REGULARITY OF CR-MAPPINGS INTO LEVI-DEGENERATE HYPERSURFACES

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Abstract. We provide regularity results for CR-maps between real hypersurfaces in complex spaces of different dimension with a *Levi-degenerate target*. We address both the real-analytic and the smooth case. Our results allow immediate applications to the study of proper holomorphic maps between Bounded Symmetric Domains.

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1. Introduction

This paper is devoted to establishing smooth and real-analytic versions of the *Schwarz reflection principle* for holomorphic maps in several complex variables. In the real-analytic version of the reflection principle, we investigate conditions under which a *CR-map* between real submanifolds in complex space (or a holomorphic map between *wedges* attached to real submanifolds) extends holomorphically to an open neighborhood of the source manifold. In the smooth version, we ask for conditions under which a CR-map between real submanifolds in complex space has higher regularity than the given one. Problems of this type have attracted considerable attention since the work of Fefferman [Fe], Lewy [Le], and Pinchuk [Pi]. In the *equidimensional* case, the reflection principle is understood quite well due to the extensive research in this direction. We refer the reader to e.g. [BER, Fr1, KL1, KL2] for detailed surveys and references related to this research, as well as for the most up-to-date results.

In this paper, we study aspects of the regularity problem for CR-mappings between CR-manifolds M and M' of *different* dimension. This has been an extensively developing direction since the pioneering work of Webster [W], Faran [Fa], and Forstnerič [Fr1]. We shall note that the case of different dimensions is far more difficult than the equi-dimensional one, and much less is known in this setting. For an overview of existing results in the real-analytic case, we refer to the recent work of Berhanu and the first author [BX1].

The regularity problem in the *smooth category* rather than in the real-analytic one (in what follows, by "smooth" we refer to the C^∞ smoothness, if not otherwise stated) seems to be even more difficult due to lack of techniques. Starting from the work of Forstnerič [Fr1] and Huang [Hu1], [Hu2], the expected type of regularity of a finitely smooth CR-map between smooth CR-manifolds is *its C^∞ smoothness at a generic point.* One of the main tools for obtaining results in this line was introduced in the work [L1, L2, L3] by the second author, which is the notion of *k-nondegeneracy* of a CR-mapping. The latter is used for studying differential systems associated with CR-mappings. In particular, this tool was applied by Berhanu and the third author for studying the situation when the target manifold is *Levi-nondegenerate*. In the work [BX1], a smooth version of the reflection principle is established for CR-mappings from an abstract CR-manifold to a strongly pseudoconvex hypersurface. In particular, it solves a conjecture formulated earlier Huang [Hu2] and also reproves a conjecture of Forstnerič [Fr1] consequently. In [BX2], this type of result is extended for CR-mapping into Levi-nondegenerate CR-submanifolds of hypersurface type with certain conditions on the signature. These results in particular show that if $F : M \to M'$ is a CR-transversal CR-mapping of class C^2 from a real-analytic (resp. smooth) strictly pseudoconvex hypersurface $M \subset \mathbb{C}^n$ into a real-analytic (resp. smooth) Levi-nondegenerate hypersurface $M' \subset \mathbb{C}^{n+1}$, then F is real-analytic (resp. smooth) on a dense open subset of M (we mention that when F is assumed to be C^∞, the result in the real-analytic case was proved in [EL]).

However, the case when the target is *Levi-degenerate* remains widely open, in both smooth and real-analytic categories, and very little is known in this setting. In the real-analytic case, a number of very interesting results in the latter direction were obtained in the recent paper of Mir [Mi1].

The main goal of this paper is to extend the reflection principle for CR-maps of real hypersurfaces in complex space to the setting when the target hypersurface $M \subset \mathbb{C}^{n+1}$ is *Levi-degenerate*, while the source $M \subset \mathbb{C}^n$ is strictly pseudoconvex.

First, we obtain in the paper the generic analyticity property (resp. the generic smoothness property) for finitely smooth CR-maps between real-analytic (resp. smooth) real hypersurfaces of different dimensions with minimal assumptions for the target. Namely, in the real-analytic case, we assume the target M' to be merely *holomorphically nondegenerate*. Clearly, for any given source, the latter assumption can not be relaxed further (see Example 1.1 below). In the smooth case, we assume the *finite nondegeneracy* of the target. For definitions of different notions of nondegeneracy, see Section 2.

Second, we establish in the paper the *everywhere analyticity* (resp. *everywhere smoothness*) of CR-maps in the case when the target belongs to the class of *uniformly* 2*-nondegenerate hypersurfaces*. The latter class of hypersurfaces is of fundamental importance in Complex Analysis and Geometry. Uniformly 2-nondegenerate hypersurfaces have been recently studied intensively (e.g. Ebenfelt [E1, E2], Kaup and Zaitsev [KaZa], Fels and Kaup [FK1, FK2], Isaev and Zaitsev [IZ13], Medori and Spiro [MS], Kim and Zaitsev [KiZa], Beloshapka and the first author [BK]). These hypersurfaces naturally occur as boundaries of Bounded Symmetric Domains (see, e.g., [KaZa], [XY] for details), and in this way CR-maps into uniformly 2-nondegenerate hypersurfaces become important for understanding proper holomorphic maps between the respective Bounded Symmetric Domains (on the latter subject, see e.g. the work of Mok [Mo1, Mo2] and references therein). Uniformly 2-nondegenerate hypersurfaces occur as well as homogeneous holomorphically nondegenerate CR-manifolds [FK1, FK2]. We also note that the study of CR-embeddings of strictly-pseudoconvex hypersurfaces into 2-nondegenerate hypersurfaces performed in the present paper is important for understanding the geometry of the latter class of CR-manifolds (see, e.g., [BK]).

We shall now formulate our main results.

Theorem 1. *Let $M \subset \mathbb{C}^n \, (n \geq 2)$ be a strongly pseudoconvex real-analytic (resp. smooth) hypersurface, and $M' \subset \mathbb{C}^{n+1}$ a uniformly $2-$nondegenerate real-analytic (resp. smooth) hypersurface. Assume that $F = (F_1, ..., F_{n+1}) : M \mapsto M'$ is a CR-transversal CR-mapping of class C^2. Then F is real-analytic (resp. smooth) everywhere on M.*

We note that Theorem 1 has direct applications to the study of rigidity of proper holomorphic maps between bounded symmetric domains (see the work [XY] of Yuan and the third author, where certain rigidity results for holomorphic proper maps from the complex unit ball to the Type IV bounded symmetric domain D_m^{IV} are obtained). We also note that Theorem 1 somehow parallels a theorem proved by Mir [Mi1] and establishing the analyticity of CR-maps (at a generic point) in the situation when the source M is real-analytic and minimal while the target is the well known uniformly 2-nondegenerate hypersurface called *the tube over the future light cone*:

$$\mathbb{T}_{n+1} = \left\{ (z_1, ..., z_{n+1}) \in \mathbb{C}^{n+1} : \; (\operatorname{Im} z_1)^2 + \cdots + (\operatorname{Im} z_n)^2 = (\operatorname{Im} z_{n+1})^2 \right\}. \tag{1.1}$$

Next, in the more general setting of M', we prove

Theorem 2. *Let $M \subset \mathbb{C}^n \, (n \geq 2)$ be a strongly pseudoconvex real-analytic (resp. smooth) hypersurface, and $M' \subset \mathbb{C}^{n+1}$ an everywhere finitely nondegenerate real-analytic (resp. smooth) hypersurface. Let $F = (F_1, ..., F_{n+1}) : M \mapsto M'$ be a CR-transversal CR-mapping of class C^2. Then F is real-analytic (resp. smooth) on a dense open subset of M.*

Finally, in the real-analytic category, we prove furthermore

Theorem 3. *Let $M \subset \mathbb{C}^n \, (n \geq 2)$ be a strongly pseudoconvex real-analytic hypersurface, and $M' \subset \mathbb{C}^{n+1}$ a holomorphically nondegenerate real-analytic hypersurface. Assume that $F = (F_1, ..., F_{n+1}) : M \mapsto M'$ is a CR-transversal CR-mapping of class C^2. Then F is real-analytic on a dense open subset of M.*

As was mentioned above, for any given M, one cannot drop the holomorphic nondegeneracy assumption when expecting the generic analyticity of CR-embeddings $F : M \mapsto M'$, $M' \in \mathbb{C}^{n+1}$ (see Example 1.1 below). The transversality assumption on F cannot be dropped either. See [BX2] for an example where F (being not transversal) is not smooth on any open subset of M. Thus, the assertion of Theorem 3 is in a sense optimal.

Example 1.1. Let $M \subset \mathbb{C}^n$, $n \geq 2$ be a strongly pseudoconvex hypersurface. Consider the holomorphically degenerate hypersurface $M' = M \times \mathbb{C} \subset \mathbb{C}^{n+1}$. Let f be a C^2 CR function on M which is not smooth on any open subset of M. Then $F(Z) := (Z, f(Z))$, $Z \in M$ is a CR-transversal map of class C^2 from M to M'. Clearly, F is not smooth on any open subset of M.

The following example shows also that one cannot expect F to be real-analytic everywhere on M in the setting of Theorems 3.

Example 1.2. Let $M \subset \mathbb{C}^2$ be the strongly pseudoconvex real hypersurface defined by

$$|z|^2 + |w|^2 + |1-w|^{10} = 1$$

near $(0,1)$, where (z,w) are the coordinates in $\mathbb{C}^2$. Let $M' \subset \mathbb{C}^3$ be the holomorphically nondegenerate real hypersurface defined by

$$|z_1|^2 + |z_2|^2 + |z_3|^4 = 1,$$

where (z_1, z_2, z_3) are the coordinates in $\mathbb{C}^3$. Consider the map

$$F = (z, w, (1-w)^{5/2})$$

from one side of $M : \{|z|^2 + |w|^2 + |1-w|^{10} < 1\}$ to $\mathbb{C}^3$. It is easy to see F extends C^2-smoothly up to M, sending M to M'. However, F is not even C^3 at the point $(0,1)$.

We, however, hope that the following is true.

Conjecture 1.3. *For any integer $n \geq 2$, there exists an integer $k = k(n)$ such that the following holds. Let $M \subset \mathbb{C}^n$ $(n \geq 2)$, $M' \subset \mathbb{C}^{n+1}$ be real-analytic (resp. smooth) hypersurfaces that are finitely nondegenerate (on some dense open subsets), and $F = (F_1, ..., F_{n+1}) : M \to M'$ is a CR-transversal CR-mapping of class C^k. Then F is real-analytic (resp. smooth) on a dense open subset of M.*

(In the real-analytic version of the Conjecture, we may replace the condition on M by its holomorphic nondegeneracy).

The paper is organized as follows. In Section 2, we present some preliminaries on the degeneracy of CR-submanifolds and CR-mappings. Section 3 is devoted to a normalization result for a CR-map between hypersurfaces satisfying the assumptions of Theorems 1-3. It will be applied in later arguments. Theorem 1-3 will be proved in Sections 4-6.

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2. Preliminaries

In this section, we recall various notions of degeneracy in CR geometry, and their relations. The following definition is introduced in [BHR].

Definition 2.1. Let M be a smooth generic submanifolds in $\mathbb{C}^N$ of CR-dimension d and CR-codimension n, and $p \in M$. Let $\rho = (\rho_1, ..., \rho_d)$ be the defining function of M near p, and choose a basis $L_1, ..., L_n$ of CR vector fields near p. For a multiindex $\alpha = (\alpha_1, ..., \alpha_n)$, write $L^\alpha = L_1^{\alpha_1} ... L_n^{\alpha_n}$. Define the increasing sequence of subspaces $E_l(p) (0 \leq l \leq k)$ of $\mathbb{C}^N$ by

$$E_l(p) = \mathrm{Span}_{\mathbb{C}}\{L^\alpha \rho_{\mu,Z}(Z, \overline{Z})|_{Z=p} : 0 \leq |\alpha| \leq l, 1 \leq \mu \leq d\}.$$

Here $\rho_{\mu,Z} = (\frac{\partial \rho_\mu}{\partial z_1}, \cdots, \frac{\partial \rho_\mu}{\partial z_N})$, and $Z = (z_1, \cdots, z_N)$ are the coordinates in $\mathbb{C}^N$. We say that M is *$k-$nondegenerate at p*, $k \geq 1$ if

$$E_{k-1}(p) \neq E_k(p) = \mathbb{C}^N.$$

We say M is $k-$degenerate at p if $E_k(p) \neq \mathbb{C}^N$.

We say M is (everywhere) finitely nondegenerate if M is $k(p)-$nondegenerate at every $p \in M$ for some integer $k(p)$ depending on p. A smooth CR-manifold M of hypersurface type is Levi-nondegenerate at $p \in M$ if and only if M is $1-$nondegenerate at p. This notion of degeneracy is then generalized to CR-mappings by the second author [La1] as follows.

Definition 2.2. Let $\widetilde{M} \subset \mathbb{C}^N, \widetilde{M}' \subset \mathbb{C}^{N'}$ be two generic CR-submanifolds of CR dimension $n,\ n'$, respectively. Let $H : \widetilde{M} \to \widetilde{M}'$ be a CR-mapping of class C^k near $p_0 \in \widetilde{M}$. Let $\rho = (\rho_1, \cdots, \rho_{d'})$ be local defining functions for $\widetilde{M}'$ near $H(p_0)$, and choose a basis $L_1, \cdots, L_n$ of CR vector fields for $\widetilde{M}$ near p_0. If $\alpha = (\alpha_1, \cdots, \alpha_n)$ is a multiindex, write $L^\alpha = L_1^{\alpha_1} \cdots L_n^{\alpha_n}$. Define the increasing sequence of subspaces $E_l(p_0)(0 \leq l \leq k)$ of $\mathbb{C}^{N'}$ by

$$E_l(p_0) = \mathrm{Span}_{\mathbb{C}}\{L^\alpha \rho_{\mu,Z'}(H(Z), \overline{H(Z)})|_{Z=p_0} : 0 \leq |\alpha| \leq l, 1 \leq \mu \leq d'\}.$$

Here $\rho_{\mu,Z'} = (\frac{\partial \rho_\mu}{\partial z'_1}, \cdots, \frac{\partial \rho_\mu}{\partial z'_{N'}})$, and $Z' = (z'_1, \cdots, z'_{N'})$ are the coordinates in $\mathbb{C}^{N'}$. We say that H *is k_0-nondegenerate at p_0* $(0 \leq k_0 \leq k)$ if

$$E_{k_0-1}(p_0) \neq E_{k_0}(p_0) = \mathbb{C}^{N'}.$$

A manifold M is k_0-nondegenerate if and only if the identity map from M to M is k_0-nondegenerate. For a real-analytic submanifold, we also introduce the notion of holomorphic degeneracy.

Definition 2.3. A real-analytic submanifold $M \subset \mathbb{C}^N$ is *holomorphically nondegenerate at $p \in M$* if there is no germ at p of a holomorphic vector field X tangent to M such that $X|_M \not\equiv 0$. We shall also say that M *is holomorphically nondegenerate* if it is so at every point of it.

We recall the following proposition about $k-$nondenegeary and holomorphic nondegeneracy. For a proof of this, see [BER].

Proposition 2.4. *Let $M \subset \mathbb{C}^N$ be a connected real-analytic generic manifold with CR dimension n. Then the following conditions are equivalent:*

- *M is holomorphically nondegenerate.*
- *M is holomorphically nondegenerate at some point $p \in M$.*
- *M is $k-$nondegenerate at some point $p \in M$ for some $k \geq 1$.*
- *There exists V, a proper real-analytic subset of M and an integer $l = l(M), 1 \leq l(M) \leq n$, such that M is $l-$nondegenerate at every $p \in M \setminus V$.*

3. Normalization

In the section, we prove an auxiliary normalization result for CR-maps (Proposition 3.3 below) in the following setting. Let $M \subset \mathbb{C}^n\,(n \geq 2)$ be a strongly pseudoconvex real-analytic (resp. smooth) hypersurface defined near a point $p_0 \in M$, and $M' \subset \mathbb{C}^{n+1}$ a real-analytic (resp. smooth) hypersurface which is Levi-degenerate at a point $q_0 \in M'$. Assume that $F = (F_1, ..., F_{n+1}) : M \mapsto M'$ is a CR-transversal CR-mapping of class C^2 near p_0 with $F(p_0) = q_0$. We assume, after a holomorphic change of coordinates in $\mathbb{C}^n$, $p_0 = 0$ and that M is defined near 0 by

$$r(Z, \overline{Z}) = -\mathrm{Im} z_n + \sum_{i=1}^{n-1} |z_i|^2 + \psi(Z, \overline{Z}), \tag{3.1}$$

where $Z = (z_1, ..., z_n)$ are the coordinates in $\mathbb{C}^n$, $\psi(Z, \overline{Z}) = O(|Z|^3)$ is real-analytic (resp. smooth) function defined near 0.

After a holomorphic change of coordinates in $\mathbb{C}^{n+1}$, we assume that $q_0 = F(p_0) = 0$ and that M' is locally defined near 0 by

$$\rho(W, \overline{W}) = -\mathrm{Im} w_{n+1} + \tilde{W} U \overline{\tilde{W}}^t + \phi(W, \overline{W}), \tag{3.2}$$

for some Hermitian $n \times n$ matrix U. Here $W = (\tilde{W}, w_{n+1}) = (w_1, ..., w_n, w_{n+1})$ are the coordinates in $\mathbb{C}^{n+1}$, $\phi(W, \overline{W}) = O(|W|^3)$ is a real-analytic (resp. smooth) function defined near 0.

If we write $F = (\tilde{F}, F_{n+1}) = (F_1, ..., F_n, F_{n+1})$, then F satisfies:

$$-\frac{F_{n+1} - \overline{F_{n+1}}}{2i} + \tilde{F} U \overline{\tilde{F}}^t + \phi(F, \overline{F}) = 0, \tag{3.3}$$

along M. Since F is CR-transversal, we get $\lambda := \frac{\partial F_{n+1}}{\partial s}|_0 \neq 0$, where we write $z_n = s + it$(cf. [BER]). Moreover, (3.3) shows that the imaginary part of F_{n+1} vanishes to second order at the origin, and so the number λ is real. By applying the change of coordinates in $\mathbb{C}^{n+1}$: $\tau(w_1, ..., w_n, w_{n+1}) = (w_1, ..., w_n, -w_{n+1})$ if necessary, we may assume that $\lambda > 0$. Let us write

$$L_j = 2i \left(\frac{\partial r}{\partial z_n} \frac{\partial}{\partial \overline{z_j}} + \frac{\partial r}{\partial \overline{z_j}} \frac{\partial}{\partial \overline{z_n}} \right), 1 \leq j \leq n-1. \tag{3.4}$$

Then $\{L_j\}_{1 \leq j \leq n-1}$ forms a basis for the CR vector fields along M near p. By applying $\overline{L_j}, \overline{L_j L_k}, 1 \leq j, k \leq n-1$ to the equation (3.3) and evaluating at 0, we get:

$$\frac{\partial F_{n+1}}{\partial z_j}(0) = 0, \frac{\partial^2 F_{n+1}}{\partial z_j \partial z_k}(0) = 0, 1 \leq j, k \leq n-1.$$

Hence we have,

$$F_{n+1}(Z) = \lambda z_n + O(|Z|^2). \tag{3.5}$$

For $1 \leq j \leq n$, we write

$$F_j = a_j z_n + \sum_{i=1}^{n-1} a_{ij} z_i + O(|Z|^2), \tag{3.6}$$

for some $a_j \in \mathbb{C}, a_{ij} \in \mathbb{C}, 1 \leq i \leq n-1, 1 \leq j \leq n$. Or equivalently,

$$(F_1, ..., F_n) = z_n(a_1, ..., a_n) + (z_1, ..., z_{n-1})A + (\hat{F}_1, ..., \hat{F}_n), \tag{3.7}$$

where $A = (a_{ij})_{1\leq i\leq n-1, 1\leq j\leq n}$ is an $(n-1) \times n$ matrix, and $\hat{F}_j = O(|Z|^2), 1 \leq j \leq n$. We plug in (3.5) and (3.7) into (3.3) to get,

$$\lambda|\widetilde{Z}|^2 + O(|\widetilde{Z}||z_n| + |z_n|^2) + o(|Z|^2) = \widetilde{z}AUA^*\overline{\widetilde{z}}^t + O(|\widetilde{Z}||z_n| + |z_n|^2) + o(|Z|^2), \tag{3.8}$$

where we write $\widetilde{Z} = (z_1, ..., z_{n-1})$. Equip $\widetilde{Z}$ with weight 1, and z_n with weight 2. We then compare terms with weight 2 at both sides of (3.8) to get:

$$\lambda \mathbf{I}_{n-1} = AUA^*. \tag{3.9}$$

As a consequence, the matrix A has full rank $(n-1), U$ has rank $(n-1)$ or n. Recall that M' is not $1-$nondegenerate at $q = 0$. We thus conclude that U has rank $(n-1)$. Moreover, note from (3.9) that U has $(n-1)$ positive eigenvalues. By a holomorphic change of coordinates in $\mathbb{C}^{n+1}$, we may assume that $U = \mathrm{diag}\{1, ..., 1, 0\}$. M' is then of the following form near 0 :

$$\rho(W, \overline{W}) = -\mathrm{Im} w_{n+1} + \sum_{j=1}^{n-1} |w_j|^2 + \phi(W, \overline{W}), \quad \phi = O(|W|^3). \tag{3.10}$$

Write $A = (B, \mathbf{b})$, where B is a $(n-1)\times(n-1)$ matrix, b is an $(n-1)-$dimensional column vector. (3.9) yields that $B\overline{B}^t = \lambda \mathbf{I}_{n-1}$. We now apply the following holomorphic change of coordinates: $\widetilde{W} = WD$ or $W = \widetilde{W}D^{-1}$, where we set

$$D = \begin{pmatrix} \frac{1}{\sqrt{\lambda}}\overline{B}^t & \mathbf{c} & \mathbf{0} \\ \mathbf{0}^t & 1 & 0 \\ \mathbf{0}^t & 0 & 1 \end{pmatrix},$$

and $\mathbf{0}$ is the $(n-1)-$dimensinal zero column vector, $\mathbf{c}$ is a $(n-1)-$dimensional column vector to be determined. We compute

$$D^{-1} = \begin{pmatrix} \frac{1}{\sqrt{\lambda}}B & \mathbf{d} & \mathbf{0} \\ \mathbf{0}^t & 1 & 0 \\ \mathbf{0}^t & 0 & 1 \end{pmatrix},$$

where $\mathbf{d} = -\frac{1}{\sqrt{\lambda}}B\mathbf{c}$.

We write the new defining function of M' and the map as $\widetilde{\rho}$ and $\widetilde{F} = (\widetilde{F}_1, ..., \widetilde{F}_{n+1})$ in the new coordinates $\widetilde{W} = (\widetilde{w}_1, ..., \widetilde{w}_{n+1})$, respectively. We have

Lemma 3.1. *$\widetilde{\rho}$ still has the form of (3.10). More precisely,*

$$\widetilde{\rho}(\widetilde{W}, \overline{\widetilde{W}}) = -\mathrm{Im}\widetilde{w}_{n+1} + \sum_{j=1}^{n-1} |\widetilde{w}_j|^2 + \widetilde{\phi}(\widetilde{W}, \overline{\widetilde{W}}),$$

where $\widetilde{\phi}(\widetilde{W}, \overline{\widetilde{W}}) = O(|\widetilde{W}|^3)$ is also a real-analytic (resp. smooth) function defined near 0.

Proof. This can be checked by a simple calculation and using the fact that

$$\begin{pmatrix} \frac{1}{\sqrt{\lambda}}B & \mathbf{d} \\ \mathbf{0}^t & 1 \end{pmatrix} \begin{pmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0}^t & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\lambda}}\overline{B}^t & \mathbf{0} \\ \overline{\mathbf{d}}^t & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{I}_n & \mathbf{0} \\ \mathbf{0}^t & 0 \end{pmatrix}.$$

□

Moreover, since $\widetilde{F} = FD$, it is easy to see that

$$\frac{\partial \widetilde{F}_i}{\partial z_j}(0) = \delta_{ij}\sqrt{\lambda},\ 1 \leq i, j \leq n-1.$$

Here we denote by δ_{ij} the Kronecker symbol that takes value 1 when $i = j$ and 0 otherwise.

Lemma 3.2. *We can choose an appropriate* $\mathbf{c}$ *such that*

$$\frac{\partial \widetilde{F}_n}{\partial z_j}(0) = 0, 1 \leq j \leq n-1. \tag{3.11}$$

Proof: *Note that* $\widetilde{F}_n = (F_1, ..., F_n) \begin{pmatrix} \mathbf{c} \\ 1 \end{pmatrix}$. *Combining this with (3.7), we obtain,*

$$\frac{\partial \widetilde{F}_n}{\partial z_j}(0) = 0, 1 \leq j \leq n-1$$

is equivalent to $A \begin{pmatrix} \mathbf{c} \\ 1 \end{pmatrix} = \mathbf{0}$, *where* $\mathbf{0}$ *is the* $(n-1)-$*dimensinal zero column vector. Recall* $A = (B, \mathbf{b})$. *We can thus choose* $\mathbf{c} = -B^{-1}\mathbf{b}$.

In the following, for brevity, we still write W, F and ρ instead of $\widetilde{W}, \widetilde{F}$ and $\widetilde{\rho}$. We summarize the considerations of this section in the following

Proposition 3.3. *Let* $M \subset \mathbb{C}^n (n \geq 2)$ *be a strongly pseudoconvex real-analytic (resp. smooth) real hypersurface,* $M' \subset \mathbb{C}^{n+1}$ *a real-analytic (resp. smooth) real hypersurface. Assume that* $F = (F_1, ..., F_{n+1}) : M \mapsto M'$ *is a CR-transversal CR-mapping of class* C^2 *near* $p_0 \in M$ *with* $F(p_0) = q_0$, *and that* M' *is Levi-degenerate at* q_0. *Then, after appropriate holomorphic changes of coordinates in* $\mathbb{C}^n$ *and* $\mathbb{C}^{n+1}$ *respectively, we have* $p_0 = 0, q_0 = 0$, *and the following normalizations hold.* M *is defined by*

$$r(Z, \overline{Z}) = -\mathrm{Im} z_n + \sum_{i=1}^{n-1} |z_i|^2 + \psi(Z, \overline{Z}), \quad \psi = O(|Z|^3) \tag{3.12}$$

near 0, *and* M' *is defined by*

$$\rho(W, \overline{W}) = -\mathrm{Im} w_{n+1} + \sum_{j=1}^{n-1} |w_j|^2 + \phi(W, \overline{W}), \quad \phi = O(|W|^3) \tag{3.13}$$

near 0, *where* $Z = (z_1, ..., z_n), W = (w_1, ..., w_{n+1})$ *are the coordinates of* $\mathbb{C}^n$ *and* $\mathbb{C}^{n+1}$, *respectively. Furthermore,* F *satisfies:*

$$\frac{\partial F_i}{\partial z_j}(0) = \delta_{ij}\sqrt{\lambda},\ 1 \leq i, j \leq n-1, \tag{3.14}$$

for some $\lambda > 0$, *and moreover,*

$$\frac{\partial F_n}{\partial z_j}(0) = 0, 1 \leq j \leq n-1; \tag{3.15}$$

$$\frac{\partial F_{n+1}}{\partial z_j}(0) = 0, 1 \leq j \leq n-1. \tag{3.16}$$

4. Proof of Theorem 1

In this section we prove Theorem 1. We first make some basic computations for the uniformly $2-$nondegenerate target hyperurface M'. For further results about normal forms along this line, see [E1]. We will write for $1 \leq k \leq n$,

$$\Lambda_k = 2i\left(\frac{\partial \rho}{\partial \overline{w}_{n+1}}\frac{\partial}{\partial \overline{w}_k} - \frac{\partial \rho}{\partial \overline{w}_k}\frac{\partial}{\partial \overline{w}_{n+1}}\right), \tag{4.1}$$

where $\{\Lambda_k\}_{1\leq k\leq n}$ forms a basis for the CR vector fields along M' near 0. Note that

$$\begin{aligned} \Lambda_k &= (1 + 2i\phi_{\overline{n+1}})\frac{\partial}{\partial \overline{w}_k} - 2i(w_k + \phi_{\overline{k}})\frac{\partial}{\partial \overline{w}_{n+1}},\ \text{if } 1 \leq k \leq n-1, \\ \Lambda_n &= (1 + 2i\phi_{\overline{n+1}})\frac{\partial}{\partial \overline{w}_n} - 2i(\phi_{\overline{n}})\frac{\partial}{\partial \overline{w}_{n+1}}. \end{aligned} \tag{4.2}$$

Here and in the following, we write for $1 \leq i, j, k \leq n+1$, $\phi_{\overline{i}} = \phi_{\overline{w}_i} = \frac{\partial \phi}{\partial \overline{w}_i}, \phi_i = \phi_{w_i} = \frac{\partial \phi}{\partial w_i}, \phi_{i\overline{j}} = \phi_{w_i\overline{w}_j} = \frac{\partial^2 \phi}{\partial w_i \partial \overline{w}_j}, \phi_{\overline{ij}k} = \phi_{\overline{w}_i\overline{w}_j w_k} = \frac{\partial^3 \phi}{\partial \overline{w}_i \partial \overline{w}_j \partial w_k}$, etc.

Recall our notation $\rho_W := (\frac{\partial \rho}{\partial w_1}, ..., \frac{\partial \rho}{\partial w_{n+1}})$. We compute

$$\rho_W(W, \overline{W}) = (\overline{w}_1 + \phi_1, ..., \overline{w}_{n-1} + \phi_{n-1}, \phi_n, \frac{i}{2} + \phi_{n+1}) \tag{4.3}$$

We thus have

$$\Lambda_1 \rho_W(W, \overline{W}) = \left(h_{11}, ..., h_{1(n+1)}\right), \tag{4.4}$$

where

$$\begin{aligned} h_{11} &= (1 + 2i\phi_{\overline{(n+1)}})(1 + \phi_{1\overline{1}}) - 2i(w_1 + \phi_{\overline{1}})\phi_{1\overline{(n+1)}}, \\ h_{12} &= (1 + 2i\phi_{\overline{(n+1)}})\phi_{2\overline{1}} - 2i(w_1 + \phi_{\overline{1}})\phi_{2\overline{(n+1)}}, \\ &\cdots, \\ h_{1(n+1)} &= (1 + 2i\phi_{\overline{(n+1)}})\phi_{(n+1)\overline{1}} - 2i(w_1 + \phi_{\overline{1}})\phi_{(n+1)\overline{(n+1)}}. \end{aligned} \tag{4.5}$$

Hence

$$\Lambda_1 \rho_W(W, \overline{W}) = (1 + O(1), O(1), ..., O(1)). \tag{4.6}$$

Here we write $O(m) = O(|W|^m)$ for any $m \geq 0$. Similarly, we have for $1 \leq k \leq n-1$,

$$\Lambda_k \rho_W(W, \overline{W}) = (O(1), ..., O(1), 1 + O(1), O(1), ..., O(1)), \tag{4.7}$$

where the term $1+O(1)$ is at the k^{th} position;

$$\Lambda_n\rho_W(W,\overline{W})=(O(1),...,O(1),\phi_{n\overline{n}}+O(2),O(1)). \tag{4.8}$$

As a consequence, we have

$$\det\begin{pmatrix}\rho_W(W,\overline{W})\\ \Lambda_1\rho_W(W,\overline{W})\\ \cdots\\ \Lambda_n\rho_W(W,\overline{W})\end{pmatrix}=\pm\frac{i}{2}\phi_{n\overline{n}}+O(2). \tag{4.9}$$

Recall that M' is uniformly $2-$nondegenerate at 0, in particular, it is $1-$degenerate at every point near 0. This implies (4.9) is identically zero near 0 along M'. Consequently, by applying $\Lambda_j, 1\le j\le n$ to (4.9) and evaluating at 0, we obtain $\phi_{\overline{jn}n}(0)=0$ for any $1\le j\le n$.

By the fact that M' is uniformly 2-nondegenerate again (see Remark 5.2 below), we have:

$$\phi_{\overline{j_0k_0}n}(0)\neq 0, \text{ for some } 1\le j_0,k_0\le n-1. \tag{4.10}$$

Consequently, if we write

$$L_{j_0}L_{k_0}\rho_W(F,\overline{F})(0):=(\nu_1,...,\nu_{n-1},\nu_n,\nu_{n+1}), \tag{4.11}$$

then ν_n is nonzero. Here L_j is as defined in (3.4). Indeed,

$$\nu_n=\frac{\partial^2\phi_n(F,\overline{F})}{\partial\overline{z}_{j_0}\partial\overline{z}_{k_0}}\bigg|_0=\sum_{i,j=1}^{n+1}\frac{\partial^2\phi_n}{\partial\overline{w}_i\partial\overline{w}_j}\bigg|_0\overline{\frac{\partial F_i}{\partial z_{j_0}}}\bigg|_0\overline{\frac{\partial F_j}{\partial z_{k_0}}}\bigg|_0=\phi_{\overline{j_0k_0}n}(0)\overline{\frac{\partial F_{j_0}}{\partial z_{j_0}}}\bigg|_0\overline{\frac{\partial F_{k_0}}{\partial z_{k_0}}}\bigg|_0\neq 0.$$

Moreover, it is easy to verify that

$$L_i\rho_W(F,\overline{F})(0)=(0,...,0,\sqrt{\lambda},0,...,0), 1\le i\le n-1, \tag{4.12}$$

where $\sqrt{\lambda}$ is at the i^{th} position, and that

$$\rho_W(F,\overline{F})(0)=(0,...,0,\frac{i}{2}). \tag{4.13}$$

Equations (4.13), (4.12) and (4.11) with $\nu_n\neq 0$ imply that F is $2-$nondegenerate at 0 in the sense of [L1, L2]. By the results of [L1, L2], F is real-analytic (resp. smooth) near 0, as required.

$\square$

5. Proof of Theorem 2

Let M' be as above and ρ as in (3.13). For any $1\le i_1\le\cdots\le i_l\le n, q\in M'$, we define,

$$\Delta_{i_1...i_l}(q)=\begin{vmatrix}\rho_W\\ \Lambda_1\rho_W\\ ...\\ \Lambda_{n-1}\rho_W\\ \Lambda_{i_1}...\Lambda_{i_l}\rho_W\end{vmatrix}(q).$$

We first prove the following lemma.

Lemma 5.1. *Let M' be as above. Assume that M' is l-nondegenerate at 0 for some $l \geq 2$. Then there exist $1 \leq i_1 \leq ... \leq i_l \leq n$, such that*

$$\Delta_{i_1 ... i_l}(0) \neq 0. \tag{5.1}$$

Proof. We note that

$$\rho_W(0) = (0, ..., 0, \frac{i}{2}), \tag{5.2}$$

$$\Lambda_j \rho_W(0) = (0, .., 0, 1, 0, ..., 0), 1 \leq j \leq n-1, \tag{5.3}$$

where 1 is at the j^{th} position. Thus $\rho_W(0), \Lambda_j \rho_W(0), 1 \leq j \leq n-1$, are linearly independent over $\mathbb{C}$. Then by the definition of l-nondegeneracy at 0, one easily sees that there exists $1 \leq i_1 \leq ... \leq i_l \leq n$ such that

$$\begin{vmatrix} \rho_W \\ \Lambda_1 \rho_W \\ ... \\ \Lambda_{n-1} \rho_W \\ \Lambda_{i_1} ... \Lambda_{i_l} \rho_W \end{vmatrix} (0) \neq 0.$$

□

Remark 5.2. In particular, when $l = 2$ in Lemma 5.1, we have there exist $1 \leq i_1 \leq i_2 \leq n$, such that $\Delta_{i_1 i_2}(0) \neq 0$. Note the n^{th} component of $\Lambda_{i_1} \Lambda_{i_2} \rho_W(0)$ is $\phi_{\overline{i_1 i_2} n}(0)$. By the form (5.2), (5.3) of $\rho_W(0)$ and $\Lambda_j \rho_W(0)$, we conclude that $\phi_{\overline{i_1 i_2} n}(0) \neq 0$.

We then prove the following proposition.

Proposition 5.3. *Let $M \subset \mathbb{C}^n (n \geq 2)$ be a strongly pseudoconvex real-analytic (resp. smooth) hypersurface, and $M' \subset \mathbb{C}^{n+1}$ be a real-analytic (resp. smooth) hypersurface. Assume that M' is either 1- or 2-nondegenerate at every point of it. Let $F = (F_1, ..., F_{n+1}) : M \mapsto M'$ be a CR-transversal CR-mapping of class C^2. Then F is real-analytic (resp. smooth) on a dense open subset of M.*

Proof. We write Ω as the open subset of M where F is real-analytic (resp. smooth). Fix any $p_0 \in M$. Write $q_0 = F(p_0) \in M'$. We will need to prove $p_0 \in \overline{\Omega}$. We assume $p_0 = 0 \in M, q_0 = 0 \in M'$. By assumption, M' is either 1-nondegenerate or 2-nondegenerate at q_0. We then split our argument in two cases.

Case I: M' is 1-nondegenerate at q_0. That is, M' is Levi-nondegenerate near q_0. Then it follows from Corollary 2.3 in [BX2] that $p_0 \in \overline{\Omega}$.

Case II: M' is 2-nondegenerate at q_0. Let O be a small neighborhood of q_0 in $\mathbb{C}^{n+1}$. Let $V = O \cap M'$. We write V_1 as the set of 1-degeneracy of M' in V. More precisely,

$$V_1 = \{q \in V : M' \text{ is 1-degenerate at } q\}.$$

If there is a sequence $\{p_i\}_{i=1}^{\infty} \subset M$ converging to p_0 such that M' is $1-$nondegenerate at each $F(p_i)$, i.e., $F(p_i) \in M \setminus V_1, i \geq 1$. Then by Case I, we have each $p_i \in \overline{\Omega}, i \geq 1$. Consequently, $p_0 \in \overline{\Omega}$.

Thus we are only left with the case that there exists a neighborhood U of p such that $F(U) \subset V_1$. We apply then normalization to M, M' and the map F as in Proposition 3.3. Since M' is 2-nondegenerate at 0, we conclude again by Lemma 5.1, $\Delta_{j_0k_0}(0) = c \neq 0$, for some $1 \leq j_0 \leq k_0 \leq n$. We then further split into the following subcases.

Case II(a): There exist some $1 \leq j_0 \leq k_0 \leq n-1$, such that, $\Delta_{j_0k_0}(0) = c \neq 0$. Consequently, we have $\phi_{\overline{j_0k_0n}}(0) \neq 0$. Then similarly as in the proof of Theorem 1, we can show that F is finitely nondegenerate. Hence again by the results of [L1, L2], F is real-analytic (resp. smooth) at 0.

Case II(b): For any $1 \leq j \leq k \leq n-1$, $\Delta_{jk}(0) = 0$. Moreover, there exists $1 \leq j_0 \leq n-1$ such that, $\Delta_{j_0n}(0) = c \neq 0$. Then by a similar argument as in Remark 5.2, we conclude $\phi_{\overline{jkn}}(0) = 0$, for any $1 \leq j \leq k \leq n-1$, and $\phi_{\overline{j_0nn}}(0) \neq 0$.

Note that $V_1 \subset \widetilde{V}_1 \bigcap M'$, where $\widetilde{V}_1$ is defined

$$\widetilde{V}_1 := \{W \in O : \varphi(W, \overline{W}) = 0\}, \tag{5.4}$$

with

$$\varphi(W, \overline{W}) = \det \begin{pmatrix} \rho_W \\ \Lambda_1 \rho_W \\ ... \\ \Lambda_{n-1}\rho_W \\ \Lambda_n \rho_W \end{pmatrix} (W, \overline{W}). \tag{5.5}$$

Then we have

Lemma 5.4. *The $\overline{w}_{j_0}$-derivative of φ is nonzero at $q_0 = 0$.*

Proof. It is equivalent to show that $\Lambda_{j_0}\varphi(0) \neq 0$. That is,

$$\Lambda_{j_0} \begin{vmatrix} \rho_W \\ \Lambda_1 \rho_W \\ ... \\ \Lambda_{n-1}\rho_W \\ \Lambda_n \rho_W \end{vmatrix} (0) \neq 0. \tag{5.6}$$

Note that

$$\Lambda_{j_0} \begin{vmatrix} \rho_W \\ \Lambda_1 \rho_W \\ ... \\ \Lambda_{n-1}\rho_W \\ \Lambda_n \rho_W \end{vmatrix} (0) =$$

$$\begin{vmatrix} \Lambda_{j_0}\rho_W \\ \Lambda_1 \rho_W \\ ... \\ \Lambda_{n-1}\rho_W \\ \Lambda_n \rho_W \end{vmatrix} (0) + \begin{vmatrix} \rho_W \\ \Lambda_{j_0}\Lambda_1 \rho_W \\ ... \\ \Lambda_{n-1}\rho_W \\ \Lambda_n \rho_W \end{vmatrix} (0) + ... + \begin{vmatrix} \rho_W \\ \Lambda_1 \rho_W \\ ... \\ \Lambda_{j_0}\Lambda_{n-1}\rho_W \\ \Lambda_n \rho_W \end{vmatrix} (0) + \begin{vmatrix} \rho_W \\ \Lambda_1 \rho_W \\ ... \\ \Lambda_{n-1}\rho_W \\ \Lambda_{j_0}\Lambda_n \rho_W \end{vmatrix} (0).$$

In the above equation, the first term is trivially zero. Then we note that in the row vector $\rho_W(0)$, or $\Lambda_i\rho_W(0), 1 \le i \le n$, the n^{th} component is zero. This is due to the fact that $\phi = O(|W|^3)$. Moreover, the n^{th} component in the row vector $\Lambda_{j_0}\Lambda_k\rho_W(0), 1 \le k \le n-1$, is $\phi_{\overline{j_0 k n}}(0)$, which is zero by the assumption. Consequently, the second term upto the n^{th} term in the above equation are all zero. We also note the last term in the equation above is just equal to $\Delta_{j_0 n}(0)$, which is nonzero. Hence the lemma is established. □

Recall that $F(U) \subset V_1 \subset \widetilde{V}_1$. We have

$$\varphi(F(Z), \overline{F(Z)}) \equiv 0, \text{ for all } Z \in U. \tag{5.7}$$

Applying L_{j_0} to the above equation and evaluating at $Z = 0$, we have,

$$L_{j_0}\varphi(F, \overline{F})|_0 = \sum_{i=1}^{n+1} \varphi_{\overline{w}_i}(F, \overline{F})|_0 L_{j_0}\overline{F}_i|_0 = 0. \tag{5.8}$$

Note that by our normalization, $L_{j_0}\overline{F}_i(0) = 0$, if $i \neq j_0$. $L_{j_0}\overline{F}_{j_0}(0) \neq 0$. Moreover, by Lemma 5.4, $\varphi_{\overline{w}_{j_0}}(0) \neq 0$. This is a contradiction to (5.8). Hence **Case II(b)** cannot happen in this setting.

Case II(c): $\Delta_{jk}(0) = \Delta_{jn}(0) = 0$, for all $1 \le j, k \le n-1$, and $\Delta_{nn}(0) \neq 0$.
We let $\widetilde{V}_1$ be defined by φ as above in (5.4), (5.5).

Lemma 5.5. *In the setting of this subcase, the $\bar{w}_n$-derivative of φ is nonzero at $q_0 = 0$.*

Proof. Similar as Lemma 5.4. □

By Lemma 5.5, we have $\varphi_{\overline{w}_n}(0) \neq 0$. Consequently, if we define $\widetilde{\varphi}(W, \overline{W}) = \overline{\varphi(W, \overline{W})}$, then

$$\widetilde{\varphi}_{w_n}(0) \neq 0. \tag{5.9}$$

Note $F(U) \subset V_1 \subset \widetilde{V}_1$. We have

$$\varphi(F(Z), \overline{F(Z)}) \equiv 0, \text{ for all } Z \in U.$$

Consequently,

$$\widetilde{\varphi}(F(Z), \overline{F(Z)}) \equiv 0. \tag{5.10}$$

Recall that for all $Z \in U$,

$$\rho(F, \overline{F}) = 0, \tag{5.11}$$

$$L_i\rho(F, \overline{F}) = 0, 1 \le i \le n-1. \tag{5.12}$$

Combining (4.12), (4.13), (5.9), we conclude that

$$\left|\begin{matrix} \rho_W(F, \overline{F}) \\ L_1\rho_W(F, \overline{F}) \\ \ldots \\ L_{n-1}\rho_W(F, \overline{F}) \\ \widetilde{\varphi}_W(F, \overline{F}) \end{matrix}\right|_{Z=0} \tag{5.13}$$

is nondegenerate. This implies that equations (5.10), (5.11), (5.12) forms a nondegenerate system for F. Then it follows that F is real-analytic (resp. smooth) at 0 by a similar argument as in [L1, L2] or [BX1, BX2]. For the convenience of the readers, we sketch a proof here for the real-analytic categary. The proof for the smooth categary is essentially the same. We assume that M is defined near 0 by $\{(z, z_n) = (z, s+it) \in U \times V : t = \phi(z, \overline{z}, s)\}$, where ϕ is a real-valued, real-analytic function with $\phi(0) = 0, d\phi(0) = 0$. Here $U \subset \mathbb{C}^{n-1}$ and $V \subset \mathbb{R}$ are sufficiently small open subsets. In the local coordinates $(z, s) \in \mathbb{C}^{n-1} \times \mathbb{R}$, we may assume that,

$$L_j = \frac{\partial}{\partial \overline{z}_j} - i\frac{\phi_{\overline{z}_j}(z, \overline{z}, s)}{1 + i\phi_s(z, \overline{z}, s)}\frac{\partial}{\partial s}, 1 \leq j \leq n-1.$$

Since ϕ is real-analytic, we can complexify in s variable and write

$$M_j = \frac{\partial}{\partial \overline{z}_j} - i\frac{\phi_{\overline{z}_j}(z, \overline{z}, s+it)}{1 + i\phi_s(z, \overline{z}, s+it)}\frac{\partial}{\partial s}, 1 \leq j \leq n-1,$$

which are holomorphic in $s+it$ and extend the vector fields L_j.

Since ϕ and L_j are real-analytic now, equations (5.10), (5.11), (5.12) implies that there is real-analytic map $\Phi(W, \overline{W}, \Theta)$ defined in a neighborhood of $\{0\} \times \mathbb{C}^q$ in $\mathbb{C}^{n+1} \times \mathbb{C}^q$, polynomial in the last q variables for some integer q such that

$$\Phi(F, \overline{F}, (L^\alpha \overline{F})_{1 \leq |\alpha| \leq 2}) = 0$$

at $(z, s) \in U \times V$. By (5.13) the matrix Φ_W is invertible at the central point 0, by the holomorphic version of the implicit function theorem(In the smooth categary, we apply the "almost holomorphic" version of the implicit function theorem, cf. [L1]), we get a holmorphic map $\Psi = (\Psi_1, ..., \Psi_{n+1})$ such that for (z, s) near the origin,

$$F_j = \Psi_j(\overline{F}, (L^\alpha \overline{F})_{1 \leq |\alpha| \leq 2}), 1 \leq j \leq n+1.$$

We now set for each $1 \leq j \leq n+1$,

$$h_j(z, s, t) = \Psi_j\left(\overline{F}(z, s, -t), (M^\alpha \overline{F})_{1 \leq |\alpha| \leq 2}(z, s, -t)\right).$$

Since M is strongly pseudoconvex, the CR functions $F_j, 1 \leq j \leq n+1$, all extends as holomorphic functions in $s+it$ to the side $t > 0$. Hence the conjugates $\overline{F}_j, 1 \leq j \leq n+1$, extends holomorphically to the side $t < 0$. It now follows that $F_j, 1 \leq j \leq n+1$, extends as holomorphic functions to a full neighborhood of the origin (See Lemma 9.2.9 in [BER]). This establishes Proposition 5.3.

□

We then prove Theorem 2.

Again we write Ω as the open subset of M where F is real-analytic (resp. smooth). Fix any $p_0 \in M$ and $q_0 = F(p_0) \in M'$. We need to show that $p_0 \in \overline{\Omega}$ to establish the theorem. Assume that $p_0 = 0, q_0 = 0$. By assumption, M' is l-nondegenerate at q_0 for some $l \geq 1$. We note that if $1 \leq l \leq 2$, it follows from Proposition 5.3 that $p_0 \in \overline{\Omega}$. We thus assume that $l \geq 3$. We will establish the result by induction.

We start with the case when $l=3$. Notice that if there is a sequence $\{p_i\}_{i=1}^{\infty} \subset M$ converging to p_0 such that M' is at most 2-nondegenerate at $F(p_i)$ for all $i \geq 1$. Then the result again follows from Proposition 5.3. Thus we are only left with the case that there exists a neighborhood U of p_0 such that $F(U) \subset V_2$, where V_2 is the set of 2-degeneracy of M' near p_0. More precisely,

$$V_2 = \{q \in V : M' \text{ is 2-degenerate at } q\},$$

for some small neighborhood $V = O \cap M'$ of q_0. Here O is a small neighborhood of q_0 in $\mathbb{C}^{n+1}$. Since M' is 3-nondegenerate at $q=0$. By Lemma 5.1, $\Delta_{i_0 j_0 k_0}(0) = c \neq 0$, for some $1 \leq i_0 \leq j_0 \leq k_0 \leq n$. We split our argument into two cases.

Case I: We first suppose that $i_0 \leq n-1$. Note that $F(U) \subset V_2 \subset \widetilde{V}_2$, where

$$\widetilde{V}_2 = \{W \in O : \varphi_2(W, \overline{W}) = 0\}.$$

Here

$$\varphi_2(W,\overline{W}) = \begin{vmatrix} \rho_W \\ \Lambda_1 \rho_W \\ ... \\ \Lambda_{n-1}\rho_W \\ \Lambda_{j_0}\Lambda_{k_0}\rho_W \end{vmatrix} (W,\overline{W}) \tag{5.14}$$

We have

Lemma 5.6. *The $\overline{w}_{i_0}$-derivative of φ_2 is nonzero at $q_0 = 0$.*

Proof. It is equivalent to show that $\Lambda_{i_0}\varphi_2(0) \neq 0$. That is,

$$\Lambda_{i_0} \begin{vmatrix} \rho_W \\ \Lambda_1 \rho_W \\ ... \\ \Lambda_{n-1}\rho_W \\ \Lambda_{j_0}\Lambda_{k_0}\rho_W \end{vmatrix} (0) \neq 0 \tag{5.15}$$

Note that

$$\Lambda_{i_0} \begin{vmatrix} \rho_W \\ \Lambda_1 \rho_W \\ ... \\ \Lambda_{n-1}\rho_W \\ \Lambda_{j_0}\Lambda_{k_0}\rho_W \end{vmatrix} (0) =$$

$$\begin{vmatrix} \Lambda_{i_0}\rho_W \\ \Lambda_1 \rho_W \\ ... \\ \Lambda_{n-1}\rho_W \\ \Lambda_{j_0}\Lambda_{k_0}\rho_W \end{vmatrix} (0) + \begin{vmatrix} \rho_W \\ \Lambda_{i_0}\Lambda_1 \rho_W \\ ... \\ \Lambda_{n-1}\rho_W \\ \Lambda_{j_0}\Lambda_{k_0}\rho_W \end{vmatrix} (0) + ... + \begin{vmatrix} \rho_W \\ \Lambda_1 \rho_W \\ ... \\ \Lambda_{i_0}\Lambda_{n-1}\rho_W \\ \Lambda_{j_0}\Lambda_{k_0}\rho_W \end{vmatrix} (0) + \begin{vmatrix} \rho_W \\ \Lambda_1 \rho_W \\ ... \\ \Lambda_{n-1}\rho_W \\ \Lambda_{i_0}\Lambda_{j_0}\Lambda_{k_0}\rho_W \end{vmatrix} (0) \tag{5.16}$$

We claim that the first term up to the n^{th} term above are all zero. Indeed, otherwise, M' is at most 2−nondegenerate at 0. This is a contraction to our assumption.

We finally note the last term in the above equation is just equal to $\Delta_{i_0 j_0 k_0}(0)$, which is nonzero. This establishes the lemma. □

Recall $F(U) \subset V_2 \subset \widetilde{V}_2$. We have

$$\varphi_2(F(Z), \overline{F(Z)}) \equiv 0, \text{ for all } Z \in U. \tag{5.17}$$

Applying L_{j_0} to the above equation and evaluating at $Z = 0$, we have,

$$L_{i_0}\varphi_2(F, \overline{F})|_0 = \sum_{i=1}^{n+1} (\varphi_2)_{\overline{w}_i}(F, \overline{F})|_0 L_{i_0}\overline{F}_i|_0 = 0. \tag{5.18}$$

Note that by our normalization, $L_{i_0}\overline{F}_i(0) = 0$, if $i \neq i_0$. $L_{i_0}\overline{F}_{i_0}(0) \neq 0$. Moreover, by Lemma 5.6, $\varphi_{\overline{w}_{i_0}}(0) \neq 0$. This is a contradiction to (5.18). Hence **Case I** cannot happen in this setting.

Case II: We are thus only left with the case if $i_0 = j_0 = k_0 = n$. Again we define

$$\widetilde{V}_2 = \{W \in O : \varphi_2(W, \overline{W}) = 0\},$$

where

$$\varphi_2(W, \overline{W}) = \begin{vmatrix} \rho_W \\ \Lambda_1 \rho_W \\ \cdots \\ \Lambda_{n-1}\rho_W \\ \Lambda_n \Lambda_n \rho_W \end{vmatrix} (W, \overline{W}). \tag{5.19}$$

By a similar argument as before, we are able to prove the following.

Lemma 5.7. *The $\overline{w}_n$-derivative of φ_2 is nonzero at* 0.

As a consequence of Lemma 5.7, if we define $\widetilde{\varphi}_2(W, \overline{W}) = \overline{\phi_2(W, \overline{W})}$, then,

$$(\widetilde{\varphi}_2)_{w_n}(0) \neq 0. \tag{5.20}$$

Note $F(U) \subset V_2 \subset \widetilde{V}_2$. We have

$$\varphi_2(F(Z), \overline{F(Z)}) \equiv 0, \text{ for all } Z \in U.$$

Consequently,

$$\widetilde{\varphi}_2(F(Z), \overline{F(Z)}) \equiv 0. \tag{5.21}$$

Recall that for all $Z \in U$,

$$\rho(F, \overline{F}) = 0, \tag{5.22}$$

$$L_i \rho(F, \overline{F}) = 0, 1 \leq i \leq n-1. \tag{5.23}$$

Note that

$$\left|\begin{array}{c} \rho_W(F,\overline{F}) \\ L_1\rho_W(F,\overline{F}) \\ \ldots \\ L_{n-1}\rho_W(F,\overline{F}) \\ (\widetilde{\varphi}_2)_W(F,\overline{F}) \end{array}\right|_{Z=0} \tag{5.24}$$

is nondegenerate. This implies that equations (5.21), (5.22), (5.23) forms a nondegenerate system for F. Then by a similar argument as in Proposition 5.3, it follows that F is real-analytic (resp. smooth) at 0.

We now consider the case when $l = 4$. Notice that if there exists a sequence $\{p_i\}_{i=1}^{\infty}$ converging to p_0 such that M' is at most $3-$nondegenerate at $F(p_i), i \geq 1$, then the conclusion is established by the argument above. Thus we only need to consider the case when there exists a neighborhood U of p_0 such that $F(U) \subset V_3$, where V_3 denotes the set of 3-degeneracy of M' near q_0. That is,

$$V_3 = \{q \in V : M' \text{ is 3-degenerate at } q\}$$

for a small neighborhood V of q_0 on M'. Since M' is 4-nondegenerate at q_0, then $\Delta_{i_0j_0k_0l_0}(0) \neq 0$, for some $1 \leq i_0 \leq j_0 \leq k_0 \leq l_0 \leq n$. By a similar argument as in the case $l = 3$, we are able to prove $i_0 = j_0 = k_0 = l_0 = n$, and then furthermore arrive at the desired conclusion.

By an inductive argument, we obtain the proof of Theorem 2 in the general case.

□

6. Proof of Theorem 3

We are now going to prove Theorem 3. Fix $p_0 \in M$ and let $q_0 = F(p_0) \in M'$. We will show below that we can apply Theorem 2 for $q_0 \in M' \setminus X$ for some complex variety X in $\mathbb{C}^{n+1}$; by the transversality of F, the set $F^{-1}(M' \setminus X)$ is open and dense in M, and the statement of Theorem 3 follows.

The following theorem gives the missing claim in the above argument. Let V be a small neighborhood of q_0 in $\mathbb{C}^{n+1}$. We first need to show that

Theorem 4. *M' is finitely nondegenerate near q_0 away from a complex analytic variety X in V.*

In order to do so, we shall first state and prove a useful general fact. For this, let $M \subset \mathbb{C}^N$ be a generic real-analytic submanifold of CR dimension n and real codimension d (i.e. $N = n + d$). We denote the set of germs at $p \in M$ of real-analytic functions on M with $\mathbb{C}\{M\}_p$. We say that an ideal $I \subset \mathbb{C}\{M\}_p$ is $\bar{\partial}_b$-closed if for any CR vector field L on M and any $f \in I$ we have that $Lf \in I$. For any ideal $I \subset \mathbb{C}\{M\}_p$, we denote by $\mathcal{V}(I)$ the germ of the real-analytic subset of M given by the vanishing of all elements of I.

Proposition 6.1. *Let $I \subset \mathbb{C}\{M\}_p$ be a ideal which is $\bar{\partial}_b$-closed. Then there exists a neighborhood U of p in $\mathbb{C}^N$ and a complex subvariety $V \subset U$ such that, in the sense of germs at p, $V \cap M = \mathcal{V}(I)$.*

Proof. We choose normal coordinates $(z,w)\in\mathbb{C}^n\times\mathbb{C}^d$ for M at p; in these coordinates, $p=0$ and M is defined by

$$w = Q(z,\bar z,\bar w),$$

where $Q=(Q^1,\dots,Q^d)$ is a holomorphic map with values in $\mathbb{C}^d$, defined in a neighborhood of $(0,0,0)\in\mathbb{C}^n\times\mathbb{C}^n\times\mathbb{C}^d$, satisfying

$$Q(z,0,\bar w)=Q(0,\bar z,\bar w)=\bar w,\quad Q(z,\bar z,\bar Q(\bar z,z,w))=w. \tag{6.1}$$

A basis of the CR vector fields on M near 0 is given by

$$L_j=\frac{\partial}{\partial\bar z_j}+\sum_{k=1}^d \bar Q^k_{\bar z_j}(\bar z,z,w)\frac{\partial}{\partial\bar w_k}.$$

As usual, we use multtiindex notation and for $\alpha=(\alpha_1,\dots,\alpha_n)$ we write $L^\alpha=L_1^{\alpha_1}\cdots L_n^{\alpha_n}$.

Let $f\in\mathbb{C}\{M\}_p$. There exists a holomorphic function $F(z,w,\chi,\tau)$ defined in a neighborhood of $(0,0,0,0)\in\mathbb{C}^n\times\mathbb{C}^d\times\mathbb{C}^n\times\mathbb{C}^d$ such that $f(z,w,\bar z,\bar w)=F(z,w,\bar z,\bar Q(\bar z,z,w))$ for $(z,w)\in M$. For any such f, we denote by $\varphi_f(z,w,\chi)$ the right hand side of the above equation. We note that $\frac{\partial^{|\alpha|}\varphi_f}{\partial\chi^\alpha}(z,w,\bar z)=\varphi_{L^\alpha f}(z,w,\bar z)$. We also note that we can write

$$\begin{aligned}F(z,w,\chi,\bar Q(\chi,z,w)) &= \sum_\alpha \frac{1}{\alpha!}\frac{\partial^{|\alpha|}}{\partial\chi^\alpha}F(z,w,\chi,\bar Q(\chi,z,w))\bigg|_{\chi=0}\chi^\alpha\\ &=\sum_\alpha\frac{1}{\alpha!}L^\alpha F(z,w,0,w)\chi^\alpha\\ &=\sum_\alpha\frac{1}{\alpha!}\frac{\partial^{|\alpha|}\varphi_f}{\partial\chi^\alpha}(z,w,0)\chi^\alpha.\end{aligned}\tag{6.2}$$

So assume that we have chosen a small neighborhood of 0, such that inside this neighborhood, $\mathcal{V}(I)$ is defined by an ideal $\tilde I$ of functions $f(z,w,\bar z,\bar w)$ extending holomorphically to a common neighborhood of $(0,0,0,0)\in\mathbb{C}^n\times\mathbb{C}^d\times\mathbb{C}^n\times\mathbb{C}^d$. We claim that $\mathcal{V}(\tilde I)=\{(z,w)\colon \varphi_f(z,w,0)=0,\ f\in\tilde I\}$.

Let $Z_0=(z_0,w_0)\in\mathcal{V}(\tilde I)$, and let $f\in\tilde I$. Then the holomorphic function $\chi\mapsto\varphi_f(z_0,w_0,\chi)$ vanishes to infinite order at $\chi=\bar z_0$; hence also $\varphi_f(z_0,w_0,0)=0$. Assume now that $\varphi_g(z_0,w_0,0)=0$ for every $g\in\tilde I$. Then by (6.2), if $f\in\tilde I$ is arbitrary, then $f(z_0,w_0,\bar z_0,\bar w_0)=0$. Hence, $\mathcal{V}(\tilde I)=\{(z,w)\colon\varphi_f(z,w,0)=0,\ f\in\tilde I\}$ as claimed. □

The proof of Theorem 4 is a combination of Proposition 6.1 with the following fact.

Lemma 6.2. *Let $X\subset M$ be the set of points p in M at which M is not finitely nondegenerate of any order k. Then X can be defined, near every point $p\in M$, by an ideal which is $\bar\partial_b$-closed.*

Proof. Let $p\in M$, and let $Z=(Z_1,\dots,Z_N)$ be coordinates near p. We note that M is k-nondegenerate if the space $E_k(p)$ has dimension N, where

$$E_0=\Gamma(M,T^0M),\quad E_k=E_{k-1}+\{\mathcal{L}_L\omega\colon\omega\in E_{k-1}, L\text{ CR}\}.$$

Here T^0M denotes the characteristic bundle of M and $\mathcal{L}$ the Lie derivative (of forms). It turns out that $E_k \subset \Gamma(M, T'M)$, where $T'M$ is the bundle of holomorphic forms on M. We have that

$$T'M = \langle dZ_1, \ldots, dZ_N \rangle.$$

We note that for

$$\omega = \sum_{j=1}^{N} \omega^j dZ_j,$$

it holds that

$$\mathcal{L}_L \omega = \sum_{j=1}^{N} (L\omega^j) dZ_j.$$

Choose a basis of characteristic forms $\theta_j = \sum_{k=1}^N \theta_j^k dZ_k$, where $j = 1, \ldots, d$. The space E_k is therefore spanned by forms of the form

$$\mathcal{L}_{L^\alpha} \theta_j = \mathcal{L}_{L_1}^{\alpha_1} \cdots \mathcal{L}_{L_n}^{\alpha_n} \theta_j = \sum_{k=1}^{N} (L^\alpha \theta_j^k) dZ_k, \quad j = 1, \ldots, d, \quad |\alpha| \le k.$$

We therefore have that M is not ℓ-nondegenerate for some $\ell \le k_0$ at p if and only if for every choice $r = (r_1, \ldots, r_N)$ of integers $r_k \in \{1, \ldots, d\}$ and for every choice of multiindeces $A = (\alpha^1, \ldots, \alpha^N)$, where $\alpha^j = (\alpha_1^j, \ldots, \alpha_N^j)$ satisfies $|A| = \max\{|\alpha^j| \colon j = 1, \ldots, N\} \le k_0$, the determinant

$$D(A, r) = \begin{vmatrix} L^{\alpha^1} \theta_{r_1}^1 & \ldots & L^{\alpha^1} \theta_{r_1}^N \\ \vdots & & \vdots \\ L^{\alpha^N} \theta_{r_N}^1 & \ldots & L^{\alpha^N} \theta_{r_N}^N \end{vmatrix}$$

vanishes at p; that is, if we denote by X_{k_0} the set of all points where p is not ℓ-nondegenerate where $\ell \le k_0$, then X_{k_0} is defined by the ideal

$$I_{k_0} = \left(\{ D(A, r) \colon |A| \le k_0 \} \right).$$

Note that $LI_k \subset I_{k+1}$. The set $X = \cap_k X_k$ is now defined by $I = \cup_k I_k$, which is $\bar{\partial}_b$-closed. $\square$

By combining Proposition 6.1 and Lemma 6.2, we obtain the result in Theorem 4. Now the proof of Theorem 3 follows by combining Theorem 4 and the argument in the beginning of the section.

$\square$

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