is that they exhibit homogeneous CR manifolds as "building blocks" of holomorphically nondegenerate, minimal, real-analytic CR manifolds since they are the disjoint union of the equivalence orbits, which are (by Theorem 1) homogeneous CR manifolds.

A very simple example of a homogeneous CR manifold in $\mathbb{C}^2(z, w)$ is given by the Lewy hypersurface $\{\operatorname{Im} w = |z|^2\}$, which is biholomorphic to the standard sphere $S^3 \subset \mathbb{C}^2$.

Example 6. Let $S = \{\operatorname{Im} w = |z|^4\}$; then S has type 4 along $\{z = 0, \operatorname{Im} w = 0\}$ and is strongly pseudoconvex elsewhere. It follows that $E_0 = \{z = 0, \operatorname{Im} w = 0\}$ is a real line. On the other hand, $S \setminus E_0$ is homogeneous: a way to verify this fact is to consider the map $\phi : \mathbb{C}^2 \to \mathbb{C}^2$, $\phi(z, w) = (z^2, w)$. The pull-back of S through ϕ is the Lewy hypersurface; since ϕ is non-singular for $z \neq 0$, then, the claim follows.

Alternatively, one can consider the group generated by $\{r_\theta\}_{\theta\in[0,2\pi)}$ and $\{d_t\}_{t\in\mathbb{R}\setminus\{0\}}$, where $r_\theta(z, w) = (e^{i\theta}z, w)$ and $d_t(z, w) = (tz, t^4w)$, which (alongside with the translation in the $\operatorname{Re} w$-direction) acts transitively on $S \setminus E_0$.

This example already shows that E_p needs not be, globally, a closed submanifold of M, since this is not the case for $S \setminus E_0$. We can also see that the function $p \to \dim E_p$ is (only) lower semicontinuous.

Example 7. By modifying the previous example, we can get rid of some of its symmetries and "break" the homogeneity of the strongly pseudoconvex part. Let $S_1 = \{\operatorname{Im} w = |z|^4 + |z|^6\}$; then the rotations r_θ are still automorphisms of S_1. A heavy, computer-assisted calculation of $\dim_{\mathbb{R}} E_q$ at $q = (i, 2)$ (as it turns out, the equation defining $\mathfrak{hol}(S_1, q)$ needs to be checked up to its 8-th jet) showed that the locus at q has dimension 2, as expected.
Let, now, $S_2 = \{\operatorname{Im} w = |z|^4 + \operatorname{Re}(z\bar{z}^3)\}$. In this case, the only rotation preserved corresponds to $\theta = \pi$, while S_2 is invariant under the dilations d_t. Similarly as before, an explicit computation (this time involving the 7-th order jet of the equation defining $\mathfrak{hol}(S_1, q)$) at $q = (i, 0)$ allows to verify that $\dim_{\mathbb{R}} E_q = 2$. We remark that in this example E_q is (globally) disconnected, since it contains (at least) the two connected components $\{(is_1, s_2)\}_{s_1>0,s_2\in\mathbb{R}}$ and $\{(is_1, s_2)\}_{s_1<0,s_2\in\mathbb{R}}$.

Example 8. Let $S = \{\operatorname{Im} w = |z|^6 + \operatorname{Re}(z\bar{z}^5)\}$. In this example, the automorphisms of S include the dilations $d_t(z, w) = (tz, t^6w)$ for $t \in \mathbb{R}$ and the rotations r_θ for $\theta = \pi, \pm\pi/2$. As in the previous example, we could explicitly check that the real dimension of E_q at the point $q = (i, 2)$ is 2. Also in this case, the locus E_q is not connected; moreover, we can see that E_q (while it is in itself a locally closed submanifold of S) is no longer an open subset of a regular submanifold of S. The closure of the locus corresponds in fact to the 2-dimensional real analytic set $\{(s_1, s_2) \cup (is_1, s_2)\}_{(s_1,s_2)\in\mathbb{R}^2}$, which is singular at 0.

Example 9. Let $S = \{\operatorname{Im} w = |z_1|^4 + |z_1z_2|^2\} \subset \mathbb{C}^3(z_1, z_2, w)$: this hypersurface is obtained by modifying the homogeneous submanifold $S' = \{\operatorname{Im} w' = |z_1'|^2 + |z_2'|^2\}$ via the map $\phi(z_1, z_2, w) = (z_1^2, z_1z_2, w)$. Since ϕ is non-singular for $\{z_1 \neq 0\}$, then, it follows that the open subset $E_5 = S \cap \{z_1 \neq 0\}$ is homogeneous. Moreover, $S \cap \{z_1 = 0\}$ is the 3-dimensional plane $\{\operatorname{Im} w = 0, z_1 = 0\}$ and so it is, by itself, an homogeneous submanifold of $\mathbb{C}^3$. Nonetheless, the equivalence locus of the point $(0, 0, 0)$ does not coincide with $S \cap \{z_1 = 0\}$, but it reduces to the line $E_1 = (0,0,s)_{s\in\mathbb{R}}$ (these are the only points in which S has type 4). The remaining set $E_3 = S \cap \{z_1 = 0, z_2 \neq 0\}$ also constitutes a locus: looking at the automorphisms $(z_1, z_2, w) \to (cz_1, cz_2, |c|^4 w)$ for $c \in \mathbb{C} \setminus \{0\}$ shows that $E_p \supset E_3$ for all $p \in E_3$, while considering the Levi form of S (which has exactly one vanishing eigenvalue only along E_3) shows that $E_p \subset E_3$ for all $p \in E_3$. To sum it up, S is the disjoint union of three loci E_1, E_3, E_5, forming an analytic (in fact algebraic) stratification.

Example 10. Let $S = \{\operatorname{Im} w = (\operatorname{Re} w)^2 e^{-\operatorname{Re} z_1} + |z_2|^2\} \subset \mathbb{C}^3$. Here the automorphisms of S include the translations along the $\operatorname{Im} z_1$-direction as well as the transformations $(z_1, z_2, w) \to (z_1 + t, e^{t/2} z_2, e^t w)$ for $t \in \mathbb{R}$. The group they generate is transitive on $E = \{z_2 = 0, w = 0\} \subset S$; once again, looking at the eigenvalues of the Levi form shows that E coincides with the equivalence locus of 0 in S. It follows that, even if S is nondegenerate, the locus E_p of some point $p \in S$ can be a complex submanifold.

From this example, as well as from the previous one, we see that S needs not be "factorized over E_p", in the sense of the existence (for some "transversal" manifold T) of a CR map from $E_p \times T$ to a neighborhood of p in S; in other words, the loci nearby may not contain submanifolds which are CR-isomorphic to E_p.

We also see that the group of automorphisms of E_p, considered in itself as a homogeneous manifold, can be much bigger than the group of automorphisms of S which preserve E_p.

7.2. **Counterexamples of class C^k.** The following construction provides examples of hypersurfaces of $\mathbb{C}^2$ for which the equivalence locus of certain points is not locally a manifold, or, more precisely, is not even locally closed. For all finite k, we could find an example of class C^k; we were not able, up to now, to find an example of class C^∞.

Let us consider the function $g : (-1,1) \to \mathbb{R}$ defined as

$$g(x) = e^{-\tan^2(\pi x/2)};$$

observe that g is real analytic on $(-1,1)$; moreover, the extension of g to the whole $\mathbb{R}$ by 0 is of class C^∞. We are going to define a function $f : [0,1] \to \mathbb{R}$, vanishing exactly on the standard Cantor set $\mathcal{C} \subset [0,1]$, by a suitable sequence of dilations and translations of the function g. More precisely, for $m \in \mathbb{N}$ define

$$a_{n,m} = \frac{n}{3^m},\ b_{n,m} = \frac{n+1}{3^m},\ m \in \mathbb{N},\ 0 \leq n \leq 3^m - 1$$

and let

$$I_m = \{i \in \{0,1,\ldots,3^m-1\} : \forall 0 \leq j < m, 1 \leq r \leq 3^{m-j-1}, i \notin \{(3r-2)3^j, \ldots, (3r-1)3^j - 1\}\},$$

$$J_m = \{0,1,\ldots,3^m-1\} \setminus I_m.$$

We will consider the following family of linear (affine) transformations of $\mathbb{R}$:

$$\psi_{n,m} : [0,1] \to [a_{n,m}, b_{n,m}], \quad \psi_{n,m}(x) = \frac{x+n}{3^m};$$

note that $\psi_{n,m}$ restricts to an automorphism of the Cantor set $\mathcal{C}$ as long as $n \in I_m$. For a choice of $h > 0$, we define

$$f(x) = \begin{cases} h^{-m} g(2 \cdot 3^m x - 2n - 1), & \text{if } x \in (a_{n,m}, b_{n,m}) \text{ with } n \in J_m, \\ 0 & \text{otherwise (i.e. } x \in \mathcal{C}). \end{cases}$$

Then f satisfies the following relation:

$$f(\psi_{n,m}(x)) = h^{-m} f(x) \ \text{ for all } m \in \mathbb{N}, n \in I_m.$$

In fact, if $x \in \mathcal{C}$ then also $\psi_{n,m}(x) \in \mathcal{C}$ and $f(x) = f(\psi_{n,m}(x)) = 0$. Otherwise, we have that $x \in (a_{p,q}, b_{p,q})$ for some $q \in \mathbb{N}$, $p \in J_q$. Then $\psi_{n,m}(x) \in (a_{p+n3^q, m+q}, b_{p+n3^q, m+q})$ and since $p + n3^q \in J_{m+q}$, we have

$$f(\psi_{n,m}(x)) = h^{-(m+q)} g(2 \cdot 3^{m+q} \left(\frac{x+n}{3^m}\right) - 2(p + n3^q) - 1) =$$

$$= h^{-m} \left(h^{-q} g(2 \cdot 3^q x - 2p - 1)\right) = h^{-m} f(x).$$

In other words, the graph $\{u = f(x)\}$ of the function f in $\mathbb{R}^2(u,x)$ is invariant under the affine transformation $(x,u) \to (\psi_{n,m}(x), h^{-m} u)$ for all $m \in \mathbb{N}$ and $n \in I_m$.

Now, for $k \in \mathbb{N}$, assume that $h > 3^k$; we will show that f is of class C^k. For every $l \in \mathbb{N}$, define a function f_l by "truncating the construction of f at the step l":

$$f_l(x) = \begin{cases} h^{-m} g(2 \cdot 3^m x - 2n - 1), & \text{if } x \in (a_{n,m}, b_{n,m}) \text{ with } m \leq l, n \in J_m, \\ 0 & \text{otherwise;} \end{cases}$$

observe that f_l is smooth for any fixed $l \in \mathbb{N}$, in particular it belongs to $C^k([0,1])$. Let $M = \|g\|_{C^k([-1,1])} = \max_{0 \leq i \leq k}(\sup_{x \in [-1,1]} |g^{(i)}(x)|)$. Then, for every $0 \leq j \leq k$ and $x \in (a_{n,m}, b_{n,m})$ $(n \in J_m)$,

$$\left| \frac{d^j}{dx^j} f(x) \right| = |h^{-m}(2 \cdot 3^m)^j g^{(j)}(2 \cdot 3^m x - 2n - 1)| \leq \left(\frac{3^k}{h} \right)^m 2^k M.$$

If $l_1, l_2 \in \mathbb{N}$, $l_2 > l_1$, then for $j \geq 0$ the j-th derivative of $f_{l_2} - f_{l_1}$ is only non vanishing on $\bigcup_{l_1 \leq m \leq l_2, n \in J_m} (a_{n,m}, b_{n,m})$, hence for $0 \leq j \leq k$

$$\left| \frac{d^j}{dx^j} (f_{l_2} - f_{l_1})(x) \right| \leq \sup_{x \in \bigcup_{m \geq l_1} (a_{n,m}, b_{n,m})} \left| \frac{d^j}{dx^j} f(x) \right| \leq \left(\frac{3^k}{h} \right)^{l_1} 2^k M.$$

Therefore $\{f_l\}_{l \in \mathbb{N}}$ is a Cauchy sequence in $C^k([0,1])$; since by construction $f_l \to f$, it follows that f is in turn of class C^k.

Now, consider in $\mathbb{C}^2$ complex coordinates $(z = x + iy, w = u + iv)$, and let S be the real, tubular hypersurface defined by

$$S = \{(z, w) \in \mathbb{C}^2 : u = f(x)\}.$$

Note that, for $m \in \mathbb{N}$ and $n \in I_m$, the (affine) complex linear transformation $(z, w) \to (\psi_{n,m}(z), h^{-m} w)$ is an automorphism of S, where $\psi_{n,m}(z) = (z+n)/3^m$. Moreover, for $t, s \in \mathbb{R}$ the translation $(z, w) \to (z+it, w+is)$ and (since g was chosen to be an even function) the reflection $(z, w) \to ((-z+1/2), w)$ are also automorphisms of S. Combining these, it is easy to see that

$$F_p = \{(a_{n,m} + iy, iv) \text{ and } (b_{n,m} + iy, iv) : y, v \in \mathbb{R}, m \in \mathbb{N}, n \in J_m\}$$

is contained in the equivalence locus E_p of any $p = (a_{n_0,m_0}, 0)$ for $m_0 \in \mathbb{N}$ and $n_0 \in J_m$. We claim that actually $F_p = E_p$.

In fact, consider the set

$$A = \{(z, w) \in S : \mathrm{Re} z \in [0,1] \setminus \mathcal{C}\};$$

then S is real analytic in a neighborhood of a point $q \in S$ if and only if $q \in A$. In particular, $\phi(A) \subset A$ for any local automorphism ϕ of S, which implies that (with p as above) $E_p \subset \mathcal{C} \times i\mathbb{R}^2 = \{(x + iy, iv) : x \in \mathcal{C}\}$. Choose, then,

$$x_1 \in \mathcal{C} \setminus \bigcup_{m \in \mathbb{N}, n \in J_m} \{a_{n,m}, b_{n,m}\}$$

and let $p_1 = (x_1, 0)$. Then p_1 doesn't belong to the boundary of any connected component of A. Since, on the other hand, p does belong to the boundary of $\{(x + iy, f(x) + iv) : x \in (a_{n_0,m_0}, b_{n_0,m_0}), y, v \in \mathbb{R}\}$ (which is a connected component of A), it follows that $p_1 \notin E_p$; hence we obtain $E_p = F_p$.

Finally, we observe that E_p is dense in $\mathcal{C} \times i\mathbb{R}^2$; since the latter is a perfect set, it follows that E_p is nowhere locally closed.

8. Global automorphisms

Our first step in studying the global automorphism group is once again a local result. We shall show that, given a point $p \in M$, the germs of biholomorphisms of M at p can be analytically parameterized in the following way:

Theorem 15. *Let M be a real-analytic CR manifold which is holomorphically nondegenerate and minimal. Then for every $p \in M$ there exists a neighbourhood U of p in M, an integer ℓ, a neighbourhood V of $j_p^\ell \mathrm{id}$, and a real-analytic map $\Psi \colon \Omega \to M$, defined on $U \times V$ such that for any germ at q of a real-analytic CR automorphism f with $j_q^\ell f \in V$ for some $q \in U \cap E_p$, we have*

$$f(z) = \Psi(z, j_q^\ell f)$$

for z near q.

Let us first see how Theorem 15 implies Theorem 7. We will follow the line of reasoning due to Baouendi, Rothschild, Winkelmann, and Zaitsev [3], and thus restrict ourselves to a sketch: In order to show that there exists a neighbourhood of $\mathrm{id} \in \mathrm{Aut}_{CR}^{\omega}(M)$ which is locally compact, we show that for a judiciously chosen neighbourhood $\mathcal{U}$ of $\mathrm{id} \in \mathrm{Aut}_{CR}^{\omega}(M)$, any sequence of CR automorphisms $(f_n)_{n\in\mathbb{N}} \subset \mathcal{U}$ contains a subsequence which converges everywhere on M to an automorphism. Let K be the compact set given in Theorem 7. For any $p \in K$, we can find a neighbourhood $U(p)$, an integer ℓ_p, and maps Ψ_p, from Theorem 3 applied to the deformation of moving the basepoint and considering the inverse, as in the proof of Theorem 2, such that

$$f(z) = \Psi_p(z, j_p f),$$

for any f with $j_p^{\ell_p} f$ in a neighbourhood V_p of $j_p^{\ell_p}\,\mathrm{id}$. By possibly shrinking U_p and V_p, we can assume that $j_q^1 \Psi_p(\cdot, \Lambda)$ is invertible for every $\Lambda \in V_p$ and $q \in U_p$.

We can cover K with finitely many such $U_{p_j} =: U_j$ and by requiring that $\mathcal{U}$ consists only of maps f for which $j_{p_j}^k f$ is sufficiently close to $j_{p_j}^k\,\mathrm{id}$ where $k = \max_j \ell_{p_j}$, we have that

$$f(z) = \Psi_{p_j}(z, j_{p_j}^k f), \quad f \in \mathcal{U}.$$

Let now (f_n) be a sequence in $\mathcal{U}$. We first claim that there exists a subsequence of (f_n) such that f_n converges on $U = \cup_j U_j \supset K$ to a map with invertible Jacobian at every point. For this, we choose a subsequence f_{n_s} such that for every j, $j_{p_j}^k f_{n_s}$ converges to a jet $\Lambda_j^0 \in V_j$; then of course the f_n converge on U (uniformly on compact subsets, in particular, on K), and the limit map has an invertible Jacobian at every point of U by our choice of U_j and V_j.

We now consider the set $\mathcal{O}$ of points $p \in M$ which have a neighbourhood on which f_n converges to a map whose Jacobian is invertible. We will show that that $\mathcal{O} = M$; one can then follow the arguments of [3] in order to finish the proof. For every $p \in M$, denote by $\tilde{E}_p$ the connected component of E_p containing p. By assumption, $\tilde{E}_p \cap K$ is not empty, and thus, $\mathcal{O} \cap \tilde{E}_p$ is also not empty. Obviously the latter set is open in $\tilde{E}_p$; we shall show that it is also closed in $\tilde{E}_p$ and thus $\mathcal{O} \cap \tilde{E}_p = \tilde{E}_p$. This implies that $\mathcal{O} = M$.

Now the assertion that $\mathcal{O} \cap \tilde{E}_p$ is closed in $\tilde{E}_p$ follows from Theorem 15 with exactly the same proof as [3, Lemma 3.3] follows from [3, Proposition 2.2.]. We thus only need to give a proof of Theorem 15.

Proof of Theorem 15. We first claim that we can find a real-analytic map $\psi(x, y, z)$ defined on $\tilde{U} \times (\tilde{U} \cap E_p) \times (\tilde{U} \cap E_p)$, where $\tilde{U}$ is some neighbourhood of p, with the following properties:

$$x \mapsto \psi(x, y, z) \text{ is CR on } U \text{ for any } (y, z) \in \tilde{U} \times \tilde{U}; \quad \psi(y, y, z) = z, \quad x = \psi(x, p, p).$$

Indeed, by Theorem 6, there exists a map φ defined on $V \times W$ for some neighbourhood V of p in M and $W \subset E_p$ of p in E_p which is a CR automorphism on V for every $q \in W$ and $\varphi(p, q) = q$. We now define

$$\psi(x, y, z) = \varphi(\varphi^{-1}(x, y), z),$$

where the inverse is understood with respect to the x variable. By shrinking V and W, respectively, we can assume that the formula on the right hand side makes sense on a set of the form $\tilde{U} \times (\tilde{U} \cap E_p) \times (\tilde{U} \cap E_p)$.

From [9], we know that there exists an integer ℓ such that for a neighbourhood V of p and a neighbourhood W of $j_p^\ell\,\mathrm{id} \in G_p^k(M)$ there is a real-analytic map Ψ_0, CR in its first variable, such that for any $f \in \mathrm{Aut}(M, p)$ with $j_p^\ell f \in W$, we have $\Psi_0(x, j_p^\ell f) = f(x)$ for every CR automorphism of M fixing p. By restricting to a probably smaller neighbourhood W, we can assume that all such automorphisms with $j_p^\ell f \in W$ are actually defined in the neighbourhood $\tilde{U}$ of p (again, after possibly shrinking $\tilde{U}$).

From φ and Ψ_0, we can manufacture a parametrization of $\mathrm{Aut}(M, q)$ for $q \in U_p$ by pushing Ψ_0; i.e., let us for simplicity define for $\Lambda \in G_q^\ell(M)$ the jet associated to it via φ at p by $T_p\Lambda = j_p^\ell \varphi(\cdot, q) \Lambda j_q^\ell \varphi^{-1}(\cdot, q) \in G_p^\ell(M)$. We can also, if we write π for the map associating to any jet its basepoint, understand $T_p\Lambda = j_p^\ell \varphi(\cdot, \pi(\Lambda)) \Lambda j_q^\ell \varphi^{-1}(\cdot, \pi(\Lambda))$ as a map defined on an open neighbourhood of $j_p^\ell\,\mathrm{id} \in G^\ell(M)$. We then set

$$\Psi_1(x, \Lambda) = \varphi^{-1}\left(\Psi_0\left(\varphi(x, q), T_p\Lambda\right), q\right),$$

which is defined on $\tilde{U} \times W$, where W is an open subset of $G^\ell(M)$ with $\pi(W) = \tilde{U}$ which contains $j_p^\ell\,\mathrm{id}$ such that for every $q \in \tilde{U} \cap E_p$, and every $F \in \mathrm{Aut}(M, q)$ with $j_q^\ell F \in W$ we have $F = \Psi_1(\cdot, j_q^\ell F)$ on $\tilde{U}$. Let us now write for a jet $\Lambda \in G^\ell(M)$ $\Lambda_0 = \Lambda(\pi(\Lambda))$ for the image of its basepoint. We now define

$$\Psi(x,\Lambda) = \Psi_0(\psi(x,\pi(\Lambda),\Lambda_0),(j^\ell_{\pi(\Lambda)}\psi(\cdot,\pi(\Lambda),\Lambda_0))^{-1}\Lambda),$$

which has the claimed properties. $\square$

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